The fragment of elementary plane Euclidean geometry based on perpendicularity alone with complexity PSPACE-complete

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A. Tarski uses in his system for the elementary geometry only the primitive concept of *point*, and the two primitive relations *betweenness* and *equidistance*. Another approach is the only primitive concept to be *line*. W. Schwabhäuser and L. Szczerba showed that *perpendicularity* together with the ternary relation of *co-punctuality* (three lines intersect at one point) are sufficient for dimension two, i.e. they may be used as a system of primitive relations for elementary plane Euclidean geometry. We consider the fragment based on perpendicularity alone. Its theory is not finitely axiomatizable, it is decidable and the complexity is PSPACE-complete. In contrast the complexity of elementary plane Euclidean geometry is exponential.

We consider first-order language $\mathcal{L}$ with predicate symbols O (meaning perpendicularity) and $=$, without constants and functional symbols. We consider the theory **OFPEG** (meaning "the orthogonal fragment of elementary plane Euclidean geometry"), containing the following formulas:

$\lambda_{\mathbf{1}}: \ \forall x \neg O(x, x)$

$\lambda_{\mathbf{2}}: \ \forall x \forall y (O(x, y) \rightarrow O(y, x))$

$\lambda_{\mathbf{3}}: \ \forall x \exists y O(x, y)$

$\lambda_{\mathbf{4}}: \ \forall x \forall y \forall z \forall t (O(x, z) \wedge O(y, z) \wedge O(x, t) \rightarrow O(y, t))$

$\lambda_{\mathbf{5n}}: \ \forall y_1 \ldots \forall y_n \exists s (\neg O(s, y_1) \wedge \ldots \wedge \neg O(s, y_n)), \ n \geq 2$

$\lambda_{\mathbf{6n}}: \ \forall y_1 \ldots \forall y_n \forall s (O(s, y_1) \wedge \ldots \wedge O(s, y_n) \rightarrow \exists t (t \neq y_1 \wedge \ldots \wedge t \neq y_n \wedge O(s, t))), \ n \geq 1$

Remark

We denote by $\mathcal{F}^2_{\mathbb{R}}$ the structure for the language $\mathcal{L}$, having universe the set of all lines in Euclidean plane. Clearly $\mathcal{F}^2_{\mathbb{R}} \models$ OFPEG.

Remark

Let φ be a first-order formula with free variables $x_1, \dots, x_n$, $\mathcal{A}$ be a structure for the language of φ and $a_1, \dots, a_n \in A$. By $\mathcal{A} \models \varphi[a_1, \dots, a_n]$ we denote that φ is true in $\mathcal{A}$ under valuation, assigning $a_1, \dots, a_n$ to $x_1, \dots, x_n$.

Remark

We will denote $a_1, \dots, a_n$ by $\overline{a}$ and $f(a_1), \dots, f(a_n)$ by $\overline{f(a)}$.

Proposition

Let $\mathcal{A}$ be a model of OFPEG. We consider the relation $\mathbf{R_1}$*, defined in the following way:*

$$xR_1y \stackrel{def}{\Longleftrightarrow} \text{for every } z,\ \neg O(x,z) \text{ or } O(y,z)$$

(Intuitively xR_1y means "x does not intersect y".)
R_1 is an equivalence relation which divides A into infinitely many infinite equivalence classes.

Definition

Let $\mathcal{A}$ be a model of *OFPEG*, R_1 be the relation from the previous proposition. We consider the equivalence classes modulo R_1 and the following relation:

$$[x]\mathbf{R_2}[y] \overset{def}{\Longleftrightarrow} O(x, y)$$

It can be easily verified that this definition is correct and the following proposition holds:

Proposition

Let $\mathcal{A}$ be a model of OFPEG. Then for every equivalence class modulo R_1 $[x]$ there is exactly one equivalence class $[y]$ such that $[x]R_2[y]$. Moreover $[x] \neq [y]$.

Proposition

Let $\mathcal{A}$ be a countable model of OFPEG, $\mathcal{B}$ be a model of OFPEG. Then $\mathcal{A}$ is elementary embedded in $\mathcal{B}$.

Proof. Since $\mathcal{A}$ is a countable model of *OFPEG*, *A* has countably many equivalence classes. Let these equivalence classes be $[a_1], [a_2], \ldots, [a_n], \ldots, [a'_1], [a'_2], \ldots, [a'_n], \ldots$, all they being different and $[a_1]R_2[a'_1], [a_2]R_2[a'_2], \ldots, [a_n]R_2[a'_n], \ldots$. There exist countably many equivalence classes of B $[b_1]$, $[b_2], \ldots, [b_n], \ldots;\ [b'_1], [b'_2], \ldots, [b'_n], \ldots$ such that all they are different and $[b_1]R_2[b'_1], [b_2]R_2[b'_2], \ldots, [b_n]R_2[b'_n], \ldots$. For every n, $[a_n]$ and $[a'_n]$ are countable; $[b_n]$ and $[b'_n]$ are infinite. Consequently for every n, there are injections $h_n : [a_n] \to [b_n]$ and $h'_n : [a'_n] \to [b'_n]$.

We define the mapping f:

$$f(a) = \begin{cases} h_n(a) & \text{if } a \in [a_n] \text{ for some } n \\ h'_n(a) & \text{if } a \in [a'_n] \text{ for some } n \end{cases}$$

We will prove that f is an elementary embedding of $\mathcal{A}$ in $\mathcal{B}$. By induction on φ we will prove that for every formula φ, if φ has free variables among $x_1, \ldots, x_n$ and $c_1, \ldots, c_n \in A$, then $\mathcal{A} \models \varphi[\overline{c}] \Leftrightarrow \mathcal{B} \models \varphi[\overline{f(c)}]$.

• The base of induction, the negation and the conjunction are trivial.

• φ is $\exists x \varphi_1$
Let the free variables of φ be among $x_1, \ldots, x_n$ and $c_1, \ldots, c_n \in A$. It can be easily verified that $\mathcal{A} \models \exists x \varphi_1[\overline{c}]$ implies $\mathcal{B} \models \exists x \varphi_1[\overline{f(c)}]$.
Let $\mathcal{B} \models \exists x \varphi_1[\overline{f(c)}]$. We will prove $\mathcal{A} \models \exists x \varphi_1[\overline{c}]$. There exists $b \in B$ such that $\mathcal{B} \models \varphi_1[b, f(c_1), \ldots, f(c_n)]$

Interesting is the case when $b \notin [b_i]$ and $b \notin [b'_i]$ for every i
Let $k_1, \ldots, k_l \in A$ ($l \leq n$) be such that for every $i = 1, \ldots, l$, $[k_i]R_2[c_j]$ for some $j \in \{1, \ldots, n\}$. There is $a \in A$ such that $a \notin [c_1], \ldots, [c_n],\ [k_1], \ldots, [k_l]$. We add to the language $\mathcal{L}$ the constants d, $d_1, \ldots, d_n$. Interpreting the new constants by b, $f(c_1), \ldots, f(c_n)$ or by $f(a)$, $f(c_1), \ldots, f(c_n)$, we obtain two structures for the extended language which we denote by $\mathcal{B}' = (\mathcal{B}, b, f(c_1), \ldots, f(c_n))$ and $\mathcal{B}'' = (\mathcal{B}, f(a), f(c_1), \ldots, f(c_n))$ correspondingly. Let $[m_1]$ be the only equivalence class for which $[b]R_2[m_1]$, and $[m_2]$ be the only equivalence class for which $[f(a)]R_2[m_2]$.

We consider Ehrenfeucht-Fraïssè's game with arbitrary finite length s and the following strategy for the second player: if the first player chooses b ($f(a)$), then the second player chooses $f(a)$ (b). Otherwise, if the first player chooses an element out of $[b] \cup [f(a)] \cup [m_1] \cup [m_2]$, then the second player chooses the same element; if the first player chooses a new element of $[b]$, then the second player chooses a new element of $[f(a)]$, different from $f(a)$ and the converse; if the first player chooses a new element of $[m_1]$, then the second player chooses a new element of $[m_2]$ and the converse; if the first player chooses already chosen in the corresponding structure element x in $[b] \cup [f(a)] \cup [m_1] \cup [m_2]$, then the second player chooses the same element which then was chosen in the other structure.

Let $e_1, \ldots, e_s$ and $e'_1, \ldots, e'_s$ be correspondingly of $\mathcal{B}'$ and of $\mathcal{B}''$ in the order of their choosing. Let
$h = \{\langle e_i, e'_i\rangle :\ i = 1, \ldots, s\} \cup \{\langle C^{\mathcal{B}'}, C^{\mathcal{B}''}\rangle :$
C - a constant of the extended language$\}$. Let $\mathcal{B}'_1$ be the substructure of $\mathcal{B}'$, generated by $e_1, \ldots, e_s$, b, $f(c_1), \ldots, f(c_n)$; $\mathcal{B}''_1$ be the substructure of $\mathcal{B}''$, generated by $e'_1, \ldots, e'_s$, $f(a)$, $f(c_1), \ldots, f(c_n)$. It can be easily verified that h is an isomorphism from $\mathcal{B}'_1$ to $\mathcal{B}''_1$. Consequently for every closed formula φ we have $\mathcal{B}' \models \varphi \Leftrightarrow \mathcal{B}'' \models \varphi$; so
$\mathcal{B} \models \varphi_1[b, f(c_1), \ldots, f(c_n)] \Leftrightarrow \mathcal{B} \models \varphi_1[f(a), f(c_1), \ldots, f(c_n)]$. But we have $\mathcal{B} \models \varphi_1[b, f(c_1), \ldots, f(c_n)]$ and thus
$\mathcal{B} \models \varphi_1[f(a), f(c_1), \ldots, f(c_n)]$. From the induction hypothesis,
$\mathcal{A} \models \varphi_1[a, c_1, \ldots, c_n]$.

Consequently f is an elementary embedding of $\mathcal{A}$ in $\mathcal{B}$. □

Corollary

The theory OFPEG is complete.

Proof. Let $\mathcal{F}^2_{\mathbb{Q}}$ be the structure for the language $\mathcal{L}$ with universe $\{a$ - line in Euclidean plane: at least 2 of the points of a are with rational coordinates $\}$. The predicate symbol O is interpreted by perpendicularity. It can be proved that $\mathcal{F}^2_{\mathbb{Q}} \models$ *OFPEG*.

Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary models of *OFPEG*. Since $\mathcal{F}^2_{\mathbb{Q}}$ is a countable model of *OFPEG*, $\mathcal{F}^2_{\mathbb{Q}}$ is elementary embedded in $\mathcal{A}$ and in $\mathcal{B}$. Consequently $\mathcal{F}^2_{\mathbb{Q}}$ is elementary equivalent to $\mathcal{A}$ and to $\mathcal{B}$ and hence $\mathcal{A}$ and $\mathcal{B}$ are elementary equivalent. Consequently *OFPEG* is complete. □

We denote by P the binary predicate, meaning parallelism and by C - the predicate, meaning that two lines intersect.

- P and C are definable by O and $=$;
- O is not definable by P, C and $=$.

Proposition

The problem of whether a closed formula in $\mathcal{L}$ logically follows from OFPEG is PSPACE-complete.

Proof

We consider
$EQ^{\infty} = \{\varphi : \ \varphi$ is a closed formula in the language $\mathcal{L}_1 = \langle\ ;\ ;=\ \rangle$ and φ is true in all infinite structures$\}$. The membership problem in EQ^{∞} is PSPACE-complete.

Let $\boldsymbol{\mathcal{A}}^{*}$ be the substructure of $\mathcal{F}^2_{\mathbb{R}}$, which is obtained by eliminating from the universe the lines parallel to or coinciding with the abscissa axis, the lines parallel to or coinciding with the ordinate axis and the lines with equation of the kind $y = bx$.

Lemma

The structure $\mathcal{A}^{*}$ is a model of OFPEG.

Proof

Lemma

For every closed formula φ in the language $\mathcal{L}$, OFPEG $\models \varphi$ iff $\mathcal{A}^* \models \varphi$.

Let $\boldsymbol{\mathcal{R}}$ be the structure for $\mathcal{L}_1$ with universe $\mathbb{R} \setminus \{0\}$.

Lemma

For any closed formula φ in $\mathcal{L}_1$, $\mathcal{R} \models \varphi$ iff $\varphi \in EQ^{\infty}$.

Proof

Let a be a line with equation $y = bx + c$. We use the following notations: $\boldsymbol{a^1} = b$, $\boldsymbol{a^2} = -\frac{1}{b}$, $\boldsymbol{a^3} = c$. It is convenient to call a^1, a^2 and a^3 *coordinates* of the line a.

To every formula φ in the language $\mathcal{L}$ we juxtapose a formula $\widehat{\varphi}$ in the language $\mathcal{L}_1$ in the following way:

1) φ - atomic

($\bullet$) If φ is $O(x_1, x_2)$, then $\widehat{\varphi}$ is $x_1^1 = x_2^2$.

($\bullet$) If φ is $x_1 = x_2$, then $\widehat{\varphi}$ is $(x_1^1 = x_2^1) \wedge (x_1^3 = x_2^3)$.

Proof

2) φ **is** $\neg\varphi'$
$\widehat{\varphi}$ is $\neg\widehat{\varphi'}$.

3) φ **is** $\varphi' \wedge \varphi''$
$\widehat{\varphi}$ is $\widehat{\varphi'} \wedge \widehat{\varphi''}$.

4) φ **is** $\exists x_n \varphi'$ **and** φ' **has free variables** $x_1, \ldots, x_n$
$\widehat{\varphi}$ is $\exists x_n^1 \exists x_n^2 \exists x_n^3 (\widehat{\varphi'} \wedge \kappa_n)$, where for every natural number n, κ_n is a formula with free variables x_1^1, $x_1^2, \ldots, x_n^1$, x_n^2, defined in the following way:

$$\kappa_n : \; x_n^1 \neq x_n^2 \wedge \bigwedge_{i<n} (x_i^1 = x_n^1 \leftrightarrow x_i^2 = x_n^2) \wedge \bigwedge_{i<n} (x_i^1 = x_n^2 \leftrightarrow x_i^2 = x_n^1)$$

Definition

Let n be a natural number, a_1^1, a_1^2, $a_1^3, \ldots, a_n^1$, a_n^2, $a_n^3 \in \mathbb{R} \setminus \{0\}$ and for every $i \in \{1, \ldots, n\}$, $\mathcal{R} \models \kappa_i[a_1^1, a_1^2, \ldots, a_i^1, a_i^2]$. We say that b_1^1, b_1^2, $b_1^3, \ldots, b_n^1$, b_n^2, b_n^3 are *corresponding* to a_1^1, a_1^2, $a_1^3, \ldots, a_n^1$, a_n^2, a_n^3 if for every $i \in \{1, \ldots, n\}$,

1) b_i^1, b_i^2, $b_i^3 \in \mathbb{R} \setminus \{0\}$

2) if $a_i^1 \notin \{a_1^1, a_1^2, \ldots, a_{i-1}^1, a_{i-1}^2\}$, then $b_i^1 \notin \{b_1^1, b_1^2, \ldots, b_{i-1}^1, b_{i-1}^2\}$

3) if $a_i^1 = a_k^1$ for some $k \in \{1, \ldots, i-1\}$, then $b_i^1 = b_k^1$

4) if $a_i^1 = a_k^2$ for some $k \in \{1, \ldots, i-1\}$, then $b_i^1 = b_k^2$

5) $b_i^2 = -\frac{1}{b_i^1}$

6) $b_i^3 = a_i^3$

Proof

Lemma

Let $b_1^1, b_1^2, b_1^3, \ldots, b_n^1, b_n^2, b_n^3$ *be corresponding to* $a_1^1, a_1^2, a_1^3, \ldots, a_n^1, a_n^2, a_n^3$*. Then for any* j *and* i*, if* $1 \leq i < j \leq n$*, then*
1) $b_j^1 = b_i^1$ *implies* $a_j^1 = a_i^1$*;*
2) $b_j^1 = b_i^2$ *implies* $a_j^1 = a_i^2$*.*

Lemma

Let $b_1^1, b_1^2, b_1^3, \ldots, b_n^1, b_n^2, b_n^3$ *be corresponding to* $a_1^1, a_1^2, a_1^3, \ldots, a_n^1, a_n^2, a_n^3$ *and* φ *be* $O(x, y)$ *or* $x = y$*. Then for any* $i, j \in \{1, \ldots, n\}$, $\mathcal{R} \models \widehat{\varphi}[a_i^1, a_i^2, a_i^3, a_j^1, a_j^2, a_j^3]$ *iff* $\mathcal{R} \models \widehat{\varphi}[b_i^1, b_i^2, b_i^3, b_j^1, b_j^2, b_j^3]$*.*

Proof

Lemma

Let b_1^1, b_1^2, $b_1^3, \ldots, b_n^1$, b_n^2, b_n^3 be corresponding to a_1^1, a_1^2, $a_1^3, \ldots, a_n^1$, a_n^2, a_n^3 and $1 \leq j_1 \leq j_2 \leq \ldots \leq j_m \leq n$. Then $b_{j_1}^1$, $b_{j_1}^2$, $b_{j_1}^3, \ldots, b_{j_m}^1$, $b_{j_m}^2$, $b_{j_m}^3$ are corresponding to $a_{j_1}^1$, $a_{j_1}^2$, $a_{j_1}^3, \ldots, a_{j_m}^1$, $a_{j_m}^2$, $a_{j_m}^3$.

Lemma

Let φ be a formula for $\mathcal{L}$ with free variables $x_1, \ldots, x_m$. Let b_1^1, b_1^2, $b_1^3, \ldots, b_n^1$, b_n^2, b_n^3 be corresponding to a_1^1, a_1^2, $a_1^3, \ldots, a_n^1$, a_n^2, a_n^3. Then for any $j_1, \ldots, j_m \in \{1, \ldots, n\}$,
$\mathcal{R} \models \widehat{\varphi}[a_{j_1}^1, a_{j_1}^2, a_{j_1}^3, \ldots, a_{j_m}^1, a_{j_m}^2, a_{j_m}^3]$ iff
$\mathcal{R} \models \widehat{\varphi}[b_{j_1}^1, b_{j_1}^2, b_{j_1}^3, \ldots, b_{j_m}^1, b_{j_m}^2, b_{j_m}^3]$.

Proof

Lemma

Let φ be a formula in $\mathcal{L}$ with free variables $x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in A^*$. Then $\mathcal{A}^* \models \varphi[a_1, \ldots, a_n]$ iff $\mathcal{R} \models \widehat{\varphi}[a_1^1, a_1^2, a_1^3, \ldots, a_n^1, a_n^2, a_n^3]$.

It can be easily verified that $\widehat{\varphi}$ can be obtained algorithmically from φ by using of memory, polynomial in the size of φ. *OFPEG* $\models \varphi$ iff $\mathcal{A}^* \models \varphi$ iff $\mathcal{R} \models \widehat{\varphi}$ iff $\widehat{\varphi} \in EQ^\infty$. Thus the membership problem in OFPEG is in PSPACE.

Proof

Lemma

For every closed formula in $\mathcal{L}_1 = \langle\ ;\ ;=\rangle$ φ_1, $\varphi_1 \in$ *EQ*$^\infty$ *iff* *OFPEG* $\models \varphi_1$.

We have also that the membership problem in *EQ*$^\infty$ is PSPACE-hard. Consequently the membership problem in *OFPEG* is PSPACE-hard.

Proposition

The theory OFPEG is not finitely axiomatizable.

Proposition

The theory OFPEG is ω-categorical.

Proposition

The theory OFPEG is not α-categorical for every uncountable cardinality α.

Thank you very much!