Conservative Extensions of Abstract Structures

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Abstract. In the present paper we investigate a relation, called conservative extension, between abstract structures $\mathfrak{A}$ and $\mathfrak{B}$, possibly with different signatures and $|\mathfrak{A}| \subseteq |\mathfrak{B}|$. We give a characterisation of this relation in terms of computable Σ_n formulae and we show that in some sense it provides a finer complexity measure than the one given by degree spectra of structures. As an application, we show that the n-th jump of a structure and its Marker's extension are conservative extensions of the original structure.

1 Introduction

We shall work with abstract structures of the form $\mathfrak{A} = (A; P_1, \dots, P_s)$, where A is countable and infinite, $P_i \subseteq A^{n_i}$ and the equality is among $P_1, \dots, P_s$. We shall use the letters $\mathfrak{A}$, $\mathfrak{B}$ to denote structures and the letters A, B to denote their domains.

Our initial motivation was to investigate the common features between the structures built in [8], namely the jump structure and the Marker's extension of a structure. It turns out that both structures relate to the initial structure in a similar way. In our terminology, the jump structure of $\mathfrak{A}$ is $(1,0)$-conservative extension of $\mathfrak{A}$ and the Marker's extension of $\mathfrak{A}$ is $(0,1)$-conservative extension of $\mathfrak{A}$. Our main results are Theorem 2 and Theorem 3 which show that a conservative extension of a structure preserves some families of sets definable with computable Σ formulae.

The main tool in our research is the enumeration of a structure. The pair $\alpha = (f_\alpha, R_\alpha)$ is called an *enumeration* of $\mathfrak{A}$ if R_α is a subset of natural numbers, f_α is a partial one-to-one mapping of $\mathbb{N}$ onto A and $f_\alpha^{-1}(\mathfrak{A})$ is computable in R_α, where $f_\alpha^{-1}(P_i) = \{\langle x_1, \dots, x_{n_i}\rangle \mid x_1, \dots, x_{n_i} \in Dom(f_\alpha) \;\&\; (f_\alpha(x_1), \dots, f_\alpha(x_{n_i})) \in P_i\}$ and $f_\alpha^{-1}(\mathfrak{A}) = f_\alpha^{-1}(P_1) \oplus \dots \oplus f_\alpha^{-1}(P_s)$. For an enumeration $\alpha = (f_\alpha, R_\alpha)$ of $\mathfrak{A}$ we denote $\alpha^{(n)} = (f_\alpha, R_\alpha^{(n)})$, where $R_\alpha^{(n)}$ is the n-th Turing jump of the set R_α. Given a set $X \subseteq A$, by $X \leq \alpha$ we shall denote that $f_\alpha^{-1}(X)$ is c.e. in R_α and by $\mathfrak{A} \leq \alpha$ we shall denote that α is an enumeration of $\mathfrak{A}$.

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We shall give an informal definition of the set of the computably infinitary Σ_n formulae in the language of $\mathfrak{A}$, denoted by Σ_n^c. The Σ_0^c and Π_0^c formulae are the finitary quantifier free formulae. A Σ_{n+1}^c formula $\varphi(\overline{x})$ is a disjunction of a c.e. set of formulae of the form $\exists \overline{y}\psi$, where ψ is a Π_n^c formula and $\overline{y}$ includes the variables of ψ which are not in $\overline{x}$. The Π_{n+1}^c formulae are the negations of the Σ_{n+1}^c formulae. We refer the reader to [1] for more background information on computably infinitary formulae.

A set $X \subseteq A$ is Σ_n^c definable in the structure $\mathfrak{A}$ if there is a Σ_n^c formula $\psi(x, \overline{y})$ and a finite number of parameters $\overline{a}$ in A such that $b \in X \leftrightarrow \mathfrak{A} \models \psi(b, \overline{a})$. We denote by $\Sigma_n^c(\mathfrak{A})$ the family of all sets Σ_n^c definable in $\mathfrak{A}$. A subset X of A is said to be *relatively intrinsically* Σ_{n+1}^0 in $\mathfrak{A}$ if for every enumeration α of $\mathfrak{A}$, $f_\alpha^{-1}(X)$ is Σ_{n+1}^0 relative to $f_\alpha^{-1}(\mathfrak{A})$ or equivalently, $f_\alpha^{-1}(X)$ is c.e. relative to $f_\alpha^{-1}(\mathfrak{A})^{(n)}$. In [2] and [3], it is shown that the relatively intrinsically Σ_{n+1}^0 in $\mathfrak{A}$ sets are exactly the Σ_{n+1}^c definable sets in $\mathfrak{A}$. We shall use this result in the following form.

Theorem 1 (Ash-Knight-Manasse-Slaman [2], Chisholm [3]) *Let $\mathfrak{A}$ be a countable structure. For every set $X \subseteq A$,*

$$X \in \Sigma_{n+1}^c(\mathfrak{A}) \text{ iff } (\forall \alpha)[\mathfrak{A} \leq \alpha \to X \leq \alpha^{(n)}].$$

2 Conservative Extensions

Let $\alpha = (f_\alpha, R_\alpha)$ and $\beta = (f_\beta, R_\beta)$ be enumerations of the countable structures $\mathfrak{A}$ and $\mathfrak{B}$ respectively. We write $\alpha \leq \beta$ if

(i) $R_\alpha \leq_T R_\beta$ and
(ii) the set $E(f_\alpha, f_\beta) = \{(x, y) \mid x \in Dom(f_\alpha)\ \&\ y \in Dom(f_\beta)\ \&\ f_\alpha(x) = f_\beta(y)\}$ is c.e. in R_β.

Definition 1 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures, possibly with different signatures and $A \subseteq B$.*

(i) $\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$ if for every enumeration β of $\mathfrak{B}$ there exists an enumeration α of $\mathfrak{A}$ such that $\alpha^{(k)} \leq \beta^{(n)}$.
(ii) $\mathfrak{A} \Leftarrow_n^k \mathfrak{B}$ if for every enumeration α of $\mathfrak{A}$ there exists an enumeration β of $\mathfrak{B}$ such that $\beta^{(k)} \leq \alpha^{(n)}$.
(iii) $\mathfrak{A} \Leftrightarrow_n^k \mathfrak{B}$ if $\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$ and $\mathfrak{A} \Leftarrow_k^n \mathfrak{B}$. We shall say that $\mathfrak{B}$ is a (k, n)-conservative extension of $\mathfrak{A}$.

The reader should be aware that the relation $\Leftrightarrow_n^k$ is not symmetric. The following theorem motivates the use of the term conservative extension, i.e. if $\mathfrak{B}$ is a (k, n)-conservative extension of $\mathfrak{A}$ then all Σ_{k+1}^c definable sets in $\mathfrak{A}$ are preserved as Σ_{n+1}^c definable sets in $\mathfrak{B}$.

Theorem 2 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A \subseteq B$. For all $k, n \in \omega$,*

(i) if $\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$ then $(\forall X \subseteq A)[X \in \Sigma_{k+1}^c(\mathfrak{A}) \to X \in \Sigma_{n+1}^c(\mathfrak{B})]$;

(ii) *if* $\mathfrak{A} \Leftarrow_k^n \mathfrak{B}$ *then* $(\forall X \subseteq A)[X \in \Sigma_{n+1}^c(\mathfrak{B}) \to X \in \Sigma_{k+1}^c(\mathfrak{A})]$*;*
(iii) *if* $\mathfrak{A} \Leftrightarrow_n^k \mathfrak{B}$ *then* $(\forall X \subseteq A)[X \in \Sigma_{k+1}^c(\mathfrak{A}) \leftrightarrow X \in \Sigma_{n+1}^c(\mathfrak{B})]$*.*

Proof. *(i)* We have that for every enumeration β of $\mathfrak{B}$, there exists an enumeration α of $\mathfrak{A}$ such that $\alpha^{(k)} \leq \beta^{(n)}$. Let X be a subset of A such that $X \in \Sigma_{k+1}^c(\mathfrak{A})$. According to Theorem 1 it is equivalent to $(\forall\alpha)[\mathfrak{A} \leq \alpha \to X \leq \alpha^{(k)}]$. We wish to show $(\forall\beta)[\mathfrak{B} \leq \beta \to X \leq \beta^{(n)}]$. Let us take an arbitrary enumeration β of $\mathfrak{B}$. Since $\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$, for some enumeration α of $\mathfrak{A}$, $\alpha^{(k)} \leq \beta^{(n)}$. It gives us that $R_\alpha^{(k)}$ is computable in $R_\beta^{(n)}$ and $E(f_\alpha, f_\beta)$ is c.e. in $R_\beta^{(n)}$. Moreover, $X \leq \alpha^{(k)}$ and then $f_\alpha^{-1}(X)$ is c.e. in $R_\beta^{(n)}$. From the equivalence

$$x \in f_\beta^{-1}(X) \leftrightarrow (\exists y)[(x,y) \in E(f_\alpha, f_\beta) \ \&\ y \in f_\alpha^{-1}(X)],$$

it follows that $f_\beta^{-1}(X)$ is c.e. in $R_\beta^{(n)}$ and then $X \leq \beta^{(n)}$ which is what we wanted to show. The proof of *(ii)* is similar to that of *(i)*. □

Remark 1. Notice that we do not have the other directions in Theorem 2. Assume $A \subseteq B$ and if $(\forall X \subseteq A)[X \in \Sigma_{n+1}^c(\mathfrak{A}) \to X \in \Sigma_{k+1}^c(\mathfrak{B})]$ then $\mathfrak{A} \Rightarrow_k^n \mathfrak{B}$. We can give a simple counterexample. Let $\mathscr{O}_A = (A; =)$ and take $\mathfrak{A} = \mathfrak{B} = \mathscr{O}_A$. It is easy to see that for every natural number n, $X \subseteq A$ is Σ_n^c-definable in $\mathscr{O}_A$ iff X is a finite or co-finite subset of A. Therefore $\Sigma_1^c(\mathscr{O}_A) = \Sigma_n^c(\mathscr{O}_A)$ and then $(\forall n)(\forall X \subseteq A)[X \in \Sigma_{n+1}^c(\mathscr{O}_A) \to X \in \Sigma_1^c(\mathscr{O}_A)]$. We conclude that $(\forall n)[\mathscr{O}_A \Rightarrow_0^n \mathscr{O}_A]$, which is evidently not true.

We shall proceed with the investigation of under what conditions we have the other directions in Theorem 2. For this purpose we shall firstly introduce some coding machinery and then the sets $K_n^{\mathfrak{A}}$.

2.1 Moschovakis' Extension

Following Moschovakis [6], we define the least acceptable extension $\mathfrak{A}^\star$ of $\mathfrak{A}$. Let 0 be an object which does not belong to A and Π be a pairing operation chosen so that neither 0 nor any element of A is an ordered pair. Let $A^\star$ be the least set containing all elements of $A_0 = A \cup \{0\}$ and closed under Π.

We associate an element $n^\star$ of $A^\star$ with each $n \in \omega$ by induction. Let $0^\star = 0$ and $(n+1)^\star = \Pi(0, n^\star)$. We denote by $\mathbb{N}^\star$ the set of all elements $n^\star$. Let L and R be the functions on $A^\star$ satisfying the following conditions:

$$\begin{aligned}&L(0) = R(0) = 0;\\&(\forall t \in A)[L(t) = R(t) = 1^\star];\\&(\forall s,t \in A^\star)[L(\Pi(s,t)) = s \ \&\ R(\Pi(s,t)) = t].\end{aligned}$$

The pairing function allows us to code finite sequences of elements. Let $\Pi_1(t_1) = t_1$ and $\Pi_{n+1}(t_1, \ldots, t_{n+1}) = \Pi(t_1, \Pi_n(t_2, \ldots, t_{n+1}))$ for every $t_1, \ldots, t_{n+1} \in A^\star$. For each predicate P_i of the structure $\mathfrak{A}$ define the respective predicate $P_i^\star$ on $A^\star$ by $P_i^\star(t) \leftrightarrow (\exists a_1, \ldots, a_{n_i} \in A)[t = \Pi_{n_i}(a_1, \ldots, a_n) \& P_i(a_1, \ldots, a_n)]$.

Definition 2 *Moschovakis' extension of* $\mathfrak{A}$ *is the structure*

$$\mathfrak{A}^\star = (A^\star; A_0, P_1^\star, \dots, P_s^\star, G_\Pi, G_L, R_R, =),$$

where G_Π*,* G_L *and* G_R *are the graphs of* Π*,* L *and* R *respectively.*

Proposition 1 *For every two structures* $\mathfrak{A}$*,* $\mathfrak{B}$ *with* $A \subseteq B$ *and* $n, k \in \omega$*,* $\mathfrak{A} \Leftrightarrow_n^k \mathfrak{B}$ *iff* $\mathfrak{A}^\star \Leftrightarrow_n^k \mathfrak{B}^\star$*. Moreover,* $\mathfrak{A} \Leftrightarrow_n^n \mathfrak{A}^\star$*.*

2.2 The set $K_n^{\mathfrak{A}}$

Let $\alpha = (f_\alpha, R_\alpha)$ be an enumeration of $\mathfrak{A}$. For every $e, x \in \omega$ and every $n \in \omega$, we define the modelling relations $\models_n$ in the following way:

$$f_\alpha \models_0 F_e(x) \leftrightarrow x \in W_e^{f_\alpha^{-1}(\mathfrak{A})}$$
$$f_\alpha \models_{n+1} F_e(x) \leftrightarrow x \in W_e^{f^{-1}(\mathfrak{A})^{(n+1)}}$$
$$f_\alpha \models_n \neg F_e(x) \leftrightarrow f_\alpha \not\models_n F_e(x)$$

Following the modelling relation, we shall define a forcing relation with conditions all finite injective mappings from $\mathbb{N}$ into the domain A of $\mathfrak{A}$. We call them *finite parts* and we shall use the letters τ, ρ, δ to denote them. Let $\Delta(A)$ be the set of all finite parts and let Fin_2 be the set of all finite functions on the natural numbers taking values in $\{0, 1\}$. Given a finite part τ and a relation $R \subseteq A^n$, we define the finite function $\tau^{-1}(R)$ in Fin_2 as follows:

$$\tau^{-1}(R)(u) \downarrow = 1 \leftrightarrow (\exists x_1, \dots, x_n \in Dom(\tau))[u = \langle x_1, \dots, x_n \rangle \ \& \ (\tau(x_1), \dots, \tau(x_n)) \in R],$$
$$\tau^{-1}(R)(u) \downarrow = 0 \leftrightarrow (\exists x_1, \dots, x_n \in Dom(\tau))[u = \langle x_1, \dots, x_n \rangle \ \& \ (\tau(x_1), \dots, \tau(x_n)) \notin R].$$

By $\tau^{-1}(\mathfrak{A})$ we shall denote the finite function $\tau^{-1}(R_1) \oplus \dots \oplus \tau^{-1}(R_s)$.

If φ is a partial function and $e \in \omega$, then by W_e^φ we shall denote the set of all x such that the computation $\{e\}^\varphi(x)$ halts successfully. We shall assume that if during a computation the oracle φ is called with an argument outside of its domain, then the computation halts unsuccessfully.

For every $e, x, n \in \omega$ and for every finite part τ, we define the forcing relations in the following way:

$$\tau \Vdash_0 F_e(x) \leftrightarrow x \in W_e^{\tau^{-1}(\mathfrak{A})},$$
$$\tau \Vdash_{n+1} F_e(x) \leftrightarrow (\exists \delta \in Fin_2)[x \in W_e^\delta \ \& \ (\forall z \in Dom(\delta))[(\delta(z) = 1 \ \& \ \tau \Vdash_n F_z(z)) \vee (\delta(z) = 0 \ \& \ \tau \Vdash_n \neg F_z(z))]],$$
$$\tau \Vdash_n \neg F_e(x) \leftrightarrow (\forall \rho \in \Delta(A))[\tau \subseteq \rho \to \rho \nVdash_n F_e(x)].$$

An enumeration α of $\mathfrak{A}$ is called *n-generic* if for every $e, x \in \omega$ and every $j < n$, $(\exists \tau \subseteq f_\alpha)[\tau \Vdash_j F_e(x) \vee \tau \Vdash_j \neg F_e(x)]$.

Lemma 1 (Truth Lemma).

(i) For every $n, e, x \in \omega$ and every finite parts $\tau \subseteq \rho$,

$$\tau \Vdash_n (\neg)F_e(x) \to \rho \Vdash_n (\neg)F_e(x).$$

(ii) For every n-generic enumeration α of $\mathfrak{A}$ and all $e, x \in \omega$,

$$f_\alpha \models_n F_e(x) \leftrightarrow (\exists \tau \subseteq f_\alpha)[\tau \Vdash_n F_e(x)].$$

(iii) For every $(n+1)$-generic enumeration α of $\mathfrak{A}$ and all $e, x \in \omega$,

$$f_\alpha \models_n \neg F_e(x) \leftrightarrow (\exists \tau \subseteq f_\alpha)[\tau \Vdash_n \neg F_e(x)].$$

For each finite part $\tau \neq \emptyset$ with $Dom(\tau) = \{x_1, \ldots, x_n\}$ and $\tau(x_i) = s_i$, we associate the element $\tau^\star = \Pi_n(\Pi(x_1^\star, s_1), \ldots, \Pi(x_n^\star, s_n))$ of $A^\star$. For $\tau = \emptyset$, let $\tau^\star = 0$. We define for every $n \in \omega$ the set

$$K_n^{\mathfrak{A}} = \{\Pi_3(\delta^\star, e^\star, x^\star) \mid (\exists \tau \in \Delta(A))[\delta \subseteq \tau \ \&\ \tau \Vdash_n F_e(x)] \ \&\ e^\star, x^\star \in \mathbb{N}^\star\}.$$

Proposition 2 *For every countable structure $\mathfrak{A}$ and every $n \in \omega$, we have $K_n^{\mathfrak{A}} \in \Sigma_{n+1}^c(\mathfrak{A}^\star)$ and $A^\star \setminus K_n^{\mathfrak{A}} \in \Sigma_{n+2}^c(\mathfrak{A}^\star)$.*

Theorem 3 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A^\star \subseteq B$ and $k, n \in \omega$. Suppose that $(\forall X \subseteq A^\star)[X \in \Sigma_{k+1}^c(\mathfrak{A}^\star) \to X \in \Sigma_{n+1}^c(\mathfrak{B})]$. Then $\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$.*

Proof. Let us fix an enumeration $\beta = (f_\beta, R_\beta)$ of $\mathfrak{B}$. We shall show that there exists an enumeration $\gamma = (f_\gamma, f_\gamma^{-1}(\mathfrak{A}))$ of $\mathfrak{A}$ such that $\gamma^{(k)} \leq \beta^{(n)}$.

Firstly, let $k = 0$. Since $A \in \Sigma_1^c(\mathfrak{A}^\star)$, $A \in \Sigma_{n+1}^c(\mathfrak{B})$ and then by Theorem 1, $f_\beta^{-1}(A)$ is c.e. in $R_\beta^{(n)}$. We can take a total enumeration f_γ of A defined as $f_\gamma = f_\beta \circ \mu$, where $\mu : \mathbb{N} \to f_\beta^{-1}(A)$ is a computable in $R_\beta^{(n)}$ bijection. Such μ exists because $f_\beta^{-1}(A)$ is c.e. in $R_\beta^{(n)}$. Clearly the set $E(f_\gamma, f_\beta)$ is c.e. in $R_\beta^{(n)}$. We have for all $P_i^{\mathfrak{A}}$ of $\mathfrak{A}$, $P_i^{\mathfrak{A}} \in \Sigma_{n+1}^c(\mathfrak{B})$ and $A^{n_i} \setminus P_i^{\mathfrak{A}} \in \Sigma_{n+1}^c(\mathfrak{B})$. Thus $f_\beta^{-1}(P_i^{\mathfrak{A}})$ and $f_\beta^{-1}(A^{n_i} \setminus P_i^{\mathfrak{A}})$ are c.e. in $R_\beta^{(n)}$. $f_\gamma^{-1}(P_i^{\mathfrak{A}})$ is c.e. in $R_\beta^{(n)}$ and since f_γ is total, $\mathbb{N} \setminus f_\gamma^{-1}(P_i^{\mathfrak{A}})$ is c.e. in $R_\beta^{(n)}$. Therefore, $f_\gamma^{-1}(\mathfrak{A}) \leq_T R_\beta^{(n)}$ and hence $\gamma \leq \beta^{(n)}$.

Let $k > 0$. We shall build a k-generic enumeration $\gamma = (f_\gamma, f_\gamma^{-1}(\mathfrak{A}))$ of $\mathfrak{A}$ such that $f_\gamma^{-1}(\mathfrak{A})^{(k)} \leq_T R_\beta^{(n)}$ and $E(f_\gamma, f_\beta)$ is c.e. in $R_\beta^{(n)}$. Before proceeding with its construction, we shall describe a way to encode finite parts $\tau \in \Delta(A)$ as natural numbers. We define a coding scheme for finite sequences of natural numbers belonging to $f_\beta^{-1}(A^\star)$ in the following way:

$$J(x, y) = f_\beta^{-1}(\Pi(f_\beta(x), f_\beta(y)));$$
$$J_1(x) = x, \quad J_{n+1}(x_1, \ldots, x_{n+1}) = J(x_1, J_n(x_2, \ldots, x_n, x_{n+1})).$$

For every natural number n, we denote $n^\sharp = f_\beta^{-1}(n^\star)$ and $\mathbb{N}^\sharp = f_\beta^{-1}(\mathbb{N}^\star)$. For finite parts $\tau \in \Delta(A)$, we associate with $\tau^\star$ the natural number $\tau^\sharp = f_\beta^{-1}(\tau^\star)$.

That is, if $\tau^\star = \Pi_n(\Pi(x_1^\star, y_1), \ldots, \Pi(x_n^\star, y_n))$ then $\tau^\sharp = J_n(J(x_1^\sharp, f_\beta^{-1}(y_1)), \ldots, J(x_n^\sharp, f_\beta^{-1}(y_n)))$. Therefore, the set $\Delta^\sharp(A) = \{\tau^\sharp \mid \tau \in \Delta(A)\}$ is c.e. in $R_\beta^{(n)}$. Let $Dom(\tau^\sharp) = \{x_1^\sharp, \ldots, x_n^\sharp\}$ and $\tau^\sharp(x_i^\sharp) = f_\beta^{-1}(y_i)$. We shall assume that $Dom(\tau^\sharp) = \emptyset$ if $\tau^\sharp = 0$. Notice that $Dom(\tau^\sharp) = \{x^\sharp \mid x \in Dom(\tau)\}$ and $f_\beta(\tau^\sharp(x^\sharp)) = \tau(x)$ for all $x \in Dom(\tau)$. There exists a partial computable in $R_\beta^{(n)}$ predicate P such that for $\tau, \delta \in \Delta(A)$, $P(\tau^\sharp, \delta^\sharp) \downarrow = 1$ iff $\tau \subseteq \delta$. We shall write $\tau^\sharp \subseteq \delta^\sharp$ instead of $P(\tau^\sharp, \delta^\sharp) \downarrow = 1$. From Proposition 2 we know that $K_{k-1}^{\mathfrak{A}}$ and $A^\star \setminus K_{k-1}^{\mathfrak{A}}$ are Σ_{k+1}^c definable in $\mathfrak{A}^\star$. This means that $K_{k-1}^{\mathfrak{A}}$ and $A^\star \setminus K_{k-1}^{\mathfrak{A}}$ are Σ_{n+1}^c definable in $\mathfrak{B}$. Thus $f_\beta^{-1}(K_{k-1}^{\mathfrak{A}})$ and $f_\beta^{-1}(A^\star \setminus K_{k-1}^{\mathfrak{A}})$ are both c.e. in $R_\beta^{(n)}$. It is not hard to see that there exists a computable function χ such that for every $\tau \in \Delta(A)$, $\tau \Vdash_{k-1} F_e(x) \leftrightarrow x \in W_{\chi(\tau^\sharp, e)}^{R_\beta^{(n)}}$.

Claim. There exists a k-generic enumeration γ of $\mathfrak{A}$ such that $f_\gamma^\sharp$ is partial computable in $R_\beta^{(n)}$, where $f_\gamma^\sharp : \mathbb{N}^\sharp \to f_\beta^{-1}(A)$ is defined as $f_\gamma^\sharp(x^\sharp) = f_\beta^{-1}(f_\gamma(x))$.

Proof. Since the set A is Σ_{k+1}^c definable in $\mathfrak{A}$, $f_\beta^{-1}(A)$ is c.e. in $R_\beta^{(n)}$. Let us fix a computable in $R_\beta^{(n)}$ bijection $\mu : \mathbb{N} \to f_\beta^{-1}(A)$. We shall describe a construction in which at each stage s we shall define a finite part $\tau_s \subseteq \tau_{s+1}$. In the end, the k-generic enumeration of $\mathfrak{A}$ will be defined as $f_\gamma = \bigcup_s \tau_s$ and $R_\gamma = f_\gamma^{-1}(\mathfrak{A})$. Let $\tau_0 = \emptyset$ and suppose we have already defined τ_s.

a) Case $s = 2r$. We make sure that f_γ is one-to-one and onto A. Let $x^\sharp$ be the least natural number not in $Dom(\tau_s^\sharp)$. Find the least p such that $\mu(p) \notin Ran(\tau_s^\sharp)$. Set $\tau_{s+1}(x) = f_\beta(\mu(p))$ and $\tau_{s+1}(z) = \tau_s(z)$ for every $z \neq x$ and $z \in Dom(\tau_s)$. Leave $\tau_{s+1}(z)$ undefined for any other z.
b) Case $s = 2\langle e, x\rangle + 1$. We satisfy the requirement that f_γ is k-generic. Check whether there exists an extension δ of τ_s such that $\delta \Vdash_{k-1} F_e(x)$. This is equivalent to asking whether $J_3(\tau_s^\sharp, e^\sharp, x^\sharp) \in f_\beta^{-1}(K_{k-1}^{\mathfrak{A}})$ or $J_3(\tau_s^\sharp, e^\sharp, x^\sharp) \in f_\beta^{-1}(A^\star \setminus K_{k-1}^{\mathfrak{A}})$. We can do this effectively using the oracle $R_\beta^{(n)}$.
If $J_3(\tau_s^\sharp, e^\sharp, x^\sharp) \in f_\beta^{-1}(A^\star \setminus K_{k-1}^{\mathfrak{A}})$, then $\tau_s \Vdash_{k-1} \neg F_e(x)$ and we set $\tau_{s+1} = \tau_s$.
If $J_3(\tau_s^\sharp, e^\sharp, x^\sharp) \in f_\beta^{-1}(K_{k-1}^{\mathfrak{A}})$, we search for $\delta^\sharp \in \Delta^\sharp(A)$ such that $\tau_s^\sharp \subseteq \delta^\sharp$ and $x \in W_{\chi(\delta^\sharp, e)}^{R_\beta^{(n)}}$. Since $J_3(\tau_s^\sharp, e^\sharp, x^\sharp) \in f_\beta^{-1}(K_{k-1}^{\mathfrak{A}})$ we know that such $\delta^\sharp$ exists and we can find it effectively in $R_\beta^{(n)}$. Set $\tau_{s+1} = \delta$.

End of construction
It follows from the construction that $f_\gamma^\sharp$ is partial computable in $R_\beta^{(n)}$. □

The equivalence $f_\gamma(x) = f_\beta(y) \leftrightarrow f_\gamma^\sharp(x^\sharp) = y$ and the fact that the graph of $f_\gamma^\sharp$ is c.e. in $R_\beta^{(n)}$ implies that the set $E(f_\gamma, f_\beta)$ is c.e. in $R_\beta^{(n)}$. Since f_γ is k-generic, we have the equivalences

$$x \in f_\gamma^{-1}(\mathfrak{A})^{(k)} \leftrightarrow f_\gamma \models_{k-1} F_x(x) \leftrightarrow (\exists \tau \subseteq f_\gamma)[\tau \Vdash_{k-1} F_x(x)]$$

$$\leftrightarrow (\exists \tau^\sharp \subseteq f_\gamma^\sharp)[x \in W^{R_\beta^{(n)}}_{\chi(\tau^\sharp, x)}].$$

$$x \notin f_\gamma^{-1}(\mathfrak{A})^{(k)} \leftrightarrow f_\gamma \models_{k-1} \neg F_x(x) \leftrightarrow (\exists \tau \subseteq f_\gamma)[\tau \Vdash_{k-1} \neg F_x(x)]$$

$$\leftrightarrow (\exists \tau^\sharp \subseteq f_\gamma^\sharp)[J_3(\tau^\sharp, x^\sharp, x^\sharp) \in f_\beta^{-1}(A^\star \setminus K^{\mathfrak{A}}_{k-1})].$$

Since $f_\beta(\tau^\sharp(x^\sharp)) = \tau(x)$, we have the equivalence:

$$\tau^\sharp \subseteq f_\gamma^\sharp \leftrightarrow (\forall x^\sharp \in Dom(\tau^\sharp))(\exists y)[(x, y) \in E(f_\gamma, f_\beta) \& \ \langle \tau^\sharp(x^\sharp), y\rangle \in f_\beta^{-1}(=^{\mathfrak{A}^\star})].$$

It means that the relation $\tau^\sharp \subseteq f_\gamma^\sharp$ is c.e. in $R_\beta^{(n)}$. It follows that $f_\gamma^{-1}(\mathfrak{A})^{(k)}$ is computable in $R_\beta^{(n)}$. We conclude that for the enumeration $\gamma = (f_\gamma, f_\gamma^{-1}(\mathfrak{A}))$ of $\mathfrak{A}$, $\gamma^{(k)} \leq \beta^{(n)}$ and hence $\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$. □

Corollary 1. *For any two countable structures $\mathfrak{A}$, $\mathfrak{B}$ with $A \subseteq B$ and $n, k \in \omega$,*

$$\mathfrak{A} \Rightarrow_n^k \mathfrak{B} \leftrightarrow (\forall X \subseteq A^\star)[X \in \Sigma^c_{k+1}(\mathfrak{A}^\star) \rightarrow X \in \Sigma^c_{n+1}(\mathfrak{B}^\star)].$$

3 Applications

3.1 Degree Spectra of Structures

In [7], Richter initiates the study of the notion of the degree spectrum of a countable structure. Here we define the degree spectrum following [9].

Definition 3 *The Turing degree spectrum of $\mathfrak{A}$ is the set $DS(\mathfrak{A}) = \{d_T(R_\alpha) \mid \mathfrak{A} \leq \alpha\}$. The k-th jump Turing degree spectrum of $\mathfrak{A}$ is the set $DS_k(\mathfrak{A}) = \{d_T(R_\alpha^{(k)}) \mid \mathfrak{A} \leq \alpha\}$.*

Here by $d_T(X)$ we denote the Turing degree of the set X. A set of Turing degrees $\mathscr{A}$ is *closed upwards* if for all Turing degrees $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \in \mathscr{A}$ & $\mathbf{a} \leq \mathbf{b} \rightarrow \mathbf{b} \in \mathscr{A}$. It is clear that for every structure $\mathfrak{A}$, its degree spectrum $DS(\mathfrak{A})$ is closed upwards.

Remark 2. Richter's definition of degree spectrum is slightly different. She defines the degree spectrum as the set of all Turing degrees $d_T(f^{-1}(\mathfrak{A}))$, where f is a *total* enumeration of the domain of $\mathfrak{A}$. Both definitions produce the same sets of Turing degrees for automorphically non-trivial structures.

Proposition 3 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A \subseteq B$.*

(i) If $\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$ then $DS_n(\mathfrak{B}) \subseteq DS_k(\mathfrak{A})$;
(ii) If $\mathfrak{A} \Leftarrow_k^n \mathfrak{B}$ then $DS_k(\mathfrak{A}) \subseteq DS_n(\mathfrak{B})$;
(iii) If $\mathfrak{A} \Leftrightarrow_n^k \mathfrak{B}$ then $DS_k(\mathfrak{A}) = DS_n(\mathfrak{B})$;

Proof. We shall prove only *(i)* since the others are similar. Let $\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$ and $\mathbf{b} \in DS_n(\mathfrak{B})$. We wish to show that $\mathbf{b} \in DS_k(\mathfrak{A})$. Since $DS_k(\mathfrak{A})$ is closed upwards, it is enough to prove that there exists a Turing degree $\mathbf{a} \in DS_k(\mathfrak{A})$ and $\mathbf{a} \leq \mathbf{b}$. Let β be an enumeration of $\mathfrak{B}$ and $d_T(R_\beta^{(n)}) = \mathbf{b}$. $\mathfrak{A} \Rightarrow_n^k \mathfrak{B}$ gives us an enumeration α of $\mathfrak{A}$ such that $\alpha^{(k)} \leq \beta^{(n)}$. For $\mathbf{a} = d_T(R_\alpha^{(k)})$ we have $\mathbf{a} \in DS_k(\mathfrak{A})$ and $\mathbf{a} \leq \mathbf{b}$. □

Remark 3. We should note that we do not have the other directions in Proposition 3. Let us define the structures $\mathscr{O}_\mathbb{N} = (\mathbb{N}; =)$ and $\mathscr{S} = (\mathbb{N}; G_{Succ}, =)$, where G_{Succ} is the graph of the successor function. It is easy to see that $DS(\mathscr{O}_\mathbb{N}) = DS(\mathscr{S})$ whereas it follows easily from Theorem 2 that $\mathscr{S} \not\Rightarrow_0^0 \mathscr{O}_\mathbb{N}$.

3.2 Jumps of Structures

In [8], the jump of the structure $\mathfrak{A}$ is defined as $\mathfrak{A}' = (\mathfrak{A}^\star, K_0^\mathfrak{A})$. It is natural to ask whether we can extend it for $n > 0$.

Definition 4 *Let $\mathfrak{A}$ be a countable structure. For every natural number n, we define the n-th jump of $\mathfrak{A}$ in the following way.*

$$\mathfrak{A}^{(0)} = \mathfrak{A} \text{ and } \mathfrak{A}^{(n+1)} = (\mathfrak{A}^\star, K_n^\mathfrak{A}).$$

Actually, the results in [8] are enough to produce a definition of the n-th jump of $\mathfrak{A}$, just let $\mathfrak{A}^{(n+1)} = (\mathfrak{A}^{(n)})'$. The difficulty with it is that we add a new relation symbol and a new layer of coding to the structure for each jump.

Using the enumeration built in Lemma 7 of [8], we can easily obtain the following useful result.

Proposition 4 *Let $\mathfrak{A}$ be a countable structure.*

(i) For every enumeration α of $\mathfrak{A}$ there exists an enumeration α_0 of $\mathfrak{A}^{(n)}$ such that $\alpha_0 \leq \alpha^{(n)}$.
(ii) For every n-generic enumeration γ of $\mathfrak{A}$ there exists an enumeration $\gamma^\star = (f_{\gamma^\star}, f_{\gamma^\star}^{-1}(\mathfrak{A}^\star))$ of $\mathfrak{A}^\star$ such that $f_\gamma^{-1}(\mathfrak{A})^{(n)} \equiv_T f_{\gamma^\star}^{-1}(\mathfrak{A}^\star)^{(n)} \equiv_T f_{\gamma^\star}^{-1}(\mathfrak{A}^{(n)})$.

Proposition 5 *For any countable structure $\mathfrak{A}$, we have*

(i) For every $n \in \omega$, $K_n^\mathfrak{A} \notin \Sigma_n^c(\mathfrak{A}^\star)$.
(ii) For every $n, k \in \omega$ with $k > 0$, $K_{k+n}^\mathfrak{A} \in \Sigma_{n+1}^c(\mathfrak{A}^{(k)})$ and $K_{k+n}^\mathfrak{A} \notin \Sigma_n^c(\mathfrak{A}^{(k)})$.

Proof. *(i)* Assume $K_n^\mathfrak{A} \in \Sigma_n^c(\mathfrak{A}^\star)$. If $n = 0$ then $K_0^\mathfrak{A}$ is definable in $\mathfrak{A}^\star$ by a finitary open formula. This means that for every enumeration α of $\mathfrak{A}^\star$, $f_\alpha^{-1}(K_0^A)$ is computable in $f_\alpha^{-1}(\mathfrak{A}^\star)$ and then $f_\alpha^{-1}(\mathfrak{A}')$ is computable in $f_\alpha^{-1}(\mathfrak{A}^\star)$. Take a 1-generic enumeration γ of $\mathfrak{A}$. Then $\gamma^\star$, as in *(ii)* of Proposition 4, is an enumeration of $\mathfrak{A}^\star$ and $f_\gamma^{-1}(\mathfrak{A})' \equiv_T f_{\gamma^\star}^{-1}(\mathfrak{A}') \leq_T f_\gamma^{-1}(\mathfrak{A})$. This is clearly a contradiction.

Let $n > 0$. Theorem 1 tells us that for every enumeration α of $\mathfrak{A}^\star$, $f_\alpha^{-1}(K_n^\mathfrak{A})$ is c.e. in $R_\alpha^{(n-1)}$ and therefore $f_\alpha^{-1}(\mathfrak{A}^{(n+1)})$ is computable in $R_\alpha^{(n)}$. Let γ be an

$(n+1)$-generic enumeration of $\mathfrak{A}$ and $\gamma^\star$ be as in *(ii)* of Proposition 4. Since $\gamma^\star$ is an enumeration of $\mathfrak{A}^\star$, $f_{\gamma^\star}^{-1}(\mathfrak{A}^{(n+1)})$ is computable in $f_{\gamma^\star}^{-1}(\mathfrak{A}^\star)^{(n)}$. But we also have $f_{\gamma^\star}^{-1}(\mathfrak{A}^\star)^{(n+1)} \leq_T f_{\gamma^\star}^{-1}(\mathfrak{A}^{(n+1)})$. Thus we reach a contradiction.

The proof of the first part of *(ii)* uses Theorem 1 and follows by induction on k. For the second part, if we assume $K_{k+n}^{\mathfrak{A}} \in \Sigma_n^c(\mathfrak{A}^{(k)})$ then by taking an $(n+k)$-generic enumeration of $\mathfrak{A}$, we argue as above to reach a contradiction.

□

Proposition 6 *For every countable structure $\mathfrak{A}$ and natural number n,*

(i) $\mathfrak{A} \Leftrightarrow_0^n \mathfrak{A}^{(n)}$;
(ii) $\mathfrak{A}^{(n)} \Rightarrow_0^0 \mathfrak{A}^{(n+1)}$ *and* $\mathfrak{A}^{(n)} \not\Leftrightarrow_0^0 \mathfrak{A}^{(n+1)}$.

Proof. *(i)* Let $n > 0$ since it is obvious for $n = 0$. $\mathfrak{A} \Rightarrow_0^n \mathfrak{A}^{(n)}$ is a direct application of Theorem 3. Now we wish to show $\mathfrak{A} \Leftarrow_n^0 \mathfrak{A}^{(n)}$. Let us take an enumeration α of $\mathfrak{A}$. From *(i)* of Proposition 4, there is an enumeration α_0 of $\mathfrak{A}^{(n)}$ such that $\alpha_0 \leq \alpha^{(n)}$.

(ii) Let $n = 0$. Clearly $\mathfrak{A} \Rightarrow_0^0 \mathfrak{A}'$. Assume $\mathfrak{A} \Leftarrow_0^0 \mathfrak{A}'$. Let $\gamma = (f_\gamma, f_\gamma^{-1}(\mathfrak{A}))$ be a 1-generic enumeration of $\mathfrak{A}$ and $\beta = (f_\beta, R_\beta)$ be an enumeration of $\mathfrak{A}'$ such that $\beta \leq \gamma$. As in the proof of Theorem 3, we use β to define a coding scheme J_n and prove that the relation $\tau^\sharp \subseteq f_\gamma^\sharp$ is c.e. in $f_\gamma^{-1}(\mathfrak{A})$. From the equivalences $x \in f_\gamma^{-1}(\mathfrak{A})' \leftrightarrow (\exists \tau^\sharp \subseteq f_\gamma^\sharp)[x \in W_{\chi(\tau^\sharp, x)}^{f_\gamma^{-1}(\mathfrak{A})}]$ and $x \notin f_\gamma^{-1}(\mathfrak{A})' \leftrightarrow (\exists \tau^\sharp \subseteq f_\gamma^\sharp)[J_3(\tau^\sharp, x^\sharp, x^\sharp) \in f_\beta^{-1}(A^\star \setminus K_0^{\mathfrak{A}})]$, it follows that $f_\gamma^{-1}(\mathfrak{A})'$ is computable in $f_\gamma^{-1}(\mathfrak{A})$. Thus we reach a contradiction.

Let $n > 0$. It is clear that $\mathfrak{A}^{(n)} \Rightarrow_0^0 \mathfrak{A}^{(n+1)}$. Assume $\mathfrak{A}^{(n)} \Leftrightarrow_0^0 \mathfrak{A}^{(n+1)}$. Theorem 2 gives us $K_{n+1}^{\mathfrak{A}} \in \Sigma_1^c(\mathfrak{A}^{(n+1)})$ iff $K_{n+1}^{\mathfrak{A}} \in \Sigma_1^c(\mathfrak{A}^{(n)})$. On the other hand, *(ii)* of Proposition 5 tells us that $K_{n+1}^{\mathfrak{A}} \in \Sigma_1^c(\mathfrak{A}^{(n+1)})$ and $K_{n+1}^{\mathfrak{A}} \notin \Sigma_1^c(\mathfrak{A}^{(n)})$. Thus our assumption is absurd and hence $\mathfrak{A}^{(n)} \not\Leftarrow_0^0 \mathfrak{A}^{(n+1)}$. □

Since $\mathfrak{A} \Leftrightarrow_n^k \mathfrak{B}$ implies $DS_k(\mathfrak{A}) = DS_n(\mathfrak{B})$, we get the following.

Corollary 2. *For every countable structure $\mathfrak{A}$, $DS(\mathfrak{A}^{(n)}) = DS_n(\mathfrak{A})$.*

Proposition 7 *For all countable structures $\mathfrak{A}$, $\mathfrak{B}$ with $A \subseteq B$ and $n, k \in \omega$,*

$$\mathfrak{A} \Leftrightarrow_n^k \mathfrak{B} \textit{ iff } \mathfrak{A}^{(k)} \Leftrightarrow_0^0 \mathfrak{B}^{(n)}.$$

Proposition 8 *Let $\mathfrak{A}$ be a countable structure, $n, k \in \omega$ and $k > 0$.*

(i) $(\forall X \subseteq A^\star)[X \in \Sigma_{n+2}^c(\mathfrak{A}^\star) \leftrightarrow X \in \Sigma_{n+1}^c(\mathfrak{A}')]$;
(ii) $(\forall X \subseteq A^\star)[X \in \Sigma_{n+2}^c(\mathfrak{A}^{(k)}) \leftrightarrow X \in \Sigma_{n+1}^c(\mathfrak{A}^{(k+1)})]$;

3.3 Marker's Extensions

In [4], Goncharov and Khoussainov adapted Marker's construction from [5] to prove that for any natural number $n \geq 1$, there exists an $\aleph_1$-categorical theory T with a computable model of a finite language whose theories are Turing equivalent to $\emptyset^{(n)}$. Building on their results, A. Soskova and I. Soskov proved a theorem for the degree spectrum of structures resembling a jump inversion theorem, namely the following theorem.

Theorem 4 (A. Soskova - I. Soskov [8]) *Let $\mathfrak{A}$ and $\mathfrak{C}$ be countable structures and $DS(\mathfrak{A}) \subseteq DS_1(\mathfrak{C})$. There exists a structure $\mathfrak{B}$ such that $DS(\mathfrak{A}) = DS_1(\mathfrak{B})$ and $DS(\mathfrak{B}) \subseteq DS(\mathfrak{C})$.*

In [10], Stukachev proves an analogue of this theorem for the semilattices of Σ-degrees of structures with arbitrary cardinalities.

Theorem 5 (Stukachev [10]) *Let $\mathfrak{A}$ be a structure such that $\mathbf{0}' \leq_{\mathbf{\Sigma}} \mathfrak{A}$. There exists a structure $\mathfrak{B}$ such that $\mathfrak{A} \equiv_{\Sigma} \mathfrak{B}'$.*

We can prove a result similar to Stukachev's.

Proposition 9 *Let $\mathfrak{A}$ be a countable structure and $\mathcal{O}_A \Rightarrow_0^k \mathfrak{A}$ for some $k \in \omega$. There exists a countable structure $\mathfrak{B}$ such that $\mathfrak{A} \Leftrightarrow_0^0 \mathfrak{B}^{(k)}$.*

Proof. We give a brief sketch of the proof for the case $k = 1$. The proof is easily generalized for $k > 1$. Following [4], let $\mathfrak{A}^{\exists}$ and $\mathfrak{A}^{\forall}$ be the respective Marker's $\exists$ and $\forall$ extensions of the structure $\mathfrak{A}$ and define $\mathfrak{B} = (\mathfrak{A}^{\exists})^{\forall}$. With almost trivial modifications of the proof of Theorem 4 from above, we can show that $\mathfrak{A} \Leftrightarrow_1^0 \mathfrak{B}$. From Proposition 7 it follows that $\mathfrak{A} \Leftrightarrow_0^0 \mathfrak{B}'$. □

Proposition 10 *Let $\mathfrak{A}$ be a countable structure and $\mathcal{O}_A \Rightarrow_0^k \mathfrak{A}$ for some $k \in \omega$. There exists a countable structure $\mathfrak{B}$ such that for every $n \in \omega$, $\mathfrak{A} \Leftrightarrow_k^n \mathfrak{B}^{(n)}$.*

Combining Proposition 10 with Theorem 2, we get the following corollary.

Corollary 3. *Let $\mathfrak{A}$ be a countable structure and $\mathcal{O}_A \Rightarrow_0^k \mathfrak{A}$ for some $k \in \omega$. There exists a countable structure $\mathfrak{B}$ such that*

$$(\forall n \in \omega)(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathfrak{A}) \leftrightarrow X \in \Sigma^c_{k+1}(\mathfrak{B}^{(n)})].$$

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