Superintuitionistic modal logics: a minimal setting à la Fischer Servi

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Outline

A word about superintuitionistic logics

A word about modal logics

Then, about intuitionistic modal logics

A word about superintuitionistic logics

What is the language of superintuitionistic logics ?

- Propositional letters

  p, q, etc

- Propositional connectives and parentheses

  $\rightarrow$, $\bot$, $\top$, $\vee$, $\wedge$, (,)

- Formulas

  $A ::= p | (A \rightarrow A) | \bot | \top | (A \vee A) | (A \wedge A)$

- Abbreviations

  $\neg A ::= (A \rightarrow \bot)$

  $(A \leftrightarrow B) ::= ((A \rightarrow B) \wedge (B \rightarrow A))$

- Notation

  FOR : the set of all formulas

A word about superintuitionistic logics

Examples of superintuitionistic logics

- **CPL** : Classical Propositional Logic
- **IPL** : Intuitionistic Propositional Logic

A word about superintuitionistic logics

Formal desiderata for superintuitionistic logics

- Any $\mathbf{L} \subseteq \mathbf{FOR}$ closed for uniform substitution and such that

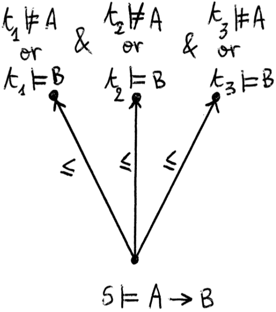
  $\mathbf{L}$ contains the standard axioms of $\mathbf{IPL}$

  $\mathbf{L}$ is closed with respect to the rule $\frac{p \quad p \rightarrow q}{q}$

A word about superintuitionistic logics

Relational semantics of superintuitionistic logics

- Kripke frame : preordered structure $(W, \leq)$
  W : nonempty set of "states" s, t, etc
  $\leq$: reflexive transitive relation on W
- Valuation on $(W, \leq)$
  V : propositional letter $p \mapsto \leq$-closed subset $V(p)$ of W
- Truth condition for $\rightarrow$



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- Valuation on $(W, \leq)$
  V : propositional letter $p \mapsto \leq$-closed subset $V(p)$ of W
- Truth conditions

$$
\begin{array}{lll}
s \models p & \Leftrightarrow & s \in V(p) \\
s \models A \to B & \Leftrightarrow & \forall t \in W,\ (s \leq t \ \Rightarrow\ t \not\models A \text{ or } t \models B) \\
s \models \bot & \Leftrightarrow & \text{never} \\
s \models \top & \Leftrightarrow & \text{always} \\
s \models A \vee B & \Leftrightarrow & s \models A \text{ or } s \models B \\
s \models A \wedge B & \Leftrightarrow & s \models A \ \&\ s \models B
\end{array}
$$

A word about superintuitionistic logics

Examples of superintuitionistic logics

- **SmL** ::= **IPL** + $(\neg q \to p) \to (((p \to q) \to p) \to p)$
- **KC** ::= **IPL** + $\neg p \vee \neg\neg p$
- **LC** ::= **IPL** + $(p \to q) \vee (q \to p)$
- **SL** ::= **IPL** + $((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p$
- **KP** ::= **IPL** + $(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$
- **WKP** ::= **IPL** + $(\neg p \to \neg q \vee \neg r) \to (\neg p \to \neg q) \vee (\neg p \to \neg r)$
- **CPL** ::= **IPL** + $p \vee \neg p$

A word about modal logics

What is the language of modal logics ?

- Propositional letters

  p, q, etc

- Propositional connectives, modal connectives and parentheses

  $\rightarrow$, $\bot$, $\top$, $\vee$, $\wedge$, $\Box$, $\Diamond$, (,)

- Formulas

  $A ::= p | (A \rightarrow A) | \bot | \top | (A \vee A) | (A \wedge A) | \Box A | \Diamond A$

- Abbreviations

  $\neg A ::= (A \rightarrow \bot)$

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- Notation

  FOR : the set of all formulas

A word about modal logics

Examples of modal logics

- **S**5 : a logic of knowledge
- **K** : the least modal logic

A word about modal logics

Formal desiderata for modal logics

- Any $\mathbf{L}\subseteq\mathbf{FOR}$ closed for uniform substitution and such that

  L contains the standard axioms of **CPL**

  L is closed with respect to the rule $\frac{p \quad p\to q}{q}$

  L contains the axiom $\Box p\leftrightarrow\neg\Diamond\neg p$

  L contains the axiom $\Diamond p\leftrightarrow\neg\Box\neg p$

  L contains the axiom $\Box p\wedge\Box q\to\Box(p\wedge q)$

  L contains the axiom $\Diamond(p\vee q)\to\Diamond p\vee\Diamond q$

  L contains the axiom $\Box\top$

  L contains the axiom $\neg\Diamond\bot$

  L contains the axiom $(\Diamond p\to\Box q)\to\Box(p\to q)$

  L contains the axiom $\Diamond(p\to q)\to(\Box p\to\Diamond q)$

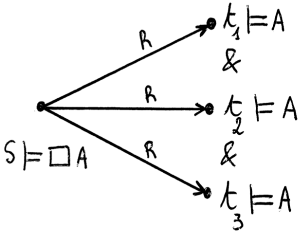
  L is closed with respect to the rule $\frac{p\to q}{\Box p\to\Box q}$

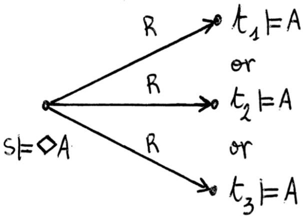
  L is closed with respect to the rule $\frac{p\to q}{\Diamond p\to\Diamond q}$

A word about modal logics

Relational semantics of modal logics

- Kripke frame : relational structure (W, R)
  - W : nonempty set of "states" s, t, etc
  - R : binary relation on W
- Valuation on (W, R)
  - V : propositional letter $p \mapsto$ subset $V(p)$ of W
- Truth conditions for $\Box$ and $\Diamond$





A word about modal logics

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\begin{array}{lcl}
s \models p & \Leftrightarrow & s \in V(p) \\
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s \models A \vee B & \Leftrightarrow & s \models A \text{ or } s \models B \\
s \models A \wedge B & \Leftrightarrow & s \models A \ \& \ s \models B \\
s \models \Box A & \Leftrightarrow & \forall t \in R(s),\ t \models A \\
s \models \Diamond A & \Leftrightarrow & \exists t \in R(s),\ t \models A
\end{array}
$$

A word about modal logics

Examples of modal logics

- **KD** $::= \mathbf{K} + \Box p \rightarrow \Diamond p$
- **KT** $::= \mathbf{K} + \Box p \rightarrow p$
- **KB** $::= \mathbf{K} + p \rightarrow \Box \Diamond p$
- **K**4 $::= \mathbf{K} + \Box p \rightarrow \Box \Box p$
- **K**5 $::= \mathbf{K} + \Diamond p \rightarrow \Box \Diamond p$
- **S**4 $::= \mathbf{K} + \Box p \rightarrow p + \Box p \rightarrow \Box \Box p$
- **S**5 $::= \mathbf{K} + \Box p \rightarrow p + \Diamond p \rightarrow \Box \Diamond p$

Then, about intuitionistic modal logics

What is the language of intuitionistic modal logics ?

- Propositional letters

  p, q, etc

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  $\rightarrow$, $\bot$, $\top$, $\vee$, $\wedge$, $\Box$, $\Diamond$, (,)

- Formulas

  $A ::= p|(A{\rightarrow}A)|\bot|\top|(A{\vee}A)|(A{\wedge}A)|\Box A|\Diamond A$

- Abbreviations

  $\neg A ::= (A{\rightarrow}\bot)$

  $(A{\leftrightarrow}B) ::= ((A{\rightarrow}B){\wedge}(B{\rightarrow}A))$

- Notations

  $\mathbf{FOR}_{\{\Box,\Diamond\}}$: the set of all formulas

  $\mathbf{FOR}_{\{\Box\}}$: the set of all formulas without $\Diamond$

  $\mathbf{FOR}_{\{\Diamond\}}$: the set of all formulas without $\Box$

  $\mathbf{FOR}_{\emptyset}$: the set of all formulas without $\Box$ and $\Diamond$

Then, about intuitionistic modal logics

Examples of intuitionistic modal logics

- **IK** : an "intuitionistic" analogue of **K**
- **WK** : a "constructive" analogue of **K**

Then, about intuitionistic modal logics

Informal desiderata for intuitionistic modal logics

Simpson (1994)

- Any $\mathbf{L} \subseteq \mathbf{FOR}_{\{\Box,\Diamond\}}$ closed for uniform substitution and such that

  $\mathbf{L} \cap \mathbf{FOR}_{\emptyset}$ is **IPL**

  L is closed with respect to the rule $\frac{p \quad p \to q}{q}$

  $\mathbf{L} + p \vee \neg p$ is a modal logic

  L possesses the disjunction property

  $\Box$ and $\Diamond$ are independent in **L**

Then, about intuitionistic modal logics

Historical perspective

- Pioneers
  Fitch (1948)
  Bull (1965, 1966)
- "Intuitionistic" contributions
  Fischer Servi (1977, 1978, 1984)
  Plotkin and Stirling (1986)
- "Constructive" contributions
  Wijesekera (1990)
  Alechina, Mendler, de Paiva and Ritter (2001)

Then, about intuitionistic modal logics

Relational semantics of intuitionistic modal logics

- Kripke frame : preordered relational structure $(W, \leq, R)$
  W : nonempty set of "states" s, t, etc
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  R : binary relation on W
- Valuation on $(W, \leq, R)$
  V : propositional letter $p \mapsto \leq$-closed subset $V(p)$ of W
- Truth conditions

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\begin{array}{ll}
s \models p & \Leftrightarrow s \in V(p) \\
s \models A \rightarrow B & \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ t \not\models A \text{ or } t \models B) \\
s \models \bot & \Leftrightarrow \text{never} \\
s \models \top & \Leftrightarrow \text{always} \\
s \models A \vee B & \Leftrightarrow s \models A \text{ or } s \models B \\
s \models A \wedge B & \Leftrightarrow s \models A \ \&\ s \models B \\
s \models \Box A & \Leftrightarrow \forall t \in R(s),\ t \models A\ ? \\
s \models \Diamond A & \Leftrightarrow \exists t \in R(s),\ t \models A\ ?
\end{array}
$$

Then, about intuitionistic modal logics

Relational semantics of intuitionistic modal logics

- Kripke frame : preordered relational structure $(W, \leq, R)$
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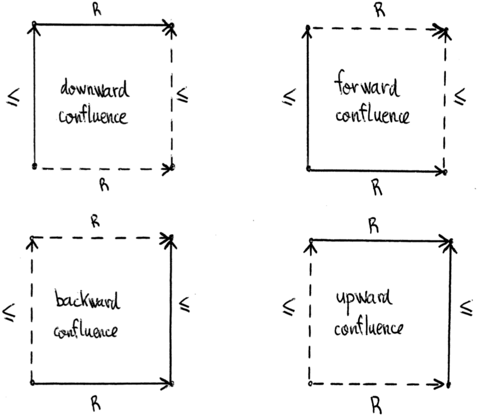

Then, about intuitionistic modal logics

Relational semantics of intuitionistic modal logics

- Kripke frame : preordered relational structure $(W, \leq, R)$
  - W : nonempty set of "states" s, t, etc
  - $\leq$: reflexive transitive relation on W
  - R : binary relation on W
- Interactions between $\leq$ and R : 4 possibilities !
  - $(W, \leq, R)$ is **downward confluent** if $\leq \circ R \subseteq R \circ \leq$
  - $(W, \leq, R)$ is **forward confluent** if $\geq \circ R \subseteq R \circ \geq$
  - $(W, \leq, R)$ is **backward confluent** if $R \circ \leq \subseteq \leq \circ R$
  - $(W, \leq, R)$ is **upward confluent** if $R \circ \geq \subseteq \geq \circ R$
- All in all
  - 2^4 choices at the level of the class of frames !

Then, about intuitionistic modal logics

Relational semantics of intuitionistic modal logics

- Kripke frame : preordered relational structure $(W, \leq, R)$
- Valuation on $(W, \leq, R)$
  V : propositional letter $p \mapsto \leq$-closed subset $V(p)$ of W
- Truth conditions for $\Box$: 2 possibilities !
  $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$
  $s \models \Box A \quad \Leftrightarrow \exists t \in W,\ (s \geq t\ \&\ \forall u \in R(t),\ u \models A)$
- Truth conditions for $\Diamond$: 2 possibilities !
  $s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$
  $s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t\ \&\ \exists u \in R(t),\ u \models A)$
- All in all
  - 2×2 choices at the level of the truth conditions !

Then, about intuitionistic modal logics

Restricting the language to $\mathbf{FOR}_{\{\Box\}}$

- Semantics : all downward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$

$$s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t\ \Rightarrow\ \forall u \in R(t),\ u \models A)$$
$$s \models \Box A \quad \Leftrightarrow \exists t \in W,\ (s \geq t\ \&\ \forall u \in R(t),\ u \models A)$$
$$s \models \Box A \quad \Leftrightarrow \forall t \in R(s),\ t \models A$$

Then, about intuitionistic modal logics

Restricting the language to $\mathbf{FOR}_{\{\Box\}}$
Božić and Došen (1984) — all down. conf. $(W, \leq, R)$

- Defined logic **HK**$\Box$
  - Least $\mathbf{L} \subseteq \mathbf{FOR}$ closed for uniform substitution and such that
    - $\mathbf{L}$ contains the standard axioms of ~~**CPL**~~ **IPL**
    - $\mathbf{L}$ is closed with respect to the rule $\frac{p \quad p \to q}{q}$
    - ~~$\mathbf{L}$ contains the axiom $\Box p \leftrightarrow \neg\Diamond\neg p$~~
    - ~~$\mathbf{L}$ contains the axiom $\Diamond p \leftrightarrow \neg\Box\neg p$~~
    - $\mathbf{L}$ contains the axiom $\Box p \wedge \Box q \to \Box(p \wedge q)$
    - ~~$\mathbf{L}$ contains the axiom $\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$~~
    - $\mathbf{L}$ contains the axiom $\Box\top$
    - ~~$\mathbf{L}$ contains the axiom $\neg\Diamond\bot$~~
    - ~~$\mathbf{L}$ contains the axiom $(\Diamond p \to \Box q) \to \Box(p \to q)$~~
    - ~~$\mathbf{L}$ contains the axiom $\Diamond(p \to q) \to (\Box p \to \Diamond q)$~~
    - $\mathbf{L}$ is closed with respect to the rule $\frac{p \to q}{\Box p \to \Box q}$
    - ~~$\mathbf{L}$ is closed with respect to the rule $\frac{p \to q}{\Diamond p \to \Diamond q}$~~

Then, about intuitionistic modal logics

Restricting the language to $\mathbf{FOR}_{\{\Diamond\}}$

- Semantics : all forward confluent $(W, \leq, R)$
  - Truth conditions for $\Diamond$

$$s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$$
$$s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$$
$$s \models \Diamond A \quad \Leftrightarrow \exists t \in R(s),\ t \models A$$

Then, about intuitionistic modal logics

Restricting the language to $\mathbf{FOR}_{\{\Diamond\}}$

Božić and Došen (1984) — all for. conf. $(W, \leq, R)$

- Defined logic $\mathbf{HK}\Diamond$
  - Least $\mathbf{L}\subseteq\mathbf{FOR}$ closed for uniform substitution and such that

    L contains the standard axioms of ~~**CPL**~~ **IPL**

    L is closed with respect to the rule $\frac{p \quad p\rightarrow q}{q}$

    ~~**L** contains the axiom $\Box p\leftrightarrow\neg\Diamond\neg p$~~

    ~~**L** contains the axiom $\Diamond p\leftrightarrow\neg\Box\neg p$~~

    ~~**L** contains the axiom $\Box p\wedge\Box q\rightarrow\Box(p\wedge q)$~~

    L contains the axiom $\Diamond(p\vee q)\rightarrow\Diamond p\vee\Diamond q$

    ~~**L** contains the axiom $\Box\top$~~

    L contains the axiom $\neg\Diamond\bot$

    ~~**L** contains the axiom $(\Diamond p\rightarrow\Box q)\rightarrow\Box(p\rightarrow q)$~~

    ~~**L** contains the axiom $\Diamond(p\rightarrow q)\rightarrow(\Box p\rightarrow\Diamond q)$~~

    ~~**L** is closed with respect to the rule $\frac{p\rightarrow q}{\Box p\rightarrow\Box q}$~~

    L is closed with respect to the rule $\frac{p\rightarrow q}{\Diamond p\rightarrow\Diamond q}$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$

- Semantics : all downward and forward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$

    $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$

    $s \models \Box A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \forall u \in R(t),\ u \models A)$

    $s \models \Box A \quad \Leftrightarrow \forall t \in R(s),\ t \models A$
  - Truth conditions for $\Diamond$

    $s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$

    $s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$

    $s \models \Diamond A \quad \Leftrightarrow \exists t \in R(s),\ t \models A$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$
Božić and Došen (1984) — all down. and for. conf.
$(W, \leq, R)$

- Defined logic $\mathbf{HK}\Box\Diamond = ?$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a first possibility

- Semantics : all forward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$

    $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$
  - Truth conditions for $\Diamond$

    $s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$

    $s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$

    $s \models \Diamond A \quad \Leftrightarrow \exists t \in R(s),\ t \models A$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a first possibility
Fischer Servi (1984) — all back. and for. conf. $(W, \leq, R)$

- Defined logic **IK**
  - Least **L**⊆**FOR** closed for uniform substitution and such that

    L contains the standard axioms of ~~**CPL**~~ **IPL**

    L is closed with respect to the rule $\frac{p \quad p \rightarrow q}{q}$

    ~~**L** contains the axiom $\Box p \leftrightarrow \neg \Diamond \neg p$~~

    ~~**L** contains the axiom $\Diamond p \leftrightarrow \neg \Box \neg p$~~

    L contains the axiom $\Box p \wedge \Box q \rightarrow \Box (p \wedge q)$

    L contains the axiom $\Diamond (p \vee q) \rightarrow \Diamond p \vee \Diamond q$

    L contains the axiom $\Box \top$

    L contains the axiom $\neg \Diamond \bot$

    L contains the axiom $(\Diamond p \rightarrow \Box q) \rightarrow \Box (p \rightarrow q)$

    L contains the axiom $\Diamond (p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q)$

    L is closed with respect to the rule $\frac{p \rightarrow q}{\Box p \rightarrow \Box q}$

    L is closed with respect to the rule $\frac{p \rightarrow q}{\Diamond p \rightarrow \Diamond q}$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a first possibility

B., Gao, Gencer and Olivetti — all for. conf. $(W, \leq, R)$

- Defined logic **FIK**
  - Least **L**⊆**FOR** closed for uniform substitution and such that

    L contains the standard axioms of ~~**CPL**~~ **IPL**

    L is closed with respect to the rule $\frac{p \quad p\rightarrow q}{q}$

    ~~**L** contains the axiom $\Box p\leftrightarrow\neg\Diamond\neg p$~~

    ~~**L** contains the axiom $\Diamond p\leftrightarrow\neg\Box\neg p$~~

    L contains the axiom $\Box p\wedge\Box q\rightarrow\Box(p\wedge q)$

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    L contains the axiom $\Box\top$

    L contains the axiom $\neg\Diamond\bot$

    ~~**L** contains the axiom $(\Diamond p\rightarrow\Box q)\rightarrow\Box(p\rightarrow q)$~~

    ~~**L** contains the axiom $\Diamond(p\rightarrow q)\rightarrow(\Box p\rightarrow\Diamond q)$~~

    L is closed with respect to the rule $\frac{p\rightarrow q}{\Box p\rightarrow\Box q}$

    L is closed with respect to the rule $\frac{p\rightarrow q}{\Diamond p\rightarrow\Diamond q}$

    — **L** contains the axiom $\Box(p\vee q)\rightarrow((\Diamond p\rightarrow\Box q)\rightarrow\Box q)$

    — **L** contains the axiom $\Box(p\rightarrow q)\rightarrow(\Diamond p\rightarrow\Diamond q)$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a second possibility

- Semantics : all $(W, \leq, R)$
  - Truth conditions for $\Box$

    $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$
  - Truth conditions for $\Diamond$

    $s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a second possibility
Wijesekera (1990) — all $(W, \leq, R)$

- Defined logic **WK**
  - Least $\mathbf{L}\subseteq\mathbf{FOR}$ closed for uniform substitution and such that
    - $\mathbf{L}$ contains the standard axioms of ~~**CPL**~~ **IPL**
    - $\mathbf{L}$ is closed with respect to the rule $\frac{p \quad p\to q}{q}$
    - ~~$\mathbf{L}$ contains the axiom $\Box p\leftrightarrow\neg\Diamond\neg p$~~
    - ~~$\mathbf{L}$ contains the axiom $\Diamond p\leftrightarrow\neg\Box\neg p$~~
    - $\mathbf{L}$ contains the axiom $\Box p\wedge\Box q\to\Box(p\wedge q)$
    - ~~$\mathbf{L}$ contains the axiom $\Diamond(p\vee q)\to\Diamond p\vee\Diamond q$~~
    - $\mathbf{L}$ contains the axiom $\Box\top$
    - $\mathbf{L}$ contains the axiom $\neg\Diamond\bot$
    - ~~$\mathbf{L}$ contains the axiom $(\Diamond p\to\Box q)\to\Box(p\to q)$~~
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    - $\mathbf{L}$ is closed with respect to the rule $\frac{p\to q}{\Box p\to\Box q}$
    - $\mathbf{L}$ is closed with respect to the rule $\frac{p\to q}{\Diamond p\to\Diamond q}$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a third possibility

- Semantics : all $(W, \leq, R)$
  - Truth conditions for $\Box$
    $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t\ \Rightarrow\ \forall u \in R(t),\ u \models A)$
  - Truth conditions for $\Diamond$
    $s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t\ \&\ \exists u \in R(t),\ u \models A)$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a third possibility
Plotkin and Stirling — all $(W, \leq, R)$

- Defined logic ?
  - **Simpson (1994):** "axiomatization is rather complicated"

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a third possibility

B., Gao, Gencer and Olivetti — all $(W,\leq,R)$

- Defined logic $\mathbf{L}_{\min}$
  - Least $\mathbf{L}\subseteq\mathbf{FOR}$ closed for uniform substitution and such that

    $\mathbf{L}$ contains the standard axioms of ~~**CPL**~~ **IPL**

    $\mathbf{L}$ is closed with respect to the rule $\frac{p \quad p\to q}{q}$

    ~~$\mathbf{L}$ contains the axiom $\Box p\leftrightarrow\neg\Diamond\neg p$~~

    ~~$\mathbf{L}$ contains the axiom $\Diamond p\leftrightarrow\neg\Box\neg p$~~

    $\mathbf{L}$ contains the axiom $\Box p\wedge\Box q\to\Box(p\wedge q)$

    $\mathbf{L}$ contains the axiom $\Diamond(p\vee q)\to\Diamond p\vee\Diamond q$

    $\mathbf{L}$ contains the axiom $\Box\top$

    $\mathbf{L}$ contains the axiom $\neg\Diamond\bot$

    ~~$\mathbf{L}$ contains the axiom $(\Diamond p\to\Box q)\to\Box(p\to q)$~~

    ~~$\mathbf{L}$ contains the axiom $\Diamond(p\to q)\to(\Box p\to\Diamond q)$~~

    $\mathbf{L}$ is closed with respect to the rule $\frac{p\to q}{\Box p\to\Box q}$

    $\mathbf{L}$ is closed with respect to the rule $\frac{p\to q}{\Diamond p\to\Diamond q}$

    — $\mathbf{L}$ contains the axiom $\Box(p\vee q)\to((\Diamond p\to\Box q)\to\Box q)$

    — $\mathbf{L}$ is closed with respect to the rule $\frac{\Diamond p\to q\vee\Box(p\to r)}{\Diamond p\to q\vee\Diamond r}$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a fourth possibility

- Semantics : all downward and forward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$

    $s \models \Box A \Leftrightarrow \forall t \in W,\ (s \leq t \Rightarrow \forall u \in R(t),\ u \models A)$

    $s \models \Box A \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \forall u \in R(t),\ u \models A)$

    $s \models \Box A \Leftrightarrow \forall t \in R(s),\ t \models A$
  - Truth conditions for $\Diamond$

    $s \models \Diamond A \Leftrightarrow \forall t \in W,\ (s \leq t \Rightarrow \exists u \in R(t),\ u \models A)$

    $s \models \Diamond A \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$

    $s \models \Diamond A \Leftrightarrow \exists t \in R(s),\ t \models A$

Then, about intuitionistic modal logics

Considering the full language $\mathbf{FOR}_{\{\Box,\Diamond\}}$, a fourth possibility
B., Gao, Gencer and Olivetti — all down. and for. conf. $(W, \leq, R)$

- Defined logic $\mathbf{HK}\Box\Diamond = \mathbf{L}_{\mathsf{min}} + \Box(p \vee q) \rightarrow \Diamond p \vee \Box q + \Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q)$

Conclusion

Fischer Servi (1984)

- Semantics : all forward and backward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$
    $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$
  - Truth conditions for $\Diamond$
    $s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$
    $s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$
    $s \models \Diamond A \quad \Leftrightarrow \exists t \in R(s),\ t \models A$
- Defined logic **IK**
- Extensions of **IK**
  - **IT** $::=$ **IK** $+ \Box p \to p + p \to \Diamond p$
  - **I**4 $::=$ **IK** $+ \Box p \to \Box\Box p + \Diamond\Diamond p \to \Diamond p$
  - **IB** $::=$ **IK** $+ p \to \Box\Diamond p + \Diamond\Box p \to p$
  - **I**5 $::=$ **IK** $+ \Box p \to p + p \to \Diamond p + \Diamond p \to \Box\Diamond p + \Diamond\Box p \to \Box p$

Conclusion

B., Gao, Gencer and Olivetti

- Semantics : all forward confluent $(W, \leq, R)$
  - Truth conditions for $\Box$
    $$s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$$
  - Truth conditions for $\Diamond$
    $$s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$$
    $$s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$$
    $$s \models \Diamond A \quad \Leftrightarrow \exists t \in R(s),\ t \models A$$
- Defined logic **FIK**

Conclusion

Wijesekera (1990)

- Semantics : all $(W, \leq, R)$
  - Truth conditions for $\Box$
    $$s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$$
  - Truth conditions for $\Diamond$
    $$s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$$
- Defined logic **WK**
- Extension of **WK**
  - **WK** $+ \neg\Diamond\top\to\Box\bot + (\Diamond\top\to\Box p)\to\Box p$

Conclusion

B., Gao, Gencer and Olivetti

- Semantics : all $(W, \leq, R)$
  - Truth conditions for $\Box$
    $s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$
  - Truth conditions for $\Diamond$
    $s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$
- Defined logic $\mathbf{L}_{\mathrm{min}}$
- Extensions of $\mathbf{L}_{\mathrm{min}}$
  - $\mathbf{L}_{\mathbf{dc}} ::= \mathbf{L}_{\mathrm{min}} + \Box(p \vee q) \rightarrow \Diamond p \vee \Box q$
  - $\mathbf{L}_{\mathbf{fc}} ::= \mathbf{L}_{\mathrm{min}} + \Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q)$
  - $\mathbf{L}_{\mathbf{bc}} ::= \mathbf{L}_{\mathrm{min}} + (\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$
  - $\mathbf{L}_{\mathbf{dfc}} ::= \mathbf{L}_{\mathrm{min}} + \Box(p \vee q) \rightarrow \Diamond p \vee \Box q + \Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q)$
  - $\mathbf{L}_{\mathbf{dbc}} ::= \mathbf{L}_{\mathrm{min}} + \Box(p \vee q) \rightarrow \Diamond p \vee \Box q + (\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$
  - $\mathbf{L}_{\mathbf{fbc}} ::= \mathbf{L}_{\mathrm{min}} + \Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q) + (\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$
  - $\mathbf{L}_{\mathbf{dfbc}} ::= \mathbf{L}_{\mathrm{min}} + \Box(p \vee q) \rightarrow \Diamond p \vee \Box q + \Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q) + (\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$

Conclusion

B., Gao, Gencer and Olivetti

- Semantics : all down. and for. conf. $(W, \leq, R)$
  - Truth conditions for $\Box$

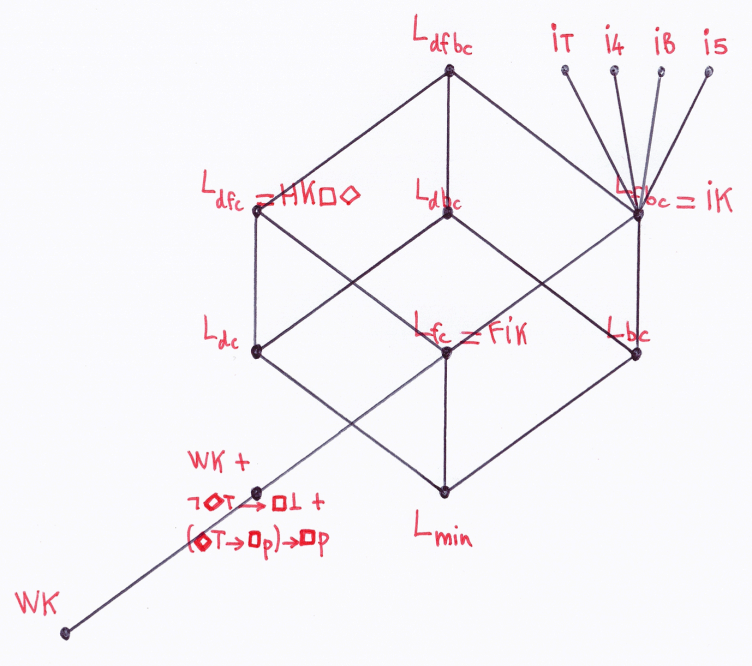
$$s \models \Box A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \forall u \in R(t),\ u \models A)$$
$$s \models \Box A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \forall u \in R(t),\ u \models A)$$
$$s \models \Box A \quad \Leftrightarrow \forall t \in R(s),\ t \models A$$

  - Truth conditions for $\Diamond$

$$s \models \Diamond A \quad \Leftrightarrow \forall t \in W,\ (s \leq t \ \Rightarrow\ \exists u \in R(t),\ u \models A)$$
$$s \models \Diamond A \quad \Leftrightarrow \exists t \in W,\ (s \geq t \ \&\ \exists u \in R(t),\ u \models A)$$
$$s \models \Diamond A \quad \Leftrightarrow \exists t \in R(s),\ t \models A$$

- Defined logic **HK**$\Box\Diamond$

Conclusion



Ldfbc
iT
i4
iB
i5
Ldfc = HK□◇
Ldbc
Lfbc = iK
Ldc
Lfc = FiK
Lbc
WK +
¬◇⊤ → □⊥ +
(◇⊤ → □p) → □p
Lmin
WK

Open problems

- Computability and complexity
- Correspondence Theory

Discussion

Extended language of intuitionistic modal logics

- Formulas

$$A ::= p|(A{\to}A)|\bot|\top|(A{\vee}A)|(A{\wedge}A)|\Box_1 A|\Diamond_1 A|\Box_2 A|\Diamond_2 A$$

Extended relational semantics of intuitionistic modal logics

- Truth conditions

$$s{\models}p \Leftrightarrow s{\in}V(p)$$
$$s{\models}A{\to}B \Leftrightarrow \forall t{\in}W,\ (s{\leq}t \ \Rightarrow\ t{\not\models}A \text{ or } t{\models}B)$$
$$\ldots$$
$$s{\models}\Box_1 A \Leftrightarrow \forall t{\in}W,\ (s{\leq}t \ \Rightarrow\ \forall u{\in}R(t),\ u{\models}A)$$
$$s{\models}\Diamond_1 A \Leftrightarrow \exists t{\in}W,\ (s{\geq}t \ \&\ \exists u{\in}R(t),\ u{\models}A)$$
$$s{\models}\Box_2 A \Leftrightarrow \exists t{\in}W,\ (s{\geq}t \ \&\ \forall u{\in}R(t),\ u{\models}A)$$
$$s{\models}\Diamond_2 A \Leftrightarrow \forall t{\in}W,\ (s{\leq}t \ \Rightarrow\ \exists u{\in}R(t),\ u{\models}A)$$

Thank you !

Questions ?

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