On Some Elementary Theories for Rotation in the Line-Based Euclidean Plane II: The Unoriented Irrational Angle

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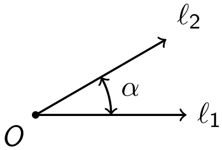
Introduction to the main problem

Based on the type of the fixed angle α and the symmetry of the relation "angle between two lines is α" we explore four theories. We provide suitable axiomatization and prove results concerning expressiveness, completeness, categoricity and complexity.

Our current case - $\mathbf{EL}^{\mathrm{u}}(\infty)$

Now we take into account the case when α is undirected irrational multiple of π.



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  $\forall x_1 \ldots \forall x_n \exists y (P(x_1,x_2) \wedge \cdots \wedge P(x_{n-1},x_n) \implies$
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- Axiom for limiting the directions for rotation δ_6 :
  $\forall x \forall y \forall z (\mathbf{R}(x,y) \wedge \mathbf{R}(x,z) \wedge \neg P(y,z) \implies \forall u (\mathbf{R}(x,u) \implies P(y,u) \vee P(z,u)))$

The distinctive axiom scheme

Axiom scheme for extensibility:

$$\delta'_{7,k} : \forall y_0 \forall y_1 \ldots \exists y_k (\mathbf{R}(y_0, y_1) \wedge \cdots \wedge \mathbf{R}(y_{k-1}, y_k) \wedge \bigwedge_{0 \leq i < j \leq k-1} \neg P(y_i, y_j))$$

$$\implies \bigwedge_{0 \leq i < k} \neg P(y_i, y_k))$$

Paths

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Definition

Let $\mathcal{M}$ be a model of $\mathbf{EL}^{\mathrm{u}}(\infty)$ and $a_0 \in M$. An *infinite path* with start a_0 will be called any infinite sequence of elements $(a_0, a_1, a_2 \ldots, a_n, \ldots)$ from M such that $\mathbf{R}(a_i, a_{i+1})$ for all $i \in \mathbb{N}$. Furthermore, no two elements in an infinite path should be parallel.

Definition

Two infinite paths with start a, say $(a, a_1, \ldots, a_n, \ldots,)$ and $(a, b_1, \ldots, b_n, \ldots)$ will be called *coinciding* if for each natural i holds that $P(a_i, b_i)$. They will be called *different* if for no i holds $P(a_i, b_i)$.

Paths

Proposition

Let $\mathcal{M} \models \mathbf{EL}^{\mathrm{u}}(\infty)$*. The following claims hold as a direct consequences of the axioms:*

1. *Any two infinite paths with the same start are either coinciding or different.*
2. *For each element a there are exactly two different infinite paths with start a and any other infinite path coincides with one of the two.*

Categoricity

Theorem

$\mathbf{EL}^{\mathrm{u}}(\infty)$ *is not* α*-categorical for any cardinal number* $\alpha \geq \omega$*.*

Proof relies on dividing the structure into equivalence classes and taking into consideration two models with different cardinal number for the classes and different cardinality for each class.

Ehrenfeucht-Fraïssé games on the models of $\mathbf{EL}^{\mathrm{u}}(\infty)$

Definition

Again, a countable model of $\mathbf{EL}^{\mathrm{u}}(\infty)$ that consists of the orbit of only one element will be called *star model*. This model is unique up to isomorphism.

The game is played on the star model and another random one.

Proposition

Let $\mathcal{M} \models \mathbf{EL}^{\mathrm{u}}(\infty)$. For all integers n Abelard has a winning strategy for the Ehrenfeucht-Fraïssé game with length n played on S and M.

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Theorem

$\mathbf{EL}^{\mathrm{u}}(\infty)$ *is complete. Furthermore,* $\mathbf{EL}^{\mathrm{u}}(\infty) = Th(\mathrm{PLE}^{\mathrm{u}}(\alpha))$ *when* α *is not co-measurable with* π.

Complexity

Using the following translation:

$\mathrm{PLE}^{\mathrm{u}}(\alpha) \models \mathbf{R}(x, y)[\![a, b]\!]$ if and only if $\mathrm{PLE}^{\mathrm{d}}(\alpha) \models R(x, y) \vee R(y, x)[\![a, b]\!]$

we can easily find for each $\mathcal{L}_2-$formula a $\mathcal{L}_1-$formula and thus the complexity will be inherited. We have proven that $\mathbf{EL}^{\mathrm{d}}(\infty)$ is PSPACE-complete, so should $\mathbf{EL}^{\mathrm{u}}(\infty)$ be.

Citations

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A special greeting for prof. Tinchev - the ∀∃ puzzle

Rules: You are given a board $4\times$ 4 that should be colored in three colors- 1,2,3. In some cells are written special formulas. Your task is to colour all cells (including those with quantifiers) according to the formulas. One cell validates the formula $\exists$ if and only if there is another cell in the same column or the same row that contains the same number. Similarly, one cell validates the formula $\forall$ if and only if all other cells in its row and column contain the same number. Inductively, one cell can validate formulas of higher rank such as $\forall\exists, \exists\forall$.

The puzzle itself

∃∀	1	2	3
	∃∀		∀
∃∀	1	∀∀∃	
3	∀¬∀	1	