Logic of Ternary Contact

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Outline

1. Logics of *n*-ary Contact
   - Objective
   - Formal System
   - Completeness of the Formal System
2. Logics of Ternary Contact
   - Formal System
   - Modal Definability
   - Unification

Outline

The Logic of *n*-ary Contact

- In general, *Region-based Theory of Space* studies "*part_of*" and "*contact*" relations between regions. Usually regions are regular closed sets, $\mathcal{RC}(\mathcal{T})$, in given topological space $\mathcal{T}$.
- If X and Y are regular closed sets, then "*part_of*" and "*contact*" are interpreted as $X \subset Y$ and $X \cap Y \neq 0$.
- Recall that regular closed sets with the set-theoretical inclusion "$\subset$" form a *complete Boolean algebra* and the *meet* and *complement* are *not* the set-theoretical intersection and complement.

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The Logic of *n*-ary Contact

- We extend the language by adding (the notion of) *n*-ary contact for any $n > 2$, interpreted as:

$$C_n(X_1, \ldots, X_n) \quad \textit{iff} \quad X_1 \cap \ldots \cap X_n \neq 0.$$

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Polytopes

- The *polytopes* are the generated by the finite intersections of half-spaces Boolean subalgebra of the Boolean algebra of the regular closed sets of $\mathbb{R}^m$.

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Language of *n*-ary Contact

- A quantifier free fragment of a first-order language.
- **Nonlogical symbols**: the Boolean constants and operations $(0, -, \cup)$.
- **Predicate symbols**: one *n*-ary symbol per every positive integer $n > 1$ $(R_2, R_3, \ldots, R_n, \ldots)$.

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Algebraic Semantics

- Boolean algebra B with n-ary relations R_n, $n > 1$, called *Boolean frame*, satisfying the following conditions:
    - If $R_n(a_1, \dots, a_n)$, then for every mapping $\sigma : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ we have $R_n(a_{\sigma(1)}, \dots, a_{\sigma(n)})$.
    - $R_n(a'_1 \cup a''_1, a_2, \dots, a_n)$ *iff* $R_n(a'_1, a_2, \dots, a_n)$ or $R_n(a''_1, a_2, \dots, a_n)$.
    - If $R_n(a_1, \dots, a_n)$, then $a_1 \neq 0$.

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Algebraic Semantics

- A Boolean frame satisfying $R_2(a, -a)$ for all $a \neq 0, 1$ is called *connected*.
- The intended models
  - Boolean subalgebras of $\mathcal{RC}(\mathbb{R}^m)$ or the polytopes of $\mathbb{R}^m$.
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Relational Semantics

- (Kripke) frames with a carrier (or set of worlds) W and n-ary relation for every $n > 1$.
- The semantics in such a structure is given in the set-theoretical Boolean algebra $B = \mathscr{P}(W)$.
- Such a semantic structure in essence contains sufficient information to generate the whole corresponding Boolean frame for the set-theoretical Boolean algebra $\mathscr{P}(W)$.

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Axiomatization

Base Axioms

- **Logical axioms**: sentential, identity and equivalence, congruence.
- **Boolean algebra axioms**: stipulating a non-degenerate Boolean algebra.
- **Proximity axioms**:
  - $R_n(x_1, ..., x_n) \Rightarrow x_1 \neq 0$
  - $R_n(x_1' \cup x_1'', x_2, ..., x_n) \Leftrightarrow R_n(x_1', x_2, ..., x_n) \vee R_n(x_1'', x_2, ..., x_n)$

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n-ary Contact Axioms

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(c1) $(\sigma : \{1, \dots, n\} \to \{1, \dots, n\})$

$$R_n(x_1, \dots, x_n) \Rightarrow R_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

(c2)

$$R_{n+1}(x_1, x_1, x_2 \dots, x_n) \Leftrightarrow R_n(x_1, x_2, \dots, x_n)$$

(c3)

$$\neg(x = 0) \Rightarrow R_2(x, x)$$

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$$\neg(x = 0) \wedge \neg(-x = 0) \Rightarrow R_2(x, -x)$$

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PRC1

$$R_3(x_1, x_2, x_3) \Rightarrow \neg(x_1 \cap x_2 = 0) \vee \neg(x_2 \cap x_3 = 0) \vee \neg(x_1 \cap x_3 = 0)$$

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Characterisation of the *n*-ary Contact

In the *regular closed sets* of the *connected* topological spaces

The following sets (logics) are equal:

- The theorems of the formal system of *n-ary (connected) Contact* with inference rules *uniform substitution* and *modus ponens* and axioms the base axioms and **(c1)** to **(c4)**.
- The formulas valid in the *Boolean frames* of the *polytopes* of $\mathbb{R}^m$ for $m \geq 2$.
- …in the *Boolean frames* of the *regular closed sets* of $\mathbb{R}^m$ for $m \geq 1$.
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As a consequence:

- The n-ary contact for $n > 2$ is not definable by 2-contact.

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Formal System

Language

- L_{R_3}: The language of the n-ary contact restricted to ternary predicate symbols. Recall:
  - A quantifier free fragment of a first-order language.
  - **Function symbols**
    The Boolean constants and operations: 0, $-$, $\cup$.
  - **Predicate symbols**
    One binary and one ternary symbols: R_2, R_3.

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Semantic Structures

Definition

Contact frame $\mathfrak{F} = \langle W, R_2, R_3 \rangle$

- W: nonempty
- R_2, R_3: binary and ternary relations on W such that
  - **(a)** If $R_n(w_1, \ldots, w_n)$, then for every mapping $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ we have $R_n(w_{\sigma(1)}, \ldots, w_{\sigma(n)})$
  - **(b)** $R_3(w_1, w_1, w_2) \leftrightarrow R_2(w_1, w_2)$
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Valuation

Definition

A *valuation on a contact frame* $\mathfrak{F}$

- A mapping $\mathcal{V}$ from the set of terms of L_{R_3} in $\mathcal{P}(W)$ such that:
  - For a variable x of L_{R_3} $\mathcal{V}(x)$ is a subset of W.
  - The values for terms of L_{R_3} are defined inductively with respect to the (standard) set-theoretical interpretation of the Boolean connectives.

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Formal System

Model

Definition

Model on a contact frame:

a pair $\langle \mathfrak{F}, \mathcal{V} \rangle$ of a contact frame $\mathfrak{F}$ and a valuation $\mathcal{V}$ on $\mathfrak{F}$.

Formal System

Truth Relation

Definition

$\langle \mathfrak{F}, \mathcal{V} \rangle \vDash \varphi$: φ is *true* in $\langle \mathfrak{F}, \mathcal{V} \rangle$

- $\langle \mathfrak{F}, \mathcal{V} \rangle \vDash \tau_1 = \tau_2$ *iff* $\mathcal{V}(\tau_1) = \mathcal{V}(\tau_2)$
- $\langle \mathfrak{F}, \mathcal{V} \rangle \vDash R_n(\tau_1, \ldots, \tau_n)$ *iff* they exist $w_1, \ldots, w_n$, such that $w_1 \in \mathcal{V}(\tau_1), \ldots, w_n \in \mathcal{V}(\tau_n)$ and $R_n(w_1, \ldots, w_n)$.
- $\langle \mathfrak{F}, \mathcal{V} \rangle \vDash \neg\varphi$ *iff* $\langle \mathfrak{F}, \mathcal{V} \rangle \nvDash \varphi$.
- $\langle \mathfrak{F}, \mathcal{V} \rangle \vDash \varphi_1 \vee \varphi_2$ *iff* $\langle \mathfrak{F}, \mathcal{V} \rangle \vDash \varphi_1$ or $\langle \mathfrak{F}, \mathcal{V} \rangle \vDash \varphi_2$.

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- $\langle \mathfrak{F}, \mathcal{V} \rangle \vDash \neg\varphi$ *iff* $\langle \mathfrak{F}, \mathcal{V} \rangle \nvDash \varphi$.
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Formal System

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$\langle \mathfrak{F}, \mathcal{V} \rangle \vDash \varphi \quad : \quad \varphi$ is *true* in $\langle \mathfrak{F}, \mathcal{V} \rangle$

- $\langle \mathfrak{F}, \mathcal{V} \rangle \vDash \tau_1 = \tau_2$ *iff* $\mathcal{V}(\tau_1) = \mathcal{V}(\tau_2)$
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Formal System

Truth Relation

Definition

$\langle \mathfrak{F}, \mathcal{V} \rangle \models \varphi$: φ is *true* in $\langle \mathfrak{F}, \mathcal{V} \rangle$

- $\langle \mathfrak{F}, \mathcal{V} \rangle \models \tau_1 = \tau_2$ *iff* $\mathcal{V}(\tau_1) = \mathcal{V}(\tau_2)$
- $\langle \mathfrak{F}, \mathcal{V} \rangle \models R_n(\tau_1, \dots, \tau_n)$ *iff* they exist $w_1, \dots, w_n$, such that $w_1 \in \mathcal{V}(\tau_1), \dots, w_n \in \mathcal{V}(\tau_n)$ and $R_n(w_1, \dots, w_n)$.
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Formal System

Validity

Definition

- $\mathfrak{F} \vDash \varphi$: φ is *valid* in $\mathfrak{F}$ if for every valuation $\mathcal{V}$ we have $\langle \mathfrak{F}, \mathcal{V} \rangle \vDash \varphi$
- $\mathcal{K} \vDash \varphi$: φ is *valid* in $\mathcal{K}$ if for every frame $\mathfrak{F}$ in $\mathcal{K}$ we have $\mathfrak{F} \vDash \varphi$,
  where $\mathcal{K}$ is a class of (contact) frames.

Formal System

Validity

Definition

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Formal System

Logic of the Ternary Contact Relational Structures

Definition

- $\mathcal{CF}^3$: the class of all contact frames.
- $\mathcal{L}(\mathcal{CF}^3) = \{\varphi \mid \mathcal{CF}^3 \vDash \varphi\}$: the *logic of the relational (ternary) contact structures*.

Formal System

Logic of the Ternary Contact Relational Structures

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- $\mathcal{CF}^3$: the class of all contact frames.
- $\mathcal{L}(\mathcal{CF}^3) = \{\varphi \mid \mathcal{CF}^3 \vDash \varphi\}$: the *logic of the relational (ternary) contact structures*.

Outline

1. Logics of *n*-ary Contact
    - Objective
    - Formal System
    - Completeness of the Formal System
2. Logics of Ternary Contact
    - Formal System
    - Modal Definability
    - Unification

Definability problems

Definition

Let $L(R_2, R_3)$ be the restriction of L_{R_3} by excluding all nonlogical functional symbols.

A contact frame $\mathfrak{F}$ can be considered as a structure for the first-order language $L(R_2, R_3)$.

The class $\mathcal{CF}^3$ can be considered as a class of structures of the first-order language $L(R_2, R_3)$.

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Definability problems

Definition

- $\models_m$ (or simply $\models$) : the *truth relation* defined above (from contact language perspective).
- $\models_{FO}$: the truth relation from *first-order language* perspective.

Definability problems

Definition

- $\vDash_m$ (or simply $\vDash$) : the *truth relation* defined above (from contact language perspective).
- $\vDash_{FO}$: the truth relation from *first-order language* perspective.

Definability problems

Definition

Modal definability

Let A be a closed formula from the first-order language $L(R_2, R_3)$.

Let φ be a formula from the ternary contact language L_{R_3}.

- φ is a *modal definition of A* in $\mathcal{CF}^3$

or (equivalently)

- A is *modally definable by* φ in $\mathcal{CF}^3$

if for every $\mathfrak{F}$ in $\mathcal{CF}^3$ we have

$$\mathfrak{F} \models_m \varphi \quad \leftrightarrow \quad \mathfrak{F} \models_{FO} A$$

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Definability problems

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Definability problems

Modal definability problem

Definition

Modal definability problem:

Input: Closed formula A of the first-order language $L(R_2, R_3)$.

Output: "A is modally definable in $\mathcal{CF}^3$"
or
"A is not modally definable in $\mathcal{CF}^3$"

Definability problems

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Definability problems

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Modal Definability Problem

Result

Modal Definability Problem Outcome

The *modal definability problem* for the class of contact frames $\mathcal{CF}^3$ is *undecidable*.

Modal Definability Problem

Approach

By "Balbiani, P., Tinchev, T.: *Undecidable problems for modal definability*", 2014, Theorem 1:

- If $\mathcal{CF}^3$ is *stable*, then the problem of the decidability of $\mathcal{Th}(\mathcal{CF}^3)$ is reducible to the *modal definability problem* for $\mathcal{CF}^3$.

Modal Definability Problem

Approach

By "Balbiani, P., Tinchev, T.: *Undecidable problems for modal definability*", 2014, Theorem 1:

- If $\mathcal{CF}^3$ is *stable*, then the problem of the decidability of $\mathcal{T}h(\mathcal{CF}^3)$ is reducible to the *modal definability problem* for $\mathcal{CF}^3$.

Modal Definability Problem

Approach

So it is sufficient to show the following:

- $\mathcal{CF}^3$ is *stable*.
- $\mathcal{Th}(\mathcal{CF}^3)$ is *undecidable*.

Modal Definability Problem

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Modal Definability Problem

Approach

So it is sufficient to show the following:

- $\mathcal{CF}^3$ is *stable*.
- $\mathcal{T}h(\mathcal{CF}^3)$ is *undecidable*.

Modal Definability Problem

$\mathcal{T}h(\mathcal{CF}^3)$ is undecidable

Let $L(R_2)$ be the restriction of $L(R_2, R_3)$ by excluding R_3.

Let $\mathcal{C}_{ref,sym}$ be the class of binary reflexive and symmetric structures (in the language of $L(R_2)$).

Modal Definability Problem

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Modal Definability Problem

$\mathcal{Th}(\mathcal{CF}^3)$ is undecidable

- For any $\langle W, R_2, R_3 \rangle$ in $\mathcal{CF}^3$ the structure $\langle W, R_2 \rangle$ is its restriction to $L(R_2)$.
- For any $\langle W, R_2, R_3 \rangle$ in $\mathcal{CF}^3$ its restriction $\langle W, R_2 \rangle$ is reflexive and symmetric.
  - Hence, in the class $\mathcal{C}_{ref,sym}$.
- Clearly, for every formula A of $L(R_2)$:

$$\langle W, R_2 \rangle \models_{FO} A \quad \leftrightarrow \quad \langle W, R_2, R_3 \rangle \models_{FO} A$$

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Modal Definability Problem

$\mathcal{Th}(\mathcal{CF}^3)$ is undecidable

- Every $\langle W, R_2 \rangle$ in $\mathcal{C}_{ref,sym}$ is a restriction of some $\langle W, R_2, R_3 \rangle$ in $\mathcal{CF}^3$. Such one is with R_3 defined as:

$$R_3(x_1, x_2, x_3) \quad \leftrightarrow \quad \begin{array}{l} x_1 = x_2 \wedge R_2(x_2, x_3) \vee \\ x_2 = x_3 \wedge R_2(x_3, x_1) \vee \\ x_3 = x_1 \wedge R_2(x_1, x_2). \end{array} .$$

- Therefore, the decidability of $\mathcal{Th}(\mathcal{C}_{ref,sym})$ is reducible to that of $\mathcal{Th}(\mathcal{CF}^3)$.

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Modal Definability Problem

$\mathcal{T}h(\mathcal{CF}^3)$ is undecidable

By "H. Rogers. *Certain logical reduction and decision problems.* Annals of Mathematics", 64, 264-284, 1956:

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Modal Definability Problem

$\mathcal{CF}^3$ is *stable*

As per "Balbiani, P., Tinchev, T.: *Undecidable problems for modal definability*", 2014:

Definition

$\mathfrak{F} \preccurlyeq \mathfrak{F}$: $\mathfrak{F}$ is *weaker* than $\mathfrak{F}'$ if for every formula φ

$$\mathfrak{F} \models_m \varphi \quad \rightarrow \quad \mathfrak{F}' \models_m \varphi$$

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$\mathfrak{F}'$ is the *relativized reduct* of $\mathfrak{F}$ with respect to the first-order formula $A(x_1, \ldots, x_n, y)$ and the list of individuals $a_1, \ldots, a_n$ of $\mathfrak{F}$ if:

- $\mathfrak{F}'$ is the restriction of $\mathfrak{F}$ to the set of all individuals b such that $\mathfrak{F} \vDash_{FO} A[a_1, \ldots, a_n, b]$.

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Modal Definability Problem

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Definition

The class of frames $\mathcal{C}$ is *stable* if they exist first-order formula $A(x_1, \dots, x_n, y)$ and sentence B such that:

(a) $\mathcal{C}$ is closed with respect to the relativized reducts of its elements with respect to A (and an arbitrary list of their individuals $a_1, \dots, a_n$).

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Modal Definability Problem

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As per "Balbiani, P., Tinchev, T.: *Undecidable problems for modal definability*", 2014:

Definition

The class of frames $\mathcal{C}$ is *stable* if they exist first-order formula $A(x_1, \ldots, x_n, y)$ and sentence B such that:

(b) For all $\mathfrak{F}_0$ in $\mathcal{C}$, there exist $\mathfrak{F}$ and $\mathfrak{F}'$ in $\mathcal{C}$ such that:

- $\mathfrak{F}_0$ is a relativized reduct of $\mathfrak{F}$ with respect to A and some list $a_1, \ldots, a_n$ of individuals of $\mathfrak{F}$.
- $\mathfrak{F} \models_{FO} B$ and $\mathfrak{F}' \not\models_{FO} B$
- $\mathfrak{F} \preccurlyeq \mathfrak{F}'$

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Modal Definability Problem

$\mathcal{CF}^3$ is *stable*

As per "Balbiani, P., Tinchev, T.: *Undecidable problems for modal definability*", 2014:

Definition

The class of frames $\mathcal{C}$ is *stable* if they exist first-order formula $A(x_1, \ldots, x_n, y)$ and sentence B such that:

(b) For all $\mathfrak{F}_0$ in $\mathcal{C}$, there exist $\mathfrak{F}$ and $\mathfrak{F}'$ in $\mathcal{C}$ such that:

- $\mathfrak{F}_0$ is a relativized reduct of $\mathfrak{F}$ with respect to A and some list $a_1, \ldots, a_n$ of individuals of $\mathfrak{F}$.
- $\mathfrak{F} \models_{FO} B$ and $\mathfrak{F}' \not\models_{FO} B$
- $\mathfrak{F} \preccurlyeq \mathfrak{F}'$

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Modal Definability Problem

$\mathcal{CF}^3$ is *stable* (a)

Observation:

Every relativized reduct of a contact frame is a contact frame.

- ▷ Directly by definition of a *contact frame*.

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Modal Definability Problem

$\mathcal{CF}^3$ is *stable* (b)

- $A(x, y) := R_2(x, y) \wedge x \neq y$
- $B := \exists x \exists y (x \neq y)$

Let $\mathfrak{F}_0 = \langle W_0, R_2^0, R_3^0 \rangle$ be in $\mathcal{CF}^3$.

Let a be an element not in W_0.

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Modal Definability Problem

$\mathcal{CF}^3$ is *stable* (b)

Let $\mathfrak{F} = \langle W, R_2, R_3 \rangle$ be defined as follows:

- $W = W_0 \cup \{a\}$
- $R_2 = R_2^0 \cup (\{a\} \times W_0) \cup (W_0 \times \{a\}) \cup \{\langle a, a \rangle\}$
- $R_3 =$

$$\begin{array}{ll} R_3^0 & \cup \\ \{a\} \times \{a\} \times W_0 & \cup \\ \{a\} \times W_0 \times \{a\} & \cup \\ W_0 \times \{a\} \times \{a\} & \cup \\ \{\langle w, w, a \rangle \mid w \in W_0\} & \cup \\ \{\langle w, a, w \rangle \mid w \in W_0\} & \cup \\ \{\langle a, w, w \rangle \mid w \in W_0\} & \cup \\ \{\langle a, a, a \rangle\} & \end{array}$$

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Let $\mathfrak{F}' = \langle W', R'_2, R'_3 \rangle$ be the single element frame with

- $W' = \{a\}$.

Modal Definability Problem

$\mathcal{CF}^3$ is *stable* (b)

We have:

- $\mathfrak{F}$ and $\mathfrak{F}'$ are *contact frames*.
- $\mathfrak{F}_0$ is a relativized reduct of $\mathfrak{F}$ with respect to $A(x, y)$ and the individual a of $\mathfrak{F}$.
- $\mathfrak{F} \models_{FO} B$ and $\mathfrak{F}' \not\models_{FO} B$

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Let us consider the mapping $f : \{0, \{a\}\} \longrightarrow \{0, W\}$ such that:

- $f(0) = 0$
  $f(\{a\}) = W$
- For an arbitrary $\mathcal{V}'$ for $\mathfrak{F}'$ define a valuation $\mathcal{V}$ for $\mathfrak{F}$:
  $\mathcal{V}(x) = f(\mathcal{V}'(x))$

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$\mathcal{CF}^3$ is *stable* (b)

Then:

- $\mathcal{V}(\tau) = f(\mathcal{V}'(\tau))$, for any term τ of L_{R_3}.
- For an arbitrary formula φ of L_{R_3}:

$$\langle \mathfrak{F}, \mathcal{V} \rangle \models_m \varphi \quad \leftrightarrow \quad \langle \mathfrak{F}', \mathcal{V}' \rangle \models_m \varphi$$

Therefore:

- $\mathfrak{F} \models_m \varphi \ \rightarrow \ \mathfrak{F}' \models_m \varphi$ i.e. $\mathfrak{F} \preccurlyeq \mathfrak{F}'$

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Outline

Unification Problems

Elementary Unification

Definition

Elementary unification problem:

Input: $\varphi[x_1, \ldots, x_n]$

Output: "*There are* terms (of L_{R_3}) $\tau_1, \ldots, \tau_n$ such that $\mathcal{CF}^3 \models \varphi[x_1/\tau_1, \ldots, x_n/\tau_n]$"

or

"*There are not* terms (of L_{R_3}) $\tau_1, \ldots, \tau_n$ such that $\mathcal{CF}^3 \models \varphi[x_1/\tau_1, \ldots, x_n/\tau_n]$"

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Unification Problems

Parametric Unification

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Parametric Unification Problem

Result

Parametric Unification Problem Outcome

The *parametric unification problem* for the class of contact frames $\mathcal{CF}^3$ is *decidable*.

Parametric Unification Problem

Approach

Assume that they exist $\tau_1, \dots, \tau_n$ (of L_{R_3}) such that

$$\mathcal{CF}^3 \vDash \varphi[p_1, \dots, p_k, x_1/\tau_1, \dots x_n/\tau_n]$$

Let φ' be $\varphi[p_1, \dots, p_k, x_1/\tau_1, \dots x_n/\tau_n]$ and φ'' be φ' with substituted all variables but the parameters $p_1, \dots, p_k$ with 1.

Let $\mathfrak{F}$ be arbitrary from $\mathcal{CF}^3$ and $\mathcal{V}$ an arbitrary valuation on $\mathfrak{F}$.

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Let $\mathfrak{F}$ be arbitrary from $\mathcal{CF}^3$ and $\mathcal{V}$ an arbitrary valuation on $\mathfrak{F}$.

Parametric Unification Problem

Approach

Let $\mathcal{V}'$:

$\mathcal{V}'(x) = \mathcal{V}(1)$ for all x in φ' other than the parameters $p_1, \ldots, p_k$.

$\mathcal{V}'(y) = \mathcal{V}(y)$ otherwise.

Then, for any $\mathfrak{F}$ in $\mathcal{CF}^3$:

$$\langle \mathfrak{F}, \mathcal{V}' \rangle \vDash \varphi' \quad \leftrightarrow \quad \langle \mathfrak{F}, \mathcal{V} \rangle \vDash \varphi''$$

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Parametric Unification Problem

Approach

Therefore, if there are $\tau_1, \ldots, \tau_n$ such that

$$\mathcal{CF}^3 \models \varphi[p_1, \ldots, p_k, x_1/\tau_1, \ldots x_n/\tau_n],$$

then there are terms $\kappa_1, \ldots, \kappa_n$ with variables only among $p_1, \ldots, p_k$ such that:

$$\mathcal{CF}^3 \models \varphi[p_1, \ldots, p_k, x_1/\kappa_1, \ldots x_n/\kappa_n]$$

Parametric Unification Problem

Approach

- The problem of parametric unification is reduced to checking if some of $(2^{2^k})^n$ (the number of distinct vectors of terms $\kappa_1, \ldots, \kappa_n$) formulas is in $\mathcal{L}(\mathcal{CF}^3)$.

It is the sufficient to show:

- $\mathcal{L}(\mathcal{CF}^3)$ is decidable.

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$\mathcal{L}(\mathcal{CF}^3)$ is decidable

Formal System $\mathcal{L}_{Cont3}$: Nonlogical Axioms

- **Logical axioms**: sentential, identity and equivalence, congruence.
- **Boolean algebra axioms**: stipulating a non-degenerate Boolean algebra.
- **Proximity axioms**:
  - $R_n(x_1, ..., x_n) \Rightarrow x_1 \neq 0$
  - $R_n(x_1' \cup x_1'', x_2, ..., x_n) \Leftrightarrow R_n(x_1', x_2, ..., x_n) \vee R_n(x_1'', x_2, ..., x_n)$

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Formal System $\mathcal{L}_{Cont3}$: Nonlogical Axioms

(c1) $(\sigma : \{1, \dots, n\} \to \{1, \dots, n\})$

$$R_n(x_1, \dots, x_n) \Rightarrow R_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

(c2)

$$R_{n+1}(x_1, x_1, x_2 \dots, x_n) \Leftrightarrow R_n(x_1, x_2, \dots, x_n)$$

(c3)

$$\neg(x = 0) \Rightarrow R_2(x, x)$$

~~(c4)~~

$$\cancel{\neg(x = 0) \wedge \neg(-x = 0) \Rightarrow R_2(x, -x)}$$

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Formal System $\mathcal{L}_{Cont3}$: Inference Rules

- **Inference rules**: *uniform substitution*, *modus ponens*.

$\mathcal{L}(\mathcal{CF}^3)$ is decidable

Formal System $\mathcal{L}_{Cont3}$ Completeness

Completeness

$\mathcal{L}_{Cont3}$ is complete with respect to $\mathcal{CF}^3$.

$\mathcal{L}(\mathcal{CF}^3)$ is decidable

Formal System $\mathcal{L}_{Cont3}$ Completeness

Correctness: Trivial verification.

Completeness:

Let us assume that $\nvdash_{\mathcal{L}_{Cont3}} \varphi$.

$\mathcal{L}(\mathcal{CF}^3)$ is decidable

Formal System $\mathcal{L}_{Cont3}$ Completeness

- Let us consider L_{R_3} as a first-order language.
- Let T be the first-order theory in L_{R_3} with no nonlogical axioms. Let Γ be the set of all nonlogical axioms of the formal system $\mathcal{L}_{Cont3}$.
- Let φ' be the *closure* of $\varphi(x_1, \dots, x_n)$.
- Let T_c be obtained from T by adding to the language L_{R_3} new constants $c_1, \dots, c_n$.

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- $T_c[\Gamma; \neg\varphi[c_1, \ldots, c_n]]$ is an extension of $T_c[\Gamma; \neg\varphi']$. $T_c[\Gamma; \neg\varphi']$ is an extension of $T[\Gamma; \neg\varphi']$.
- If $T[\Gamma; \neg\varphi']$ is inconsistent, then such is $T_c[\Gamma; \neg\varphi[c_1, \ldots, c_n]]$.
  By the *Hilbert-Ackermann* theorem there is a *quasi-tautology* $\neg\psi'_1 \vee \ldots \vee \neg\psi'_n \vee \neg\neg\varphi[c_1, \ldots, c_n]$, where ψ'_i are instances of formulas from Γ.
- Hence, there is a quasi-tautology $\neg\psi_1 \vee \ldots \vee \neg\psi_n \vee \varphi$, where ψ_i are formulas from Γ and thus

$$\vdash_{\mathcal{L}_{Cont3}} \varphi.$$

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- A first-order model $\mathfrak{B}$ of $T[\Gamma; \neg\varphi']$ is a Boolean frame in which there is a valuation $\mathcal{V}$ such that $\langle \mathfrak{B}, \mathcal{V} \rangle \models \neg\varphi$.
- The set of values of $\mathcal{V}$ applied on the variables of φ generates a finite Boolean subalgebra of the universe of $\mathfrak{B}$, hence, a finite subframe $\mathfrak{B}_0$ of $\mathfrak{B}$ a model of $\mathcal{L}_{Cont3}$.
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- There is a finite frame $\mathfrak{F}$ such that:

$$\mathfrak{B}_0 \simeq \mathcal{B}(\mathfrak{F})$$

- Moreover, in $\mathfrak{F}$ are valid the formulas valid in $\mathfrak{B}_0$ and there is a valuation $\mathcal{V}'$ such that $\langle \mathfrak{F}, \mathcal{V}' \rangle \vDash \neg\varphi$.
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- Remark that, since $\mathfrak{F}$ was finite we have also demonstrated the *finite frame property* of $\mathcal{L}_{Cont3}$, respectively of $\mathcal{L}(\mathcal{CF}^3)$.
- Since $\mathcal{L}_{Cont3}$ is with decidable axiomatization $\mathcal{L}(\mathcal{CF}^3)$ is *recursively enumerable*.
- By the *finite frame property* of $\mathcal{L}(\mathcal{CF}^3)$ it follows the set of formulas refutable in the class $\mathcal{CF}^3$ is also *recursively enumerable*.

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Discussion

Questions?

Thank you for your attention!