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Periods Are $\mathcal{M}^2$-Computable Real Numbers

Master’s Thesis

by

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Contents

1 Introduction

In [KZ01], Maxim Kontsevich and Don Zagier introduced the notion of periods:

Definition 1. A period is a complex number whose real and imaginary parts are values of absolutely convergent integral of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

They denote the set of all periods by $\mathcal{P}$ and pose the following "**Problem 3.** Exhibit at least one number which does not belong to $\mathcal{P}$". In [Yo08], Masahiko Yoshinaga gives an answer to this problem by constructing a computable real number which can not be a period. Along the way he proves that every period is an elementary real number (i.e. $\mathcal{E}^3$-computable). A year later Katrin Tent and Martin Ziegler proved in [TZ09] that periods are lower elementary real numbers (i.e. $\mathcal{L}^2$-computable). The purpose of this thesis is to show that periods are $\mathcal{M}^2$-computable real numbers.

I would like to express my sincere gratitude to my supervisor, Assoc. Prof. Ivan Georgiev, for steering me toward such an interesting research problem. I am glad I had the opportunity to work with him and solve the problem. I would also like to express my sincere thanks to the members of the Department of Mathematical Logic and Its Applications for their support and trust. Without them, I wouldn't have been able to succeed.

2 The classes $\mathcal{M}^2$, $\mathcal{L}^2$, $\mathcal{E}^2$, $\mathcal{E}^3$

Definition 2. 2.1. We denote $\mathcal{T}_m = \{f \mid f : \mathbb{N}^m \to \mathbb{N}\}$, $m \in \mathbb{N}$, and $\mathcal{T} = \bigcup_{m\in\mathbb{N}} \mathcal{T}_m$.

2.2. The following functions in $\mathcal{T}$ are called the initial functions:

- The projection functions, $(x_1, \ldots, x_n) \mapsto x_k$, $(n, k \in \mathbb{N}\ \&\ 1 \leq k \leq n)$.
- The successor function, $x \mapsto x + 1$.
- The product function, $(x, y) \mapsto xy$.
- The modified subtraction function, $(x, y) \mapsto \max(x - y, 0)$.
- The quotient function, $(x, y) \mapsto \left\lfloor \dfrac{x}{y+1} \right\rfloor$.

2.3. The smallest subclass of $\mathcal{T}$, which contains the initial functions and is closed under composition and:

- bounded minimisation $\big(f \mapsto \lambda\overline{x}, y.\mu_{z\leq y}[f(\overline{x}, z) = 0]\big)$, is denoted by $\mathcal{M}^2$.
- bounded summation $\big(f \mapsto \lambda\overline{x}, y.\sum_{z\leq y} f(\overline{x}, z)\big)$, is denoted by $\mathcal{L}^2$ (the class of lower elementary functions).
- limited primitive recursion, is denoted by $\mathcal{E}^2$ (the second Grzegorczyk class).

- bounded summation $\left(f \mapsto \lambda\overline{x}, y. \sum_{z \leq y} f(\overline{x}, z)\right)$ and bounded product $\left(f \mapsto \lambda\overline{x}, y. \prod_{z \leq y} f(\overline{x}, z)\right)$, is denoted by $\mathcal{E}^3$ (the class of elementary functions, i.e. the third Grzegorczyk class).

It is known that

$$\mathcal{M}^2 \subseteq \mathcal{L}^2 \subseteq \mathcal{E}^2 \subsetneq \mathcal{E}^3,$$

but whether the first and the second of these inclusions is proper is an open question.

Definition 3. 3.1. A name of a real number ξ is any triple $(f, g, h) \in \mathcal{T}_1^3$ such that for all $t \in \mathbb{N}$,

$$\left|\xi - \frac{f(t) - g(t)}{h(t) + 1}\right| < \frac{1}{t + 1}.$$

3.2. For a class $\mathcal{F} \subseteq \mathcal{T}$ of functions, a real number ξ is called $\mathcal{F}$-computable iff there exists a triple $(f, g, h) \in \mathcal{F}^3$ which is a name of ξ.

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3 An increasing ω-sequence of compacts covering an open set

Definition 4. Let O be an open subset of $\mathbb{R}^n$ $(n \geq 1)$. For every $e \in \mathbb{R}_{\geq 1}$ we define the set

$$O_e = \left\{\overline{x} \in O \;\middle|\; \|\overline{x}\|_\infty \leq e \;\&\; \mathsf{dist}(\overline{x}, \mathbb{R}^n \setminus O) \geq \frac{1}{e}\right\},$$

where $\|\bullet\|_\infty$ is the maximum norm

$$\|(x_1, \ldots, x_n)\|_\infty = \max\left\{|x_1|, \ldots, |x_n|\right\}$$

and

$$\mathsf{dist}(\overline{x}, \mathbb{R}^n \setminus O) = \inf\left\{\|\overline{x} - \overline{y}\|_\infty \mid \overline{y} \in \mathbb{R}^n \setminus O\right\}.$$

Remark 1. Note that if we denote by

$$B(\overline{x}, r) = \left\{\overline{y} \in \mathbb{R}^n \mid \|\overline{x} - \overline{y}\|_\infty < r\right\}$$

the open ball of radius $r \in \mathbb{R}_{>0}$ centred on $\overline{x} \in \mathbb{R}^n$, then

$$B\left(\overline{x}, \frac{1}{e}\right) \subseteq O \;\leftrightarrow\; \mathsf{dist}(\overline{x}, \mathbb{R}^n \setminus O) \geq \frac{1}{e}.$$

Lemma 1. Let O be an open subset of $\mathbb{R}^n$ $(n \geq 1)$.

1.1. O_e is a compact for all $e \in \mathbb{R}_{\geq 1}$.

1.2. $e \leq e' \;\rightarrow\; O_e \subseteq O_{e'}$ for all $e, e' \in \mathbb{R}_{\geq 1}$.

1.3. $\bigcup_{e \in \mathbb{N}^+} O_e = O$.

Lemma 2. Let O be an open subset of $\mathbb{R}^n$ $(n \geq 1)$ and $\alpha, \beta : O \to \mathbb{R}$ be continuous functions with $\alpha < \beta$ on O. Then the following set is open

$$U = \big\{(\overline{x}, y) \in \mathbb{R}^{n+1} \mid \overline{x} \in O \;\&\; \alpha(\overline{x}) < y < \beta(\overline{x})\big\}.$$

Further, suppose there exist strictly increasing functions $d_\alpha, d_\beta : \mathbb{N} \to \mathbb{N}$ and functions $f_\alpha, f_\beta : \mathbb{Q}^n \times \mathbb{N} \to \mathbb{Q}$ such that the following conditions hold

$$(\forall e \in \mathbb{N}^+)(\forall \overline{x} \in O_e)(\forall \overline{a} \in \mathbb{Q}^N)\Big[\|\overline{x} - \overline{a}\|_\infty < \frac{1}{d_\alpha(e)} \;\rightarrow\; |\alpha(\overline{x}) - f_\alpha(\overline{a}, e)| < \frac{1}{e}\Big]$$

and

$$(\forall e \in \mathbb{N}^+)(\forall \overline{x} \in O_e)(\forall \overline{a} \in \mathbb{Q}^N)\Big[\|\overline{x} - \overline{a}\|_\infty < \frac{1}{d_\beta(e)} \;\rightarrow\; |\beta(\overline{x}) - f_\beta(\overline{a}, e)| < \frac{1}{e}\Big].$$

Then

$$(\forall e \in \mathbb{R}_{\geq 1})(\exists e'' \in \mathbb{R}_{\geq 1}, e < e'')\big(\forall (\overline{x}, y) \in \mathbb{R}^{n+1}\big)\big[\overline{x} \in O_e \;\&\; y \in \big(\alpha(\overline{x}), \beta(\overline{x})\big)_e \;\rightarrow\; \big(\overline{x}, y\big) \in U_{e''}\big].$$

Proof. *Let's define the function $d : \mathbb{N} \to \mathbb{N}$ by*

$$d(e) = \max\big(d_\alpha(e), d_\beta(e)\big).$$

More precisely, we will show that

$$(\forall e, e', e'' \in \mathbb{R}_{\geq 1})\Big[2e \leq e' \;\&\; \frac{d(2\lceil e' \rceil)}{2} \leq e'' \;\&\; \overline{x} \in O_e \;\&\; y \in \big(\alpha(\overline{x}), \beta(\overline{x})\big)_e \;\rightarrow\; (\overline{x}, y) \in U_{e''}\Big].$$

Let $e, e', e'' \in \mathbb{R}_{\geq 1}$, $2e \leq e'$, $\frac{d(2\lceil e' \rceil)}{2} \leq e''$, $\overline{x} \in O_e$ and $y \in \big(\alpha(\overline{x}), \beta(\overline{x})\big)_e$. Thus $\frac{1}{e'} + \frac{1}{e''} \leq \frac{1}{e}$ and

$$y \in \Big[\alpha(\overline{x}) + \frac{1}{e'} + \frac{1}{e''}, \beta(\overline{x}) - \frac{1}{e'} - \frac{1}{e''}\Big].$$

We wish to show that $(\overline{x}, y) \in U_{e''}$, i.e.

$$(\overline{x}, y) \in U \;\&\; \|(\overline{x}, y)\|_\infty \leq e'' \;\&\; \mathit{dist}\big((\overline{x}, y), \mathbb{R}^{n+1} \setminus U\big) \geq \frac{1}{e''},$$

that is,

$$\overline{x} \in O \;\&\; y \in \big(\alpha(\overline{x}), \beta(\overline{x})\big) \;\&\; \|\overline{x}\|_\infty \leq e'' \;\&\; |y| \leq e'' \;\&\; B\Big((\overline{x}, y), \frac{1}{e''}\Big) \subseteq U.$$

The effort is on proving that $B\Big((\overline{x}, y), \frac{1}{e''}\Big) \subseteq U$. Let $(\overline{x}', y') \in B\Big((\overline{x}, y), \frac{1}{e''}\Big)$ (i.e. $\|\overline{x} - \overline{x}'\|_\infty < \frac{1}{e''}$ and $|y - y'| < \frac{1}{e''}$). Thus we wish to see that $(\overline{x}', y') \in U$, i.e. $\overline{x}' \in O$ and $\alpha(\overline{x}') < y' < \beta(\overline{x}')$.

Next, let's see that $\overline{x}' \in O_{e'}$. *From* $\|\overline{x} - \overline{x}'\|_\infty < \frac{1}{e''} \leq \frac{1}{e}$ *and* $B\Big(\overline{x}, \frac{1}{e}\Big) \subseteq O$ *(as dist*$(\overline{x}, \mathbb{R}^n \setminus O) \geq \frac{1}{e}$*) we conclude that* $\overline{x}' \in O$. *We have*

$$\|\overline{x}'\|_\infty = \|\overline{x} + (\overline{x}' - \overline{x})\|_\infty \leq \|\overline{x}\|_\infty + \|\overline{x}' - \overline{x}\|_\infty \leq e + \frac{1}{e''} < e + 1 \leq 2e \leq e'.$$

We will show that

$$(\forall \overline{z} \in \mathbb{R}^n \setminus O)\Big[\|\overline{x}' - \overline{z}\|_\infty \geq \frac{1}{e'}\Big].$$

From here we can conclude that dist$(\overline{x}', \mathbb{R}^n \setminus O) \geq \frac{1}{e'}$. *Therefore* $\overline{x}' \in O_{e'}$. *Let* $\overline{z} \in \mathbb{R}^n \setminus O$. *Hence* $\|\overline{x} - \overline{z}\|_\infty \geq \frac{1}{e}$ *(as dist*$(\overline{x}, \mathbb{R}^n \setminus O) \geq \frac{1}{e}$*). By the triangle inequality we have*

$$\|\overline{x} - \overline{z}\|_\infty \leq \|\overline{x} - \overline{x}'\|_\infty + \|\overline{x}' - \overline{z}\|_\infty.$$

Thus

$$\|\overline{x}' - \overline{z}\|_\infty \geq \|\overline{x} - \overline{z}\|_\infty - \|\overline{x} - \overline{x}'\|_\infty > \frac{1}{e} - \frac{1}{e''} \geq \frac{1}{e'}.$$

Further, since

$$\|\overline{x} - \overline{x}'\|_\infty < \frac{1}{e''} \leq \frac{2}{d(2\lceil e' \rceil)}$$

let $\overline{a} \in \mathbb{Q}^n$ *be such that*

$$\|\overline{x} - \overline{a}\|_\infty < \frac{1}{d(2\lceil e' \rceil)} \leq \frac{1}{d_\alpha(2\lceil e' \rceil)}$$

and

$$\|\overline{x}' - \overline{a}\|_\infty < \frac{1}{d(2\lceil e' \rceil)} \leq \frac{1}{d_\alpha(2\lceil e' \rceil)}.$$

From here, $\overline{x} \in O_e \subseteq O_{2\lceil e' \rceil}$ *and* $\overline{x}' \in O_{e'} \subseteq O_{2\lceil e' \rceil}$ *we obtain*

$$\big|\alpha(\overline{x}) - f_\alpha(\overline{a}, 2\lceil e' \rceil)\big| < \frac{1}{2\lceil e' \rceil}$$

and

$$\big|\alpha(\overline{x}') - f_\alpha(\overline{a}, 2\lceil e' \rceil)\big| < \frac{1}{2\lceil e' \rceil}$$

respectively. Therefore

$$\begin{aligned}\big|\alpha(\overline{x}) - \alpha(\overline{x}')\big| &= \big|\alpha(\overline{x}) - f_\alpha(\overline{a}, 2\lceil e' \rceil) + f_\alpha(\overline{a}, 2\lceil e' \rceil) - \alpha(\overline{x}')\big| \leq \\ &\leq \big|\alpha(\overline{x}) - f_\alpha(\overline{a}, 2\lceil e' \rceil)\big| + \big|f_\alpha(\overline{a}, 2\lceil e' \rceil) - \alpha(\overline{x}')\big| < \\ &< \frac{1}{2\lceil e' \rceil} + \frac{1}{2\lceil e' \rceil} = \frac{1}{\lceil e' \rceil}.\end{aligned}$$

In the same manner we can see that

$$\big|\beta(\overline{x}) - \beta(\overline{x}')\big| < \frac{1}{\lceil e' \rceil}.$$

Finally, we consequently obtain

$$\alpha(\overline{x}') < \alpha(\overline{x}) + \frac{1}{\lceil e' \rceil} \leq \alpha(\overline{x}) + \frac{1}{e'} \leq y - \frac{1}{e''} < y' < y + \frac{1}{e''} \leq \beta(\overline{x}) - \frac{1}{e'} \leq \beta(\overline{x}) - \frac{1}{\lceil e' \rceil} < \beta(\overline{x}').$$

Hence $\alpha(\overline{x}') < y' < \beta(\overline{x}')$ *and* $(\overline{x}', y') \in U$.

4 Parametrically MSO-computable functions

Definition 5. For $k, m \in \mathbb{N}$, a (k,m)-operator F is a total mapping $F : \mathcal{T}_1^k \to \mathcal{T}_m$. An operator is (k,m)-operator for some $k, m \in \mathbb{N}$.

Remark 2. Next, we recall a higher-order counterpart for the class $\mathcal{M}^2$.

Definition 6. The class **MSO** (of $\mathcal{M}^2$-substitutional operators) is the smallest class of operators such that:

(i) For all m, n, i with $1 \leq i \leq m$, the (n,m)-operator F defined by $F\big(\overline{f}^n\big)(\overline{x}^m) = x_i$ belongs to **MSO**.

(ii) For any n, m and $k \in \{1, \ldots, n\}$, if F_0 is an (n,m)-operator which belongs to **MSO**, then the (n,m)-operator F defined by

$$F\big(\overline{f}^n\big)(\overline{x}^m) = f_k\big(F_0\big(\overline{f}^n\big)(\overline{x}^m)\big)$$

also belongs to **MSO**.

(iii) For any n, m, k and $a \in \mathcal{T}_k \cap \mathcal{M}^2$, if $F_1, \ldots, F_k$ are (n,m)-operators which belong to **MSO**, then so does the (n,m)-operator F defined by

$$F\big(\overline{f}^n\big)(\overline{x}^m) = a(F_1\big(\overline{f}^n\big)(\overline{x}^m), \ldots, F_k\big(\overline{f}^n\big)(\overline{x}^m)\big).$$

Remark 3. Our main reference for the properties of the class **MSO** is [G20].

Let O be an open subset of $\mathbb{R}^n$ $(n \geq 1)$ and $\theta : O \to \mathbb{R}$ be a function. Intuitively, if θ is uniformly **MSO**-computable on O, then there exist operators in **MSO** which approximate the value $\theta(\overline{x})$ for any $\overline{x} \in O$. On the other hand, if θ is parametrically **MSO**-computable on O, then there exist operators in **MSO** which for any fixed $e \in \mathbb{N}^+$ approximate the value $\theta(\overline{x})$ for any $\overline{x}$ in the compact O_e.

Definition 7. Let D be a subset of $\mathbb{R}^n$ $(n \geq 1)$. We call a function $\theta : D \to \mathbb{R}$ uniformly **MSO**-computable iff there exist $(3n,1)$-operators $F, G, H \in$ **MSO** such that for all $(x_1, \ldots, x_n) \in D$ and any names $(p_1, q_1, r_1), \ldots, (p_n, q_n, r_n) \in \mathcal{T}_1^3$ of $x_1, \ldots, x_n$ respectively, the triple

$$\begin{aligned}\big(&F(p_1, q_1, r_1, \ldots, p_n, q_n, r_n),\\ &G(p_1, q_1, r_1, \ldots, p_n, q_n, r_n),\\ &H(p_1, q_1, r_1, \ldots, p_n, q_n, r_n)\big)\end{aligned}$$

is a name of $\theta(x_1, \ldots, x_n)$.

Definition 8. Let O be an open subset of $\mathbb{R}^n$ $(n \geq 1)$. We call a function $\theta : O \to \mathbb{R}$ parametrically MSO-computable iff there exist $(3n+3,1)$-operators $F, G, H \in$ **MSO** such that for all $e \in \mathbb{N}^+$, $(x_1, \dots, x_n) \in O_e$ and any names $(p_1, q_1, r_1), \dots, (p_n, q_n, r_n), (p_{n+1}, q_{n+1}, r_{n+1}) \in \mathcal{T}_1^3$ of $x_1, \dots, x_n, e$ respectively, the triple

$$\begin{aligned}\big(&F(p_1, q_1, r_1, \dots, p_n, q_n, r_n, p_{n+1}, q_{n+1}, r_{n+1}),\\ &G(p_1, q_1, r_1, \dots, p_n, q_n, r_n, p_{n+1}, q_{n+1}, r_{n+1}),\\ &H(p_1, q_1, r_1, \dots, p_n, q_n, r_n, p_{n+1}, q_{n+1}, r_{n+1})\big)\end{aligned}$$

is a name of $\theta(x_1, \dots, x_n)$.

Remark 4. If θ is uniformly **MSO**-computable on O, then it is also parametrically **MSO**-computable on O. On the contrary, if θ is parametrically **MSO**-computable on O, then for any $e \in \mathbb{N}^+$ the restriction $(\theta \upharpoonright O_e) : O_e \to \mathbb{R}$ is uniformly **MSO**-computable on O_e.

Remark 5. Recall that if (p, q, r) is a name of $e \in \mathbb{N}$, then $e = \Big\lfloor \dfrac{|p(1) - q(1)|}{r(1)+1} + \dfrac{1}{2} \Big\rfloor$.

Remark 6. The function $x \mapsto \dfrac{1}{x}$ is not uniformly **MSO**-computable on the interval $(0,1)$, because it is not uniformly continuous on that interval. On the other hand, the function $x \mapsto \dfrac{1}{x}$ is parametrically **MSO**-computable on the interval $(0,1)$ via the $(6,1)$-operators

$$\begin{aligned}
F(p_1, q_1, r_1, p_2, q_2, r_2)(t) &= r_1\Big(\Big\lfloor \frac{|p_2(1) - q_2(1)|}{r_2(1)+1} + \frac{1}{2}\Big\rfloor^2 (t+1) + \Big\lfloor \frac{|p_2(1) - q_2(1)|}{r_2(1)+1} + \frac{1}{2}\Big\rfloor - 1\Big) + 1,\\
G(p_1, q_1, r_1, p_2, q_2, r_2)(t) &= 0,\\
H(p_1, q_1, r_1, p_2, q_2, r_2)(t) &= p_1\Big(\Big\lfloor \frac{|p_2(1) - q_2(1)|}{r_2(1)+1} + \frac{1}{2}\Big\rfloor^2 (t+1) + \Big\lfloor \frac{|p_2(1) - q_2(1)|}{r_2(1)+1} + \frac{1}{2}\Big\rfloor - 1\Big)\\
&\quad - q_1\Big(\Big\lfloor \frac{|p_2(1) - q_2(1)|}{r_2(1)+1} + \frac{1}{2}\Big\rfloor^2 (t+1) + \Big\lfloor \frac{|p_2(1) - q_2(1)|}{r_2(1)+1} + \frac{1}{2}\Big\rfloor - 1\Big) - 1.
\end{aligned}$$

In particular, if $e \in \mathbb{N}^+$, $x \in (0,1)_e$ and $(p_1, q_1, r_1), (p_2, q_2, r_2)$ are names of x and e respectively, then

$$\begin{aligned}
&F(p_1, q_1, r_1, p_2, q_2, r_2)(t) = r_1\big(e^2(t+1) + e - 1\big) + 1,\\
&G(p_1, q_1, r_1, p_2, q_2, r_2)(t) = 0,\\
&H(p_1, q_1, r_1, p_2, q_2, r_2)(t) = p_1\big(e^2(t+1) + e - 1\big) - q_1\big(e^2(t+1) + e - 1\big) - 1.
\end{aligned}$$

Note that $(0,1)_1 = \emptyset$ and for all $e \in \mathbb{N}$ with $e \geq 2$ we have $(0,1)_e = \Big[\dfrac{1}{e}, 1 - \dfrac{1}{e}\Big]$.

Proposition 1. Let O be an open subset of $\mathbb{R}^n$ $(n \geq 1)$ and the functions $\alpha, \beta : O \to \mathbb{R}$ be parametrically **MSO**-computable. Then the following functions are also parametrically **MSO**-computable:

1.1. $\dfrac{p}{q} \cdot \alpha : O \to \mathbb{R}$, $\quad \Big(\dfrac{p}{q} \cdot \alpha\Big)(\overline{x}) = \dfrac{p}{q} \cdot \alpha(\overline{x})$, for all $p, q \in \mathbb{N}^+$,

1.2. $\alpha + \beta : O \to \mathbb{R}$, $\quad (\alpha + \beta)(\overline{x}) = \alpha(\overline{x}) + \beta(\overline{x})$, 1.3. $\alpha-\beta:O\to\mathbb{R},\quad (\alpha-\beta)(\overline{x})=\alpha(\overline{x})-\beta(\overline{x}),$

1.4. $\alpha\cdot\beta:O\to\mathbb{R},\quad (\alpha\cdot\beta)(\overline{x})=\alpha(\overline{x}).\beta(\overline{x})$, here we additionally assume that α and β are bounded.

Proof. *Let α and β be parametrically **MSO**-computable via the triples (F_1,G_1,H_1) and (F_2,G_2,H_2) respectively.*

Since

$$\frac{p}{q}\cdot\frac{F_1-G_1}{H_1+1}=\frac{p\cdot F_1-p\cdot G_1}{\big(q\cdot H_1+(q-1)\big)+1}$$

we define

$$F_{\frac{p}{q}\cdot\alpha}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=(p\cdot F_1)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})\Big(\Big\lceil\frac{p}{q}\Big\rceil.(t+1)-1\Big),$$
$$G_{\frac{p}{q}\cdot\alpha}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=(p\cdot G_1)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})\Big(\Big\lceil\frac{p}{q}\Big\rceil.(t+1)-1\Big),$$
$$H_{\frac{p}{q}\cdot\alpha}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=\big(q\cdot H_1+(q-1)\big)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})\Big(\Big\lceil\frac{p}{q}\Big\rceil.(t+1)-1\Big).$$

As

$$\frac{F_1-G_1}{H_1+1}+\frac{F_2-G_2}{H_2+1}=\frac{\big(F_1(H_2+1)+F_2(H_1+1)\big)-\big(G_1(H_2+1)+G_2(H_1+1)\big)}{(H_1H_2+H_1+H_2)+1}$$

it is appropriate to define

$$F_{\alpha+\beta}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=\big(F_1(H_2+1)+F_2(H_1+1)\big)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(2t+1),$$
$$G_{\alpha+\beta}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=\big(G_1(H_2+1)+G_2(H_1+1)\big)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(2t+1),$$
$$H_{\alpha+\beta}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=\big(H_1H_2+H_1+H_2\big)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(2t+1).$$

Further, since

$$\frac{F_1-G_1}{H_1+1}-\frac{F_2-G_2}{H_2+1}=\frac{\big(F_1(H_2+1)+G_2(H_1+1)\big)-\big(F_2(H_1+1)+G_1(H_2+1)\big)}{(H_1H_2+H_1+H_2)+1}$$

we put

$$F_{\alpha-\beta}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=\big(F_1(H_2+1)+G_2(H_1+1)\big)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(2t+1),$$
$$G_{\alpha-\beta}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=\big(F_2(H_1+1)+G_1(H_2+1)\big)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(2t+1),$$
$$H_{\alpha-\beta}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=\big(H_1H_2+H_1+H_2\big)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(2t+1).$$

Suppose that $M_\alpha,M_\beta>0$ bound α and β respectively. Since

$$\frac{F_1-G_1}{H_1+1}\cdot\frac{F_2-G_2}{H_2+1}=\frac{(F_1F_2+G_1G_2)-(F_1G_2+F_2G_1)}{(H_1H_2+H_1+H_2)+1}$$

we define

$$F_{\alpha\cdot\beta}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=\big(F_1F_2+G_1G_2\big)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})\big(k(t)\big),$$
$$G_{\alpha\cdot\beta}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=\big(F_1G_2+F_2G_1\big)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})\big(k(t)\big),$$
$$H_{\alpha\cdot\beta}(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})(t)=\big(H_1H_2+H_1+H_2\big)(p_1,q_1,r_1,\ldots,p_{n+1},q_{n+1},r_{n+1})\big(k(t)\big)$$

where

$$k(t) = (t+1)\lceil M_\alpha + M_\beta + 1\rceil - 1.$$

Indeed, let $e \in \mathbb{N}^+$, $\overline{x} = (x_1, \ldots, x_n) \in O_e$ *and* $(p_1, q_1, r_1), \ldots, (p_n, q_n, r_n), (p_{n+1}, q_{n+1}, r_{n+1}) \in \mathcal{T}_1^3$ *be names of* $x_1, \ldots, x_n, e$ *respectively. Let's denote*

$$\overline{(p_i, q_i, r_i)} = (p_1, q_1, r_1, \ldots, p_n, q_n, r_n, p_{n+1}, q_{n+1}, r_{n+1}).$$

Suppose $t \in \mathbb{N}$. *We consequently obtain*

$$\begin{aligned}
&\left|(\alpha\cdot\beta)(\overline{x}) - \frac{F_{\alpha\cdot\beta}(\overline{p_i, q_i, r_i})(t) - G_{\alpha\cdot\beta}(\overline{p_i, q_i, r_i})(t)}{H_{\alpha\cdot\beta}(\overline{p_i, q_i, r_i})(t) + 1}\right| = \\
=&\Bigg|\alpha(\overline{x})\cdot\beta(\overline{x}) - \underbrace{\frac{F_1(\overline{p_i, q_i, r_i})\big(k(t)\big) - G_1(\overline{p_i, q_i, r_i})\big(k(t)\big)}{H_1(\overline{p_i, q_i, r_i})\big(k(t)\big) + 1}}_{=A}\cdot\underbrace{\frac{F_2(\overline{p_i, q_i, r_i})\big(k(t)\big) - G_2(\overline{p_i, q_i, r_i})\big(k(t)\big)}{H_2(\overline{p_i, q_i, r_i})\big(k(t)\big) + 1}}_{=B}\Bigg| \\
=&\big|A\cdot B - \alpha(\overline{x})\cdot\beta(\overline{x})\big| \\
=&\big|A\cdot B - \alpha(\overline{x})\cdot\beta(\overline{x}) - (\alpha(\overline{x}) - A)(\beta(\overline{x}) - B) + (\alpha(\overline{x}) - A)(\beta(\overline{x}) - B)\big| \\
=&\big|A\cdot B - \underline{\alpha(\overline{x})\cdot\beta(\overline{x})} - \underline{\underline{\alpha(\overline{x})\cdot\beta(\overline{x})}} + \underline{\alpha(\overline{x})\cdot B} + \underline{\underline{\beta(\overline{x})\cdot A}} - A\cdot B + (\alpha(\overline{x}) - A)(\beta(\overline{x}) - B)\big| \\
=&\big|\alpha(\overline{x})(B - \beta(\overline{x})) + \beta(\overline{x})(A - \alpha(\overline{x})) + (\alpha(\overline{x}) - A)(\beta(\overline{x}) - B)\big| \leq \\
\leq&|\alpha(\overline{x})|\cdot|B - \beta(\overline{x})| + |\beta(\overline{x})|\cdot|A - \alpha(\overline{x})| + |\alpha(\overline{x}) - A|\cdot|\beta(\overline{x}) - B)| < \\
<&M_\alpha\cdot\frac{1}{k(t)+1} + M_\beta\cdot\frac{1}{k(t)+1} + \frac{1}{k(t)+1}\cdot\frac{1}{k(t)+1} < \\
<&M_\alpha\cdot\frac{1}{k(t)+1} + M_\beta\cdot\frac{1}{k(t)+1} + \frac{1}{k(t)+1} \\
=&\frac{1}{k(t)+1}\cdot(M_\alpha + M_\beta + 1) \\
=&\frac{1}{(t+1)\lceil M_\alpha + M_\beta + 1\rceil}\cdot(M_\alpha + M_\beta + 1) \leq \\
\leq&\frac{1}{t+1}.
\end{aligned}$$

5 Semialgebraic sets

Definition 9. We say that a subset of $\mathbb{R}^n$ $(n \geq 1)$ is semialgebraic if it is a Boolean combination of sets of the form

$$\big\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid p(x_1, \ldots, x_n) > 0\big\},$$

where $p \in \mathbb{Z}[X_1, \ldots, X_n]$. A function is called semialgebraic if its graph is semialgebraic.

Remark 7. By definition, semialgebraic sets are closed under finite union, finite intersection and taking complements. They are also closed under projection. Furthermore, by quantifier elimination, the semialgebraic sets are exactly the definable sets in the ordered field $\mathbb{R}$.

Proposition 2. ([BCR98], Proposition 2.2.4.) Let $\varphi(\mathsf{x}_1,\ldots,\mathsf{x}_n)$ $(n \geq 1)$ be a first-order formula of the language $\{0,1,+,-,\cdot,<\}$ of ordered fields, without parameters, with free variables $\mathsf{x}_1,\ldots,\mathsf{x}_n$. Then the set

$$\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid \varphi[\![x_1,\ldots,x_n]\!]\}$$

is semialgebraic.

Remark 8. All the semialgebraic sets we are considering are definable without parameters.

Proposition 3. ([BCR98], Proposition 2.6.1.) Let $a \in \mathbb{R}$ and f be a semialgebraic function from $(a,+\infty) \subseteq \mathbb{R}$ to $\mathbb{R}$. There exist $r \in (a,+\infty)$, $m \in \mathbb{N}^+$, such that, for every $x \geq r$, we have $|f(x)| < x^m$, i.e.

$$\big(\exists r \in (a,+\infty)\big)(\exists m \in \mathbb{N}^+)(\forall x)\big[r \leq x \;\rightarrow\; |f(x)| < x^m\big].$$

---

6 Continuous semialgebraic functions defined on open semialgebraic sets are parametrically MSO-computable

Definition 10. A name of a rational number a is any triple $(p,q,r) \in \mathbb{N}^3$ such that $a = \dfrac{p-q}{r+1}$.

Definition 11. A partial function $f : \mathbb{Q}^n \longrightarrow\!\!\!\!\circ\;\, \mathbb{Q}$ $(n \geq 1)$ is called $\mathcal{M}^2$-computable iff there are functions $f_1, f_2, f_3 : \mathbb{N}^{3n} \to \mathbb{N}$ in $\mathcal{M}^2$ such that for all $(a_1,\ldots,a_n) \in \mathrm{dom}(f)$ and for any names $(p_1,q_1,r_1),\ldots,(p_n,q_n,r_n) \in \mathbb{N}^3$ of $a_1,\ldots,a_n$ respectively, it holds that

$$f\Big(\underbrace{\frac{p_1-q_1}{r_1+1}}_{=a_1},\ldots,\underbrace{\frac{p_n-q_n}{r_n+1}}_{=a_n}\Big) = \frac{f_1(p_1,q_1,r_1,\ldots,p_n,q_n,r_n) - f_2(p_1,q_1,r_1,\ldots,p_n,q_n,r_n)}{f_3(p_1,q_1,r_1,\ldots,p_n,q_n,r_n)+1}.$$

Definition 12. A relation $R \subseteq \mathbb{Q}^n$ $(n \geq 1)$ is called $\mathcal{M}^2$-computable iff its characteristic function is $\mathcal{M}^2$-computable.

Lemma 3. Let $R \subseteq \mathbb{R}^n$ $(n \geq 1)$ be a semialgebraic relation. Then the restriction $R \restriction \mathbb{Q}^n$ is $\mathcal{M}^2$-computable.

Proof. *Let R be definable by the formula $\varphi(x_1, \ldots, x_n)$. By quantifier elimination φ is equivalent to a quantifier-free formula. Further, φ is equivalent to a Boolean combination of formulas of the form $p_1 < p_2$ or $p_1 = p_2$, where both p_1 and p_2 are polynomials with natural coefficients in variables $x_1, \ldots, x_n$. Hence the restriction $R \upharpoonright \mathbb{Q}^n$ is definable by a Boolean combination of formulas of the form $q_1 < q_2$ or $q_1 = q_2$, where both q_1 and q_2 are polynomials with natural coefficients in variables*

$$x_{1,1}, x_{1,2}, x_{1,3}, \ldots, x_{n,1}, x_{n,2}, x_{n,3}.$$

Indeed, we obtain $q_1 < q_2$ from $p_1 < p_2$ by the following steps:

3.1. *in the inequality $p_1 < p_2$ we replace any of the variables $x_1, \ldots, x_n$ by $\dfrac{x_{1,1} - x_{1,2}}{x_{1,3} + 1}, \ldots, \dfrac{x_{n,1} - x_{n,2}}{x_{n,3} + 1}$ respectively, obtaining $p_1' < p_2'$;*

3.2. *we rewrite the inequality $p_1' < p_2'$ so that each side becomes a polynomial with non-negative coefficients, preserving the original relation, obtaining $q_1 < q_2$.*

We apply the same steps to obtain the equality $q_1 = q_2$ from $p_1 = p_2$. Consequently, $R \upharpoonright \mathbb{Q}^n$ is an $\mathcal{M}^2$-computable relation.

Lemma 4. Let O be an open semialgebraic subset of $\mathbb{R}^n$ $(n \geq 1)$ and $\theta : O \to \mathbb{R}$ be a continuous semialgebraic function. Then there exist a strictly increasing function $d : \mathbb{N} \to \mathbb{N}$ in $\mathcal{M}^2$ and an $\mathcal{M}^2$-computable function $f : \mathbb{Q}^n \times \mathbb{N} \to \mathbb{Q}$ such that

$$(\forall e \in \mathbb{N}^+)(\forall \overline{x} \in O_e)(\forall \overline{a} \in \mathbb{Q}^n)\Big[\|\overline{x} - \overline{a}\|_\infty < \frac{1}{d(e)} \ \to\ |\theta(\overline{x}) - f(\overline{a}, e)| < \frac{1}{e}\Big].$$

- $(\exists m_1 \in \mathbb{N}^+)(\exists r_1 \in \mathbb{R}_{\geq 1})(\forall e \in \mathbb{R}_{\geq 1}, e \geq r_1)(\forall \overline{x}, \overline{y} \in O_{2e})\Big[\|\overline{x} - \overline{y}\| < \dfrac{1}{e^{m_1}} \ \to\ |\theta(\overline{x}) - \theta(\overline{y})| \leq \dfrac{1}{2e}\Big].$

Proof. *Note that the family $\{O_e \mid e \in \mathbb{R}_{\geq 1}\}$ is uniformly definable, i.e. there is a formula $\varphi(e, x_1, \ldots, x_n)$ such that for all $e \in \mathbb{R}_{\geq 1}$ the set O_e is definable by $\varphi(e, x_1, \ldots, x_n)$. Since θ is continuous on O_{2e} (as it is continuous on O) and the set O_{2e} is compact it follows that θ is uniformly continuous on O_{2e}. Therefore the set*

$$A(e) = \Big\{d \in \mathbb{R}_{>0} \ \Big|\ (\forall \overline{x}, \overline{y} \in O_{2e})\Big[\|\overline{x} - \overline{y}\| < \frac{1}{d} \ \to\ |\theta(\overline{x}) - \theta(\overline{y})| \leq \frac{1}{2e}\Big]\Big\}$$

is non-empty and semialgebraic for every $e \in \mathbb{R}_{\geq 1}$ (as O, O_{2e} and θ are semialgebraic). We have that

$$(\forall e \in \mathbb{R}_{\geq 1})(\forall d, d' \in \mathbb{R}_{>0})\big[d \in A(e) \ \&\ d \leq d' \ \to\ d' \in A(e)\big]$$

and

$$(\forall e \in \mathbb{R}_{\geq 1})(\forall d \in \mathbb{R}_{>0})\big[\inf A(e) < d \ \to\ d \in A(e)\big].$$

The function $g : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$ defined by $g(e) = \inf A(e)$ is semialgebraic. Therefore by Proposition 3 let $r_1 \geq 1$ be a real number and m_1 be a positive natural number such that

$$(\forall e \in \mathbb{R}_{\geq 1}, e \geq r_1)[g(e) < e^{m_1}].$$

Hence

$$(\forall e \in \mathbb{R}_{\geq 1}, e \geq r_1)(\forall \overline{x}, \overline{y} \in O_{2e})\Big[\|\overline{x} - \overline{y}\| < \frac{1}{e^{m_1}} \ \to\ |\theta(\overline{x}) - \theta(\overline{y})| \leq \frac{1}{2e}\Big]. \qquad (r_1, m_1)$$

- $(\exists m_2 \in \mathbb{N}^+)(\exists r_2 \in \mathbb{R}_{\geq 1})(\forall e \in \mathbb{R}_{\geq 1}, e \geq r_2)(\forall \overline{x} \in O_{2e})\big[|\theta(\overline{x})| \leq e^{m_2}\big].$

Proof. *Since θ is continuous on the compact O_{2e}, the set*

$$B(e) = \big\{d \in \mathbb{R}_{\geq 0} \mid (\forall \overline{x} \in O_{2e})\big[|\theta(\overline{x})| \leq d\big]\big\}$$

is nonempty and semialgebraic for all $e \in \mathbb{R}_{\geq 1}$. Thus the function $h : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$ defined by $h(e) = \inf B(e)$ is semialgebraic. By virtue of Proposition 3 let $r_2 \geq 1$ be a real number and m_2 be a positive natural number such that

$$(\forall e \in \mathbb{R}_{\geq 1}, e \geq r_2)[h(e) < e^{m_2}].$$

Thus

$$(\forall e \in \mathbb{R}_{\geq 1}, e \geq r_2)(\forall \overline{x} \in O_{2e})\big[|\theta(\overline{x})| < e^{m_2}\big].$$

Further, let's define $e_0 = \lceil \max(r_1, r_2) \rceil$. The function

$$d : \mathbb{N} \to \mathbb{N}^+, \quad d(e) = (e + e_0)^{m_1+1} \quad (\text{thus } d(e) \geq (e+1)^2 > 2e)$$

is strictly increasing and it is in $\mathcal{M}^2$. Since O, O_{2e} and θ are each semialgebraic, the relations $R' \subseteq \mathbb{R}^{n+2}$ and $S' \subseteq \mathbb{R}^{n+1}$ defined by

$$R'(\overline{a}, e, b) \ \leftrightarrow \ \Big(b \leq \theta(\overline{a}) < b + \frac{1}{2e}\Big)$$

and

$$S'(\overline{a}, e) \ \leftrightarrow \ \big(\overline{a} \in O_{2e} \ \& \ e \geq r_2\big)$$

are also semialgebraic. Let's denote $R = R' \cap \mathbb{Q}^{n+2}$ and $S = S' \cap \mathbb{Q}^{n+1}$. By Lemma 3 we obtain that R and S are $\mathcal{M}^2$-computable relations. Hence the function $f_0 : \mathbb{Q}^n \times \mathbb{N} \to \mathbb{Q}$ given by

$$f_0(\overline{a}, e) = \begin{cases} 0 & \text{if } \overline{a} \notin O_{2e} \ \vee \ e < r_2 \\ -e^{m_2} + \dfrac{\mu_{i \leq 4e^{m_2+1}}\big[-e^{m_2} + \frac{i}{2e} \leq \theta(\overline{a}) < -e^{m_2} + \frac{i+1}{2e}\big]}{2e} & \text{if } \overline{a} \in O_{2e} \ \& \ e \geq r_2 \end{cases}$$

$$= \begin{cases} 0 & \text{if } \neg S(\overline{a}, e) \\ \dfrac{\mu_{i \leq 4e^{m_2+1}}\big[R(\overline{a}, e, -e^{m_2} + \frac{i}{2e})\big] - 2e^{m_2+1}}{2e} & \text{if } S(\overline{a}, e) \end{cases}$$

is $\mathcal{M}^2$-computable. Thus if $\overline{a} \in \mathbb{Q}^n \cap O_{2e}$, $e \in \mathbb{N}$ and $e \geq r_2$, then $f_0(\overline{a}, e)$ is the unique

$$b \in \Big\{-e^{m_2}, -e^{m_2} + \frac{1}{2e}, \ldots, e^{m_2} - \frac{1}{2e}, e^{m_2}\Big\}$$

such that $\theta(\overline{a}) \in \Big[b, b + \dfrac{1}{2e}\Big)$ and so

$$\big|\theta(\overline{a}) - f_0(\overline{a}, e)\big| < \frac{1}{2e}.$$

Consequently, the function $f : \mathbb{Q}^n \times \mathbb{N} \to \mathbb{Q}$ defined by

$$f(\overline{a}, e) = f_0(\overline{a}, e + e_0)$$

is also $\mathcal{M}^2$-computable. We will show that the functions d and f have the required properties.

- $(\forall e \in \mathbb{N}^+)(\forall \overline{x} \in O_e)(\forall \overline{a} \in \mathbb{Q}^n)\Big[\|\overline{x} - \overline{a}\|_\infty < \dfrac{1}{d(e)} \ \to \ |\theta(\overline{x}) - f(\overline{a}, e)| < \dfrac{1}{e}\Big].$

Proof. *Let $e \in \mathbb{N}^+$, $\overline{x} \in O_e$, $\overline{a} \in \mathbb{Q}^n$ and $\|\overline{x} - \overline{a}\|_\infty < \dfrac{1}{d(e)}$. We need several auxiliary statements.*

4.1. $\overline{x} \in O_e \ \& \ \|\overline{x} - \overline{a}\|_\infty < \dfrac{1}{d(e)} \ \to \ \overline{a} \in O_{2e}$.

Proof. *Since $\|\overline{x} - \overline{a}\|_\infty < \dfrac{1}{d(e)} \leq \dfrac{1}{e}$ and $B\Big(\overline{x}, \dfrac{1}{e}\Big) \subseteq O$ (as dist$(\overline{x}, \mathbb{R}^n \setminus O) \geq \dfrac{1}{e}$), we have $\overline{a} \in O$. Further,*

$$\|\overline{a}\|_\infty = \|(\overline{a} - \overline{x}) + \overline{x}\|_\infty \leq \|\overline{a} - \overline{x}\|_\infty + \|\overline{x}\|_\infty \leq \frac{1}{d(e)} + e < 1 + e \leq 2e.$$

Now we want to show that dist$(\overline{a}, \mathbb{R}^n \setminus O) \geq \dfrac{1}{2e}$. In order to do that, we will show that

$$(\forall \overline{y} \in \mathbb{R}^n \setminus O)\Big[\|\overline{a} - \overline{y}\|_\infty \geq \frac{1}{2e}\Big].$$

Let $\overline{y} \in \mathbb{R}^n \setminus O$. Thus $\|\overline{x} - \overline{y}\|_\infty \geq \dfrac{1}{e}$ (as dist$(\overline{x}, \mathbb{R}^n \setminus O) \geq \dfrac{1}{e}$). By the triangle inequality we have

$$\|\overline{x} - \overline{y}\|_\infty \leq \|\overline{x} - \overline{a}\|_\infty + \|\overline{a} - \overline{y}\|_\infty$$

and so

$$\|\overline{a} - \overline{y}\|_\infty \geq \|\overline{x} - \overline{y}\|_\infty - \|\overline{x} - \overline{a}\|_\infty \geq \frac{1}{e} - \frac{1}{d(e)} > \frac{1}{2e}$$

as

$$\frac{1}{e} - \frac{1}{d(e)} > \frac{1}{2e} \ \leftrightarrow \ d(e) > 2e.$$

4.2. $\big|\theta(\overline{x}) - \theta(\overline{a})\big| < \dfrac{1}{2e}$.

Proof. *We have*

$$\|\overline{x} - \overline{a}\|_\infty < \frac{1}{d(e)} = \frac{1}{(e + e_0)^{m_1+1}} < \frac{1}{(e + e_0)^{m_1}}$$

and $\overline{x}, \overline{a} \in O_{2e} \subseteq O_{2(e+e_0)}$. Thus by property (r_1, m_1) we can conclude that

$$\big|\theta(\overline{x}) - \theta(\overline{a})\big| < \frac{1}{2(e + e_0)} < \frac{1}{2e}.$$

4.3. $\left|\theta(\overline{a}) - f(\overline{a}, e)\right| < \dfrac{1}{2e}$.

Proof. *Since* $\overline{a} \in O_{2e} \subseteq O_{2(e+e_0)}$ *and*

$$e + e_0 > e_0 = \lceil \max(r_1, r_2) \rceil \geq \max(r_1, r_2) \geq r_2$$

by the definition of f *we can see that*

$$\left|\theta(\overline{a}) - f(\overline{a}, e)\right| = \left|\theta(\overline{a}) - f_0(\overline{a}, e + e_0)\right| < \frac{1}{2(e + e_0)} < \frac{1}{2e}.$$

Finally, let's check the inequality $\left|\theta(\overline{x}) - f(\overline{a}, e)\right| < \dfrac{1}{e}$. *We consequently obtain that*

$$\begin{aligned} \left|\theta(\overline{x}) - f(\overline{a}, e)\right| &= \left|\theta(\overline{x}) - \theta(\overline{a}) + \theta(\overline{a}) - f(\overline{a}, e)\right| \leq \\ &\leq \left|\theta(\overline{x}) - \theta(\overline{a})\right| + \left|\theta(\overline{a}) - f(\overline{a}, e)\right| < \\ &< \frac{1}{2e} + \frac{1}{2e} = \frac{1}{e}. \end{aligned}$$

Lemma 5. Let O be an open subset of $\mathbb{R}^n$ $(n \geq 1)$ and $\theta : O \to \mathbb{R}$ be a function. Further, suppose there exist a strictly increasing function $d : \mathbb{N} \to \mathbb{N}^+$ in $\mathcal{M}^2$ and an $\mathcal{M}^2$-computable function $f : \mathbb{Q}^n \times \mathbb{N} \to \mathbb{Q}$ such that

$$(\forall e \in \mathbb{N}^+)(\forall \overline{x} \in O_e)(\forall \overline{a} \in \mathbb{Q}^n)\Big[\|\overline{x} - \overline{a}\|_\infty < \frac{1}{d(e)} \ \rightarrow \ |\theta(\overline{x}) - f(\overline{a}, e)| < \frac{1}{e}\Big].$$

Then θ is parametrically **MSO**-computable.

Proof. *Suppose that* $e \in \mathbb{N}^+$, $\overline{x} = (x_1, \ldots, x_n) \in O_e$, $(p_1, q_1, r_1), \ldots, (p_n, q_n, r_n), (p_{n+1}, q_{n+1}, r_{n+1}) \in \mathcal{T}_1^3$ *are names of* $x_1, \ldots, x_n, e$ *respectively, and* $t \in \mathbb{N}$. *Let's denote*

$$p_i' = p_i'(e, t) = p_i(d(\max(e, t + 1))),$$

$$q_i' = q_i'(e, t) = q_i(d(\max(e, t + 1))),$$

$$r_i' = r_i'(e, t) = r_i(d(\max(e, t + 1))),$$

and

$$a_i = a_i(e, t) = \frac{p_i' - q_i'}{r_i' + 1} = \frac{p_i(d(\max(e, t + 1))) - q_i(d(\max(e, t + 1)))}{r_i(d(\max(e, t + 1))) + 1}$$

for $i = 1, 2, \ldots, n$. *Thus*

$$|x_i - a_i| < \frac{1}{d(\max(e, t + 1))}$$

for $i = 1, 2, \ldots, n$. *From here,* $\overline{x} \in O_e \subseteq O_{\max(e,t+1)}$ *and* $\overline{a} = (a_1, \ldots, a_n) \in \mathbb{Q}^n$ *we can see that*

$$\left|\theta(\overline{x}) - f\big(a_1, \ldots, a_n, \max(e, t + 1)\big)\right| < \frac{1}{\max(e, t + 1)} \leq \frac{1}{t + 1}.$$

Let f be $\mathcal{M}^2$-computable via the functions $f_1, f_2, f_3 : \mathbb{N}^{3n+3} \to \mathbb{N}$. It follows that

$$f\big(a_1, \dots, a_n, \max(e, t+1)\big) =$$

$$= \frac{f_1\big(p'_1, q'_1, r'_1, \dots, p'_n, q'_n, r'_n, \max(e,t+1), 0, 0\big) - f_2\big(p'_1, q'_1, r'_1, \dots, p'_n, q'_n, r'_n, \max(e,t+1), 0, 0\big)}{f_3\big(p'_1, q'_1, r'_1, \dots, p'_n, q'_n, r'_n, \max(e,t+1), 0, 0\big) + 1}.$$

Hence it is appropiate to define

$$\Gamma_i(\underbrace{p_1, q_1, r_1}_{\text{a name of } x_1}, \dots, \underbrace{p_n, q_n, r_n}_{\text{a name of } x_n}, \underbrace{p_{n+1}, q_{n+1}, r_{n+1}}_{\text{a name of } e})(t) =$$

$$= f_i\big(p'_1, q'_1, r'_1, \dots, p'_n, q'_n, r'_n, \max(e, t+1), 0, 0\big) =$$

$$= f_i\big(\underbrace{p'_1(e,t), q'_1(e,t), r'_1(e,t)}_{\substack{\text{the corresponding} \\ \text{name of } a_1}}, \dots, \underbrace{p'_n(e,t), q'_n(e,t), r'_n(e,t)}_{\substack{\text{the corresponding} \\ \text{name of } a_n}}, \max(e, t+1), 0, 0\big)$$

*for $i \in \{1, 2, 3\}$. Consequently the operators $\Gamma_1, \Gamma_2, \Gamma_3$ belong to the class **MSO** and θ is parametrically **MSO**-computable via the triple $(\Gamma_1, \Gamma_2, \Gamma_3)$.*

Remark 9. We should have written $\left\lfloor \dfrac{|p_{n+1}(1) - q_{n+1}(1)|}{r_{n+1}(1) + 1} + \dfrac{1}{2} \right\rfloor$ in place of e earlier in the definition of the operator Γ_i.

Theorem 1. Let O ba an open semialgebraic subset of $\mathbb{R}^n$ $(n \geq 1)$ and $\theta : O \to \mathbb{R}$ be a continuous semialgebraic function. Then θ is parametrically **MSO**-computable.

Proof. *A direct consequence of Lemma 4 and Lemma 5.*

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7 Integration of parametrically MSO-computable functions

Remark 10. The following theorem is due to Ivan Georgiev and it is our main reference for the complexity of integration. In [G20] it is proved for $l = 1$. The proof remains practically the same for $l > 1$.

Theorem 2. ([G20], Theorem 6.1.) Let α, β be $\mathcal{M}^2$-computable real numbers, $\alpha < \beta$, $D \subseteq \mathbb{R}^l$ $(l \geq 1)$ be a set (of parameters) and $\theta : [\alpha, \beta] \times D \to \mathbb{R}$ be a uniformly **MSO**-computable function. Let there exist $A \in \mathbb{R}_{>0}$, such that for every fixed $(\xi_1, \dots, \xi_l) \in D$ the function $\theta_{(\xi_1, \dots, \xi_l)} : [\alpha, \beta] \to \mathbb{R}$, defined by

$\theta_{(\xi_1,\ldots,\xi_l)}(x) = \theta(x,\xi_1,\ldots,\xi_l)$, has a (complex) analytic continuation $\Theta_{(\xi_1,\ldots,\xi_l)} : [\alpha,\beta]\times[-A,A] \to \mathbb{C}$. Let there also exist a polynomial P in l variables with natural coefficients, such that

$$\big(\forall(\xi_1,\ldots,\xi_l)\in D\big)\big(\forall x\in[\alpha,\beta]\big)\big(\forall B\in[-A,A]\big)\big[|\Theta_{(\xi_1,\ldots,\xi_l)}(x+iB)| \le P\big(|\xi_1|,\ldots,|\xi_l|\big)\big].$$

Then the function $I : D\to\mathbb{R}$ defined by

$$I(\xi_1,\ldots,\xi_l) = \int_\alpha^\beta \theta(x,\xi_1,\ldots,\xi_l)dx$$

is uniformly **MSO**-computable.

Definition 13. Let O be a bounded open subset of $\mathbb{R}^n$ $(n\ge 1)$. We call a function $\theta : O\to\mathbb{R}$ restricted analytic iff there exists a positive real number A such that θ has a bounded (complex) analytic continuation to the set

$$\big\{(x_1+iy,\ldots,x_n+iy)\in\mathbb{C}^n \mid (x_1,\ldots,x_n)\in O \;\&\; y\in[-A,A]\big\}.$$

Corollary 1. Let α,β be $\mathcal{M}^2$-computable real numbers, $\alpha<\beta$, and the real function $\theta : (\alpha,\beta)\to\mathbb{R}$ be restricted analytic and parametrically **MSO**-computable. Then the definite integral

$$\int_\alpha^\beta \theta(x)dx$$

is an $\mathcal{M}^2$-computable real number.

Proof. *We are looking for functions $p,q,r\in\mathcal{T}_1\cap\mathcal{M}^2$ such that*

$$\Big|\int_\alpha^\beta \theta(x)dx - \frac{p(t)-q(t)}{r(t)+1}\Big| < \frac{1}{t+1}$$

for all $t\in\mathbb{N}$.

Let the real number $M_\theta>0$ bound θ. For $e,t\in\mathbb{N}$ with $e>\max\big(M_\theta\cdot 3(t+1),\frac{1}{\beta-\alpha}\big)$ we have

$$\Big|\int_\alpha^{\alpha+\frac{1}{e}} \theta(x)dx\Big| \le \int_\alpha^{\alpha+\frac{1}{e}} |\theta(x)|dx \le \frac{M_\theta}{e} < \frac{1}{3(t+1)}$$

and

$$\Big|\int_{\beta-\frac{1}{e}}^{\beta}\theta(x)dx\Big| \le \int_{\beta-\frac{1}{e}}^{\beta}|\theta(x)|dx \le \frac{M_\theta}{e} < \frac{1}{3(t+1)}.$$

For $e \in \mathbb{N}^+$ with

$$\alpha + \frac{1}{e} < \beta - \frac{1}{e} \quad \Big(\leftrightarrow \frac{2}{\beta-\alpha} < e\Big),$$

we have

$$(\alpha,\beta)_e = [-e,e] \cap \Big[\alpha+\frac{1}{e}, \beta-\frac{1}{e}\Big].$$

Hence for $e > \max\big(|\alpha|, |\beta|, \dfrac{2}{\beta-\alpha}\big)$ we have

$$[-e,e] \supseteq [\alpha,\beta] \supseteq \Big[\alpha+\frac{1}{e}, \beta-\frac{1}{e}\Big]$$

and whence

$$(\alpha,\beta)_e = \Big[\alpha+\frac{1}{e}, \beta-\frac{1}{e}\Big].$$

The function $e : \mathbb{N} \to \mathbb{N}^+$ defined by

$$e = e(t) = 1 + \max\Big(\lceil|\alpha|\rceil, \lceil|\beta|\rceil, \Big\lceil\frac{2}{\beta-\alpha}\Big\rceil, \lceil M_\theta\rceil \cdot 3(t+1)\Big)$$

is in $\mathcal{M}^2$. We consequently obtain that

$$\Big|\int_{\alpha}^{\beta}\theta(x)dx - \frac{p(t)-q(t)}{r(t)+1}\Big| = \Big|\int_{\alpha}^{\alpha+\frac{1}{e(t)}}\theta(x)dx + \Big(\int_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}}\theta(x)dx - \frac{p(t)-q(t)}{r(t)+1}\Big) + \int_{\beta-\frac{1}{e(t)}}^{\beta}\theta(x)dx\Big|$$

$$\le \Big|\int_{\alpha}^{\alpha+\frac{1}{e(t)}}\theta(x)dx\Big| + \Big|\int_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}}\theta(x)dx - \frac{p(t)-q(t)}{r(t)+1}\Big| + \Big|\int_{\beta-\frac{1}{e(t)}}^{\beta}\theta(x)dx\Big|$$

$$< \frac{1}{3(t+1)} + \Big|\int_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}}\theta(x)dx - \frac{p(t)-q(t)}{r(t)+1}\Big| + \frac{1}{3(t+1)}$$

$$= \frac{2}{3(t+1)} + \Big|\int_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}}\theta(x)dx - \frac{p(t)-q(t)}{r(t)+1}\Big|.$$

So we wish to show that

$$\Big|\int_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}}\theta(x)dx - \frac{p(t)-q(t)}{r(t)+1}\Big| \le \frac{1}{3(t+1)}.$$

We will prove that the function $I:\mathbb{N}\to\mathbb{R}$ defined by

$$I(t)=\int\limits_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}}\theta(x)dx$$

is uniformly $\mathcal{M}^2$-computable. Applying the linear change of variables

$$x=\frac{(\beta-\frac{1}{e(t)})-(\alpha+\frac{1}{e(t)})}{2}\cdot u+\frac{(\beta-\frac{1}{e(t)})+(\alpha+\frac{1}{e(t)})}{2}=\frac{\beta-\alpha-\frac{2}{e(t)}}{2}\cdot u+\frac{\beta+\alpha}{2}$$

we obtain

$$\begin{aligned}I(t)=\int\limits_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}}\theta(x)dx&=\frac{\beta-\alpha-\frac{2}{e(t)}}{2}\cdot\int\limits_{-1}^{1}\underbrace{\theta\Big(\frac{\beta-\alpha-\frac{2}{e(t)}}{2}\cdot u+\frac{\beta+\alpha}{2}\Big)}_{=\theta_1(u,t)}du\\&=\frac{\beta-\alpha-\frac{2}{e(t)}}{2}\cdot\underbrace{\int\limits_{-1}^{1}\theta_1(u,t)du}_{=J(t)}\\&=\frac{\beta-\alpha-\frac{2}{e(t)}}{2}\cdot J(t).\end{aligned}$$

Since α and β are $\mathcal{M}^2$-computable real numbers and the function $e:\mathbb{N}\to\mathbb{N}^+$ is in $\mathcal{M}^2$, it suffices to show that the function $J:\mathbb{N}\to\mathbb{R}$ defined by

$$J(t)=\int\limits_{-1}^{1}\theta_1(u,t)du$$

*is uniformly **MSO**-computable. In order to do that, we will apply Theorem 2 (with parameters from $\mathbb{N}$) to the function (the integrand) $\theta_1:[-1,1]\times\mathbb{N}\to\mathbb{R}$ defined by*

$$\theta_1(u,t)=\theta\Big(\frac{\beta-\alpha-\frac{2}{e(t)}}{2}\cdot u+\frac{\beta+\alpha}{2}\Big).$$

*Since θ is restricted analytic, let A be a positive real number for which θ has a bounded (complex) analytic continuation Θ to the set $(\alpha,\beta)\times[-A,A]$. Let Θ be bounded by M_Θ. In particular, for every fixed $t\in\mathbb{N}$ the continuation Θ is defined on the set $\big[\alpha+\frac{1}{e(t)},\beta-\frac{1}{e(t)}\big]\times[-A,A]$. Consequently, for every fixed $t\in\mathbb{N}$ the function $\theta_{1,t}:[-1,1]\to\mathbb{R}$, defined by $\theta_{1,t}(u)=\theta_1(u,t)$, has a (complex) analytic continuation to the set $[-1,1]\times[-A,A]$, which is bounded by M_Θ. It remains to show that θ_1 is uniformly **MSO**-computable.*

---

*Let $\theta:(\alpha,\beta)\to\mathbb{R}$ be parametrically **MSO**-computable via the triple $(F_\theta,G_\theta,H_\theta)$. Since α and β are $\mathcal{M}^2$-computable real numbers and the function $\lambda t.\frac{2}{e(t)}:\mathbb{N}\to\mathbb{Q}$ is $\mathcal{M}^2$-computable (for the function $e:\mathbb{N}\to\mathbb{N}^+$ is in $\mathcal{M}^2$), the function $\Delta:[-1,1]\times\mathbb{N}\to[\alpha,\beta]$ defined by*

$$\Delta(u,t)=\frac{\beta-\alpha-\frac{2}{e(t)}}{2}\cdot u+\frac{\beta+\alpha}{2}$$

*is uniformly **MSO**-computable. Thus let Δ be uniformly **MSO**-computable via the triple (P, Q, R).*

Hence the $(6,1)$-operators $F_{\theta_1}, G_{\theta_1}$ and H_{θ_1} defined as follows

$$F_{\theta_1}(\underbrace{p_1, q_1, r_1}_{\substack{\text{a name of}\\ u\in[-1,1]}}, \underbrace{p_2, q_2, r_2}_{\substack{\text{a name of}\\ t\in\mathbb{N}}}) = F_\theta\Big(\underbrace{P(p_1,q_1,r_1,p_2,q_2,r_2), Q(p_1,q_1,r_1,p_2,q_2,r_2), R(p_1,q_1,r_1,p_2,q_2,r_2)}_{\text{the corresponding name of } \frac{\beta-\alpha-\frac{2}{e(t)}}{2}\cdot u+\frac{\beta+\alpha}{2}\in[\alpha+\frac{1}{e(t)},\beta-\frac{1}{e(t)}]},$$
$$\lambda m.e\Big(\underbrace{\Big\lfloor\frac{|p_2(1)-q_2(1)|}{r_2(1)+1}+\frac{1}{2}\Big\rfloor}_{=t}\Big), \lambda m.0, \lambda m.0\Big),$$

$$G_{\theta_1}(p_1,q_1,r_1,p_2,q_2,r_2) = G_\theta\Big(P(p_1,q_1,r_1,p_2,q_2,r_2), Q(p_1,q_1,r_1,p_2,q_2,r_2), R(p_1,q_1,r_1,p_2,q_2,r_2),$$
$$\lambda m.e\Big(\Big\lfloor\frac{|p_2(1)-q_2(1)|}{r_2(1)+1}+\frac{1}{2}\Big\rfloor\Big), \lambda m.0, \lambda m.0\Big),$$

$$H_{\theta_1}(p_1,q_1,r_1,p_2,q_2,r_2) = H_\theta\Big(P(p_1,q_1,r_1,p_2,q_2,r_2), Q(p_1,q_1,r_1,p_2,q_2,r_2), R(p_1,q_1,r_1,p_2,q_2,r_2),$$
$$\lambda m.e\Big(\Big\lfloor\frac{|p_2(1)-q_2(1)|}{r_2(1)+1}+\frac{1}{2}\Big\rfloor\Big), \lambda m.0, \lambda m.0\Big)$$

*are in the class **MSO** and $\theta_1 : [-1,1]\times\mathbb{N}\to\mathbb{R}$ is uniformly **MSO**-computable via the triple $(F_{\theta_1}, G_{\theta_1}, H_{\theta_1})$.*

---

*Consequently, the function $I:\mathbb{N}\to\mathbb{R}$ is also uniformly **MSO**-computable. By Remark 4.3 in [G20] let the functions $f, g, h \in \mathcal{T}_2\cap\mathcal{M}^2$ be such that for any $t\in\mathbb{N}$ the triple*

$$\big(\lambda n.f(t,n), \lambda n.g(t,n), \lambda n.h(t,n)\big)$$

is a name of the definite integral

$$I(t) = \int_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}} \theta(x)dx.$$

In particular, we have

$$\Big|\int_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}} \theta(x)dx - \frac{f(t,3t+2)-g(t,3t+2)}{h(t,3t+2)+1}\Big| < \frac{1}{(3t+2)+1} = \frac{1}{3t+3}$$

for any $t\in\mathbb{N}$. Hence the triple

$$(p,q,r) = \big(\lambda t.f(t,3t+2), \lambda t.g(t,3t+2), \lambda t.h(t,3t+2)\big)$$

is a name of $\int_\alpha^\beta \theta(x)dx$ and so $\int_\alpha^\beta \theta(x)dx$ is an $\mathcal{M}^2$-computable real number.

Corollary 2. Let α and β be $\mathcal{M}^2$-computable real numbers, $\alpha < \beta$, O be a bounded open subset of $\mathbb{R}^n$ $(n \geq 1)$ and suppose that the function $\theta : O \times (\alpha, \beta) \to \mathbb{R}$ is restricted analytic and parametrically **MSO**-computable. Then the function $I : O \to \mathbb{R}$ defined by

$$I(\overline{x}) = \int_{\alpha}^{\beta} \theta(\overline{x}, y) dy$$

is restricted analytic and uniformly **MSO**-computable.

Proof. *Let* $M_\theta \in \mathbb{R}_{>0}$ *bound* θ*. The function* $e : \mathbb{N} \to \mathbb{N}^+$ *defined by*

$$e(t) = 1 + \max\Big(\lceil|\alpha|\rceil, \lceil|\beta|\rceil, \Big\lceil\frac{2}{\beta - \alpha}\Big\rceil, \lceil M_\theta\rceil.3(t+1)\Big)$$

is in $\mathcal{M}^2$*.*

We consequently obtain

$$|I(\overline{x}) - ?| = \Big|\int_{\alpha}^{\beta} \theta(\overline{x}, y) dy - ?\Big| =$$

$$= \Big|\int_{\alpha}^{\alpha + \frac{1}{e(t)}} \theta(\overline{x}, y) dy + \Big(\int_{\alpha + \frac{1}{e(t)}}^{\beta - \frac{1}{e(t)}} \theta(\overline{x}, y) dy - ?\Big) + \int_{\beta - \frac{1}{e(t)}}^{\beta} \theta(\overline{x}, y) dy\Big| \leq$$

$$\leq \Big|\int_{\alpha}^{\alpha + \frac{1}{e(t)}} \theta(\overline{x}, y) dy\Big| + \Big|\int_{\alpha + \frac{1}{e(t)}}^{\beta - \frac{1}{e(t)}} \theta(\overline{x}, y) dy - ?\Big| + \Big|\int_{\beta - \frac{1}{e(t)}}^{\beta} \theta(\overline{x}, y) dy\Big| \leq$$

$$< \frac{1}{3t+3} + \Big|\int_{\alpha + \frac{1}{e(t)}}^{\beta - \frac{1}{e(t)}} \theta(\overline{x}, y) dy - ?\Big| + \frac{1}{3t+3} =$$

$$= \frac{2}{3t+3} + \Big|\int_{\alpha + \frac{1}{e(t)}}^{\beta - \frac{1}{e(t)}} \theta(\overline{x}, y) dy - ?\Big|.$$

So we wish the following inequality to hold

$$\Big|\int_{\alpha + \frac{1}{e(t)}}^{\beta - \frac{1}{e(t)}} \theta(\overline{x}, y) dy - ?\Big| \leq \frac{1}{3t+3}.$$

We will show that the function $J : O \times \mathbb{N} \to \mathbb{R}$ defined by

$$J(\overline{x}, t) = \int\limits_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}} \theta(\overline{x}, y)dy$$

*is uniformly **MSO**-computable. If J is unifomly **MSO**-computable via the triple (F_J, G_J, H_J), then the function $I : O \to \mathbb{R}$ will be uniformly **MSO**-computable via the triple (F_I, G_I, H_I), where*

$$\begin{aligned}
F_I(p_1, q_1, r_1, \ldots, p_n, q_n, r_n)(t) &= F_J(p_1, q_1, r_1, \ldots, p_n, q_n, r_n, \lambda m.t, \lambda m.0, \lambda m.0)(3t + 2), \\
G_I(p_1, q_1, r_1, \ldots, p_n, q_n, r_n)(t) &= G_J(p_1, q_1, r_1, \ldots, p_n, q_n, r_n, \lambda m.t, \lambda m.0, \lambda m.0)(3t + 2), \\
H_I(p_1, q_1, r_1, \ldots, p_n, q_n, r_n)(t) &= H_J(p_1, q_1, r_1, \ldots, p_n, q_n, r_n, \lambda m.t, \lambda m.0, \lambda m.0)(3t + 2).
\end{aligned}$$

The restricted analyticity of I follows from the restricted analyticity of θ and from classical results in complex analysis.

Applying the linear change of variables

$$y = \frac{(\beta - \frac{1}{e(t)}) - (\alpha + \frac{1}{e(t)})}{2} \cdot u + \frac{(\beta - \frac{1}{e(t)}) + (\alpha + \frac{1}{e(t)})}{2} = \frac{\beta - \alpha - \frac{2}{e(t)}}{2} \cdot u + \frac{\beta + \alpha}{2}$$

we obtain

$$\begin{aligned}
J(\overline{x}, t) = \int\limits_{\alpha+\frac{1}{e(t)}}^{\beta-\frac{1}{e(t)}} \theta(\overline{x}, y)dy &= \frac{\beta - \alpha - \frac{2}{e(t)}}{2} \cdot \int\limits_{-1}^{1} \underbrace{\theta\Big(\overline{x}, \frac{\beta - \alpha - \frac{2}{e(t)}}{2} \cdot u + \frac{\beta + \alpha}{2}\Big)}_{=\theta_1(\overline{x}, t, u)} du \\
&= \underbrace{\frac{\beta - \alpha - \frac{2}{e(t)}}{2}}_{=J_0(t)} \cdot \underbrace{\int\limits_{-1}^{1} \theta_1(\overline{x}, t, u)du}_{=J_1(\overline{x}, t)} \\
&= J_0(t) \cdot J_1(\overline{x}, t).
\end{aligned}$$

Since the function $J_0 : \mathbb{N} \to \mathbb{R}$ defined by

$$J_0(t) = \frac{\beta - \alpha - \frac{2}{e(t)}}{2}$$

*is uniformly **MSO**-computable and in view of the fact that multiplication preserves uniform **MSO**-computability, it remains to show that the function $J_1 : O \times \mathbb{N} \to \mathbb{R}$ defined by*

$$J_1(\overline{x}, t) = \int\limits_{-1}^{1} \theta_1(\overline{x}, t, u)du$$

*is uniformly **MSO**-computable. In order to do that, we will apply Theorem 2 (with parameters from $O \times \mathbb{N}$) to the function $\theta_1 : O \times \mathbb{N} \times [-1,1] \to \mathbb{R}$ defined by*

$$\theta_1(\overline{x}, t, u) = \theta\Big(\overline{x}, \frac{\beta - \alpha - \frac{2}{e(t)}}{2} \cdot u + \frac{\beta + \alpha}{2}\Big).$$

Since $\theta : O \times (\alpha, \beta) \to \mathbb{R}$ is restricted analytic, let A be a positive real number for which θ has a bounded (complex) analytic continuation Θ to the set

$$\big\{(x_1 + iz, \ldots, x_n + iz, y + iz) \in \mathbb{C}^{n+1} \mid (x_1, \ldots, x_n) \in O \;\&\; y \in (\alpha, \beta) \;\&\; z \in [-A, A]\big\},$$

*Let Θ be bounded by M_Θ. In particular, for every fixed $(\overline{x}, t) \in O \times \mathbb{N}$ the continuation Θ is defined on $\big[\alpha + \frac{1}{e(t)}, \beta - \frac{1}{e(t)}\big] \times [-A, A]$. Consequently, for every fixed $(\overline{x}, t) \in O \times \mathbb{N}$ the function $\theta_{1,(\overline{x},t)} : [-1,1] \to \mathbb{R}$, defined by $\theta_{1,(\overline{x},t)}(u) = \theta_1(\overline{x}, t, u)$, has a (complex) analytic continuation to the set $[-1,1] \times [-A, A]$, which is bounded by M_Θ. It remains to show that θ_1 is uniformly **MSO**-computable.*

---

*Suppose the function $\theta : O \times (\alpha, \beta) \to \mathbb{R}$ is parametrically **MSO**-computable via the $(3n+6, 1)$-operators $(F_\theta, G_\theta, H_\theta)$. Since α and β are $\mathcal{M}^2$-computable real numbers and the function $\lambda t.\frac{2}{e(t)} : \mathbb{N} \to \mathbb{Q}$ is uniformly **MSO**-computable (as the function $e : \mathbb{N} \to \mathbb{N}^+$ is in $\mathcal{M}^2$), let the function $\Delta : \mathbb{N} \times [-1,1] \to \mathbb{R}$ defined by*

$$\Delta(t, u) = \frac{\beta - \alpha - \frac{2}{e(t)}}{2} \cdot u + \frac{\beta + \alpha}{2}$$

*be uniformly **MSO**-computable via the $(6,1)$-operators (P, Q, R). Thus the function $\theta_1 : O \times \mathbb{N} \times [-1,1] \to \mathbb{R}$ is uniformly **MSO**-computable via the $(3n+6, 1)$-operators $(F_{\theta_1}, G_{\theta_1}, H_{\theta_1})$ defined by*

$$F_{\theta_1}(\underbrace{p_1, q_1, r_1, \ldots, p_n, q_n, r_n}_{\text{a name of } \overline{x} \in O}, \underbrace{p_{n+1}, q_{n+1}, r_{n+1}}_{\text{a name of } t \in \mathbb{N}}, \underbrace{p_{n+2}, q_{n+2}, r_{n+2}}_{\text{a name of } u \in [-1,1]}) =$$

$$= F_\theta\big(\underbrace{p_1, q_1, r_1, \ldots, p_n, q_n, r_n}_{\text{the same name of } \overline{x} \in O},$$

$$\underbrace{P(p_{n+1}, q_{n+1}, r_{n+1}, p_{n+2}, q_{n+2}, r_{n+2}), Q(p_{n+1}, q_{n+1}, r_{n+1}, p_{n+2}, q_{n+2}, r_{n+2}), R(p_{n+1}, q_{n+1}, r_{n+1}, p_{n+2}, q_{n+2}, r_{n+2})}_{\text{the corresponeding name of } \frac{\beta - \alpha - \frac{2}{e(t)}}{2} \cdot u + \frac{\beta + \alpha}{2} \in (\alpha, \beta)_{e(t)}},$$

$$\underbrace{\lambda m.e\Big(\Big\lfloor \frac{|p_{n+1}(1) - q_{n+1}(1)|}{r_{n+1}(1) + 1} + \frac{1}{2} \Big\rfloor\Big), \lambda m.0, \lambda m.0}_{\text{the corresponding name of } e(t) \in \mathbb{N}}\big).$$

The operators G_{θ_1} and H_{θ_1} are defined in the corresponding way.

Corollary 3. Let O be a bounded open semialgebraic subset of $\mathbb{R}^n$ $(n \geq 1)$, $\alpha, \beta : O \to \mathbb{R}$ be restricted analytic semialgebraic functions with $\alpha < \beta$ on O. Denote

$$U = \big\{(\overline{x}, y) \in \mathbb{R}^{n+1} \mid \overline{x} \in O \;\&\; \alpha(\overline{x}) < y < \beta(\overline{x})\big\}$$

and suppose that $\theta : U \to \mathbb{R}$ is restricted analytic and parametrically **MSO**-computable function on U. Then the function $I : O \to \mathbb{R}$ defined by

$$I(\overline{x}) = \int_{\alpha(\overline{x})}^{\beta(\overline{x})} \theta(\overline{x}, y) dy$$

is restricted analytic and parametrically **MSO**-computable.

Remark 11. Note that if α and β are uniformly **MSO**-computable, then so is the function I.

Proof. *For any fixed $\overline{x} \in O$ we apply the linear change of variables*

$$y = \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})$$

to the given integral and we obtain

$$\begin{aligned} I(\overline{x}) = \int_{\alpha(\overline{x})}^{\beta(\overline{x})} \theta(\overline{x}, y) dy &= \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot \int_0^3 \underbrace{\theta\Big(\overline{x}, \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})\Big)}_{=\theta_1(\overline{x}, u)} du \\ &= \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot \underbrace{\int_0^3 \theta_1(\overline{x}, u) du}_{=J(\overline{x})} \\ &= \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot J(\overline{x}). \end{aligned}$$

Remark 12. Any bounded open interval (a, b) with $\mathcal{M}^2$-computable endpoints $a, b \in \mathbb{R}$, $a < b$, and $2 < (b - a)$, is suitable for the linear change of the variable y, because in this case

$$(\forall e \in \mathbb{R}_{\geq 1}) \Big[a + \frac{1}{e} < b - \frac{1}{e} \Big].$$

The set $(0, 3)$ is just one fixed interval with that property.

---

*By virtue of Theorem 1 and Proposition 1 the function $\frac{\beta - \alpha}{3} : O \to \mathbb{R}$ is parametrically **MSO**-computable. It is also restricted analytic. In order to see that the function $I : O \to \mathbb{R}$ is restricted analytic and parametrically **MSO**-computable, it is enough (having in mind the same Proposition 1) to show that the function $J : O \to \mathbb{R}$ defined by*

$$J(\overline{x}) = \int_0^3 \theta_1(\overline{x}, u) du$$

*is restricted analytic and parametrically **MSO**-computable (actually, J is uniformly **MSO**-computable). Further, by Corollary 2 it suffices to show that the function $\theta_1 : O \times (0, 3) \to \mathbb{R}$, defined by*

$$\theta_1(\overline{x}, u) = \theta\Big(\overline{x}, \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})\Big)$$

*is restricted analytic and parametrically **MSO**-computable. Note that θ_1 is restricted analytic, since it is a composition of restricted analytic functions.*

---

*Since $\alpha, \beta : O \to \mathbb{R}$ are parametrically **MSO**-computable, let the function $\Delta : O \times (0,3) \to \mathbb{R}$ defined by*

$$\Delta(\overline{x}, u) = \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})$$

*be parametrically **MSO**-computable via the $(3n+6,1)$-operators (P, Q, R). Note that for all $e \in \mathbb{N}^+$ we have*

$$\big(O \times (0,3)\big)_e = O_e \times (0,3)_e.$$

For each $e \in \mathbb{R}_{\geq 1}$ the following set is non-empty

$$A(e) = \Big\{ e' \in \mathbb{R} \;\Big|\; e \leq e' \;\&\; \big(\forall (\overline{x}, u) \in \mathbb{R}^{n+1}\big) \Big[(\overline{x}, u) \in \big(O \times (0,3)\big)_e \;\to\; \Big(\overline{x}, \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})\Big) \in U_{e'} \Big] \Big\}.$$

Indeed, let $e \in \mathbb{R}_{\geq 1}$. For

$$u \in (0,3)_e = [-e, e] \cap \Big[0 + \frac{1}{e}, 3 - \frac{1}{e}\Big] \subseteq \Big[\frac{1}{e}, 3 - \frac{1}{e}\Big]$$

we see that

$$\alpha(\overline{x}) + \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3e} \leq \underbrace{\frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})}_{=y} \leq \beta(\overline{x}) - \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3e}.$$

Let m_e be the least value of the function $(\beta - \alpha)$ on O_e (the function $(\beta - \alpha)$ is continuous on the compact O_e). This number is positive (as $\alpha < \beta$ on O). We have

$$\alpha(\overline{x}) + \frac{m_e}{3e} \leq \alpha(\overline{x}) + \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3e} \leq \underbrace{\frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})}_{=y} \leq \beta(\overline{x}) - \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3e} \leq \beta(\overline{x}) - \frac{m_e}{3e}.$$

Let M_α and M_β bound the functions α and β respectively. Taking

$$e' = \max\Big(e, \lceil M_\alpha \rceil, \lceil M_\beta \rceil, \Big\lceil \frac{3e}{m_e} \Big\rceil\Big)$$

we obtain

$$\alpha(\overline{x}) + \frac{1}{e'} \leq \underbrace{\frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})}_{=y} \leq \beta(\overline{x}) - \frac{1}{e'}$$

and so

$$\underbrace{\frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})}_{=y} \in \Big[\alpha(\overline{x}) + \frac{1}{e'}, \beta(\overline{x}) - \frac{1}{e'}\Big] = \big(\alpha(\overline{x}), \beta(\overline{x})\big)_{e'}$$

for all $\overline{x} \in O_e \subseteq O_{e'}$. Thus we have

$$\overline{x} \in O_{e'} \;\&\; \underbrace{\frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})}_{=y} \in \big(\alpha(\overline{x}), \beta(\overline{x})\big)_{e'}.$$

Now we refer to Lemma 4 and Lemma 2 (with e replaced by e').

The function $g : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 1}$ defined by

$$g(e) = \inf A(e)$$

is semialgebraic (here we use the semialgebraicity of the set O and the functions α and β). If $e \in \mathbb{R}_{\geq 1}$, then e is a lower bound of the set $A(e)$ and so $1 \leq e \leq g(e)$. Further, we have

$$(\forall e, e', e'' \in \mathbb{R}_{\geq 1})\big[e' \in A(e) \ \& \ e' \leq e'' \ \to \ e'' \in A(e)\big]$$

and thus

$$(\forall e, e'' \in \mathbb{R}_{\geq 1})\big[g(e) < e'' \ \to \ e'' \in A(e)\big].$$

By Proposition 3 let $r \geq 1$ be a real number and k be a positive natural number such that

$$(\forall e \in \mathbb{R}_{\geq 1})\big[r \leq e \ \to \ g(e) < e^k\big].$$

Therefore

$$(\forall e \in \mathbb{R}_{\geq 1})\big[r \leq e \ \to \ e^k \in A(e)\big].$$

Hence

$$(\forall e \in \mathbb{N}, r \leq e)\big(\forall (\overline{x}, u) \in \mathbb{R}^{n+1}\big)\Big[(\overline{x}, u) \in \big(O \times (0,3)\big)_e \ \to \ \Big(\overline{x}, \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})\Big) \in U_{e^k}\Big].$$

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*It follows that the computation of $\Big(\theta_1 \restriction \big(O \times (0,3)\big)_e\Big)$ can be performed by $(\theta \restriction U_{\max(\lceil r \rceil, e)^k})$ for each $e \in \mathbb{N}^+$. Indeed, let $\theta : U \to \mathbb{R}$ be parametrically **MSO**-computable via the $(3n+6,1)$-operators $(F_\theta, G_\theta, H_\theta)$. Then the function $\theta_1 : O \times (0,3) \to \mathbb{R}$ is parametrically **MSO**-computable via the $(3n+6,1)$-operators $(F_{\theta_1}, G_{\theta_1}, H_{\theta_1})$ defined as follows:*

$$F_{\theta_1}(\overline{p_i, q_i, r_i}) =$$

$$= F_{\theta_1}(\underbrace{\underbrace{p_1, q_1, r_1, \ldots, p_n, q_n, r_n}_{\text{a name of } \overline{x} \in O_e}, \underbrace{p_{n+1}, q_{n+1}, r_{n+1}}_{\text{a name of } u \in (0,3)_e}}_{\text{a name of } (\overline{x}, u) \in O_e \times (0,3)_e = (O \times (0,3))_e}, \underbrace{p_{n+2}, q_{n+2}, r_{n+2}}_{\text{a name of } e \in \mathbb{N}^+}) =$$

$$= F_\theta\big(\underbrace{\underbrace{p_1, q_1, r_1, \ldots, p_n, q_n, r_n}_{\text{the same name of } \overline{x}}, \underbrace{P(\overline{p_i, q_i, r_i}), Q(\overline{p_i, q_i, r_i}), R(\overline{p_i, q_i, r_i})}_{\text{the corresponding name of } \left(\frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})\right) \in (\alpha(\overline{x}), \beta(\overline{x}))}}_{\text{the corresponding name of } \left(\overline{x}, \frac{\beta(\overline{x}) - \alpha(\overline{x})}{3} \cdot u + \alpha(\overline{x})\right) \in U_{\max(\lceil r \rceil, e)^k}}, \underbrace{\lambda m. \max(\lceil r \rceil, e)^k, \lambda m. 0, \lambda m. 0}_{\text{the corresponding name of } \max(\lceil r \rceil, e)^k \in \mathbb{N}^+}\big).$$

We define the other two operators G_{θ_1} and H_{θ_1} in the same way.

Remark 13. We should have written $\Big\lfloor \dfrac{|p_{n+2}(1) - q_{n+2}(1)|}{r_{n+2}(1) + 1} + \dfrac{1}{2} \Big\rfloor$ in place of e in $\max(\lceil r \rceil, e)$.

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8 The volumes of open restricted analytic cells are $\mathcal{M}^2$-computable real numbers

Definition 14. A 1-dimensional open restricted analytic cell is a bounded open interval with algebraic endpoints. An (n+1)-dimensional open restricted analytic cell ($n \geq 1$) is a set of the form

$$\{(\overline{x}, y) \in \mathbb{R}^{n+1} \mid \overline{x} \in O \ \&\ \alpha(\overline{x}) < y < \beta(\overline{x})\}$$

for some n-dimensional open restricted analytic cell O and restricted analytic semialgebraic functions $\alpha, \beta : O \to \mathbb{R}$ with $\alpha < \beta$ on O.

Remark 14. Note that every n-dimensional open restricted analytic cell is open, bounded, semialgebraic, Jordan measurable, and it has positive measure.

Definition 15. Let $n, i \in \mathbb{N}^+$ and $i \leq n$. $\pi_i : \mathbb{R}^n \to \mathbb{R}^i$ is the projection function on the first i-coordinates, i.e.

$$\pi_i(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_i).$$

Remark 15. If $i < n$ and

$$C = \{(\overline{x}, y) \in \mathbb{R}^n \mid \overline{x} \in O \ \&\ \alpha(\overline{x}) < y < \beta(\overline{x})\},$$

then $\pi_{n-1}[C] = O$.

Definition 16. Let $n, i \in \mathbb{N}^+$, $1 \leq i < n$ and C be an n-dimensional open restricted analytic cell. The fiber over a point $\overline{x}^{n-i} \in \pi_{n-i}[C]$ is the set

$$C_{\overline{x}^{n-i}} = \{\overline{y}^i \in \mathbb{R}^i \mid (\overline{x}^{n-i}, \overline{y}^i) \in C\}.$$

Lemma 6. Let C be an n-dimensional open restricted analytic cell ($n \geq 2$). For $i \in \mathbb{N}$ with $1 \leq i < n$ we define the function $V_i : \pi_{n-i}[C] \to \mathbb{R}$ by

$$V_i(\overline{x}^{n-i}) = vol(C_{\overline{x}^{n-i}}) = \int\limits_{C_{\overline{x}^{n-i}}} 1 d\overline{y}^i.$$

The functions $V_1, V_2, \ldots, V_{n-1}$ are restricted analytic and parametrically **MSO**-computable.

Proof. *We proceed by induction on $i \in \{1, 2, \ldots, n-1\}$. Let $i = 1$. We consider the function $V_1 : \pi_{n-1}[C] \to \mathbb{R}$ defined by*

$$V_1(\overline{x}^{n-1}) = vol(C_{\overline{x}^{n-1}}).$$

Since C is an n-dimensional open restricted analytic cell, it has the form

$$C = \big\{(\overline{x}^{n-1}, y) \in \mathbb{R}^n \mid \overline{x}^{n-1} \in \pi_{n-1}[C] \ \&\ \alpha_{n-1}(\overline{x}^{n-1}) < y < \beta_{n-1}(\overline{x}^{n-1})\big\}$$

*for some restricted analytic semialgebraic functions $\alpha_{n-1}, \beta_{n-1} : \pi_{n-1}[C] \to \mathbb{R}$ with $\alpha_{n-1} < \beta_{n-1}$ on $\pi_{n-1}[C]$. By virtue of Theorem 1 the functions α_{n-1} and β_{n-1} are parametrically **MSO**-computable. For every $\overline{x}^{n-1} \in \pi_{n-1}[C]$ we have*

$$C_{\overline{x}^{n-1}} = \{y \in \mathbb{R} \mid (\overline{x}^{n-1}, y) \in C\} = \big(\alpha_{n-1}(\overline{x}^{n-1}), \beta_{n-1}(\overline{x}^{n-1})\big)$$

and

$$V_1(\overline{x}^{n-1}) = vol(C_{\overline{x}^{n-1}}) = vol\Big(\big(\alpha_{n-1}(\overline{x}^{n-1}), \beta_{n-1}(\overline{x}^{n-1})\big)\Big) = \beta_{n-1}(\overline{x}^{n-1}) - \alpha_{n-1}(\overline{x}^{n-1}).$$

*Consequently, the function V_1 is restricted analytic and parametrically **MSO**-computable (as a difference of such functions).*

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*Further, let $i \in \mathbb{N}$, $1 < i < n-1$ and the function $V_i : \pi_{n-i}[C] \to \mathbb{R}$ be restricted analytic and parametrically **MSO**-computable. By definition we have*

$$V_{i+1} : \pi_{n-i-1}[C] \to \mathbb{R}, \quad V_{i+1}(\overline{x}^{n-i-1}) = vol(C_{\overline{x}^{n-i-1}}) = \int\limits_{C_{\overline{x}^{n-i-1}}} 1 d(y, \overline{z}^i).$$

As $\pi_{n-i}[C]$ is an $(n-i)$-dimensional open restricted analytic cell we have

$$\pi_{n-i}[C] = \big\{(\overline{x}^{n-i-1}, y) \in \mathbb{R}^{n-i} \mid \overline{x}^{n-i-1} \in O \ \&\ \alpha(\overline{x}^{n-i-1}) < y < \beta(\overline{x}^{n-i-1})\big\}$$

*for some $(n-i-1)$-dimensional open restricted analytic cell O and restricted analytic semialgebraic functions $\alpha, \beta : O \to \mathbb{R}$ with $\alpha < \beta$ on O. Note that $O = \pi_{n-i-1}[C]$. By virtue of Theorem 1 the functions α and β are parametrically **MSO**-computable. The fiber over a point $\overline{x}^{n-i-1} \in O$ has the form*

$$\begin{aligned}
C_{\overline{x}^{n-i-1}} &= \big\{(y, \overline{z}^i) \in \mathbb{R}^{i+1} \mid (\overline{x}^{n-i-1}, y, \overline{z}^i) \in C\big\} \\
&= \big\{(y, \overline{z}^i) \in \mathbb{R}^{i+1} \mid (\overline{x}^{n-i-1}, y) \in \pi_{n-i}[C] \ \&\ \overline{z}^i \in C_{(\overline{x}^{n-i-1}, y)}\big\} \\
&= \big\{(y, \overline{z}^i) \in \mathbb{R}^{i+1} \mid \overline{x}^{n-i-1} \in O \ \&\ \alpha(\overline{x}^{n-i-1}) < y < \beta(\overline{x}^{n-i-1}) \ \&\ \overline{z}^i \in C_{(\overline{x}^{n-i-1}, y)}\big\} \\
&= \big\{(y, \overline{z}^i) \in \mathbb{R}^{i+1} \mid \alpha(\overline{x}^{n-i-1}) < y < \beta(\overline{x}^{n-i-1}) \ \&\ \overline{z}^i \in C_{(\overline{x}^{n-i-1}, y)}\big\}.
\end{aligned}$$

Therefore by Fubini's Theorem we can see that

$$\begin{aligned}
V_{i+1}(\overline{x}^{n-i-1}) = vol(C_{\overline{x}^{n-i-1}}) = \int\limits_{C_{\overline{x}^{n-i-1}}} 1 d(y, \overline{z}^i) &= \int\limits_{\alpha(\overline{x}^{n-i-1})}^{\beta(\overline{x}^{n-i-1})} \Big(\int\limits_{C_{(\overline{x}^{n-i-1}, y)}} 1 d\overline{z}^i \Big) dy \\
&= \int\limits_{\alpha(\overline{x}^{n-i-1})}^{\beta(\overline{x}^{n-i-1})} vol(C_{(\overline{x}^{n-i-1}, y)}) dy \\
&= \int\limits_{\alpha(\overline{x}^{n-i-1})}^{\beta(\overline{x}^{n-i-1})} V_i(\overline{x}^{n-i-1}, y) dy
\end{aligned}$$

*for each $\overline{x}^{n-i-1} \in O = \pi_{n-i-1}[C]$. Consequently, by Corollary 3 (with $U = \pi_{n-i}[C]$ and $\theta = V_i$) we can conclude that the function $V_{n-i-1} : O \to \mathbb{R}$ is restricted analytic and parametrically **MSO**-computable.*

Corollary 4. The volumes of open restricted analytic cells are $\mathcal{M}^2$-computable real numbers.

Proof. *Let C be an n-dimensional open restriced analytic cell ($n \geq 1$). Since $\pi_1[C] \subseteq \mathbb{R}$ and $\pi_1[C]$ is an open restricted analytic cell, we have $\pi_1[C] = (\alpha, \beta)$ for some algebraic real numbers α, β with $\alpha < \beta$. By Lemma 6 we know that the function $V_{n-1} : (\alpha, \beta) \to \mathbb{R}$ defined by*

$$V_{n-1}(y) = vol(C_y) = \int\limits_{C_y} 1 d\overline{z}^{n-1}$$

*is restricted analytic and parametrically **MSO**-computable. As C is an n-dimensional open restricted analytic cell, it can be written in the form*

$$C = \{(y, \overline{z}^{n-1}) \in \mathbb{R}^n \mid y \in (\alpha, \beta) \;\&\; \overline{z}^{n-1} \in C_y\}.$$

Therefore by Fubini's Theorem we have

$$vol(C) = \int_C 1 d\overline{x}^n = \int\limits_{\alpha}^{\beta} \Big(\int\limits_{C_y} 1 d\overline{z}^{n-1} \Big) dy = \int\limits_{\alpha}^{\beta} vol(C_y) dy = \int\limits_{\alpha}^{\beta} V_{n-1}(y) dy.$$

As algebraic numbers α and β are $\mathcal{M}^2$-computable. Hence by Corollary 1 the volume of C is an $\mathcal{M}^2$-computable real number.

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9 The volumes of bounded semialgebraic sets are $\mathcal{M}^2$-computable real numbers

Definition 17. Analytic Cells are non-empty semialgebraic sets defined inductively as follows:

(i) The analytic cells in $\mathbb{R}$ are points $\{c\}$ and open intervals (α, β), $-\infty \leq \alpha < \beta \leq +\infty$.

Let $C \subseteq \mathbb{R}^n$ ($n \geq 1$) be an analytic cell and $\alpha, \beta : C \to \mathbb{R}$ be analytic semialgebraic functions such that $\alpha < \beta$ on C. Then the sets:

(ii) $(\alpha, \beta) = \{(\overline{x}, y) \in C \times \mathbb{R} \mid \alpha(\overline{x}) < y < \beta(\overline{x})\}$ (a cylinder);

(iii) $(-\infty, \alpha) = \{(\overline{x}, y) \in C \times \mathbb{R} \mid -\infty < y < \alpha(\overline{x}))\}$;

(iv) $(\beta, +\infty) = \{(\overline{x}, y) \in C \times \mathbb{R} \mid \beta(\overline{x}) < y < +\infty\}$;

(v) graph$(f) = \{(\overline{x}, y) \in C \times \mathbb{R} \mid y = \alpha(\overline{x})\}$, and

(vi) $C \times \mathbb{R}$

are analytic cells in $\mathbb{R}^{n+1}$.

Remark 16. Every bounded analytic cell is a Jordan measurable set.

Definition 18. An analytic cell decomposition of $\mathbb{R}^n$ $(n \geq 1)$ is defined by induction on n:

18.1. An analytic cell decomposition of $\mathbb{R}$ is a finite collection of open intervals and points:

$$\{(-\infty, a_1), \{a_1\}, (a_1, a_2), \{a_2\}, \dots, (a_{k-1}, a_k), \{a_k\}, (a_k, \infty)\},$$

where $a_1 < a_2 < \cdots < a_k$ are points in $\mathbb{R}$.

18.2. Assuming that the class of analytic cell decompositions of $\mathbb{R}^{n-1}$ $(n \geq 2)$ has been defined, an analytic cell decomposition of $\mathbb{R}^n$ is a finite partition P of $\mathbb{R}^n$ into analytic cells such that the set

$$\pi(P) = \{\pi(C) \; | C \in P\}$$

is an analytic cell decomposition of $\mathbb{R}^{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection on the first $(n-1)$ coordinates.

Definition 19. We say that an analytic cell decomposition P of $\mathbb{R}^n$ partitions a set $S \subseteq \mathbb{R}^n$ if S is a finite union of disjoint cells in P.

Theorem 3. ([HP17], Theorem 1.1) (Analytic Cell Decomposition) Let $S_1, \dots, S_k$ $(k \geq 1)$ be semialgebraic subsets of $\mathbb{R}^n$. Then there is an analytic cell decomposition of $\mathbb{R}^n$ partitioning each S_i.

Remark 17. Additionally, the functions defining the cells in the analytic cell decomposition can be chosen to be restricted analytic.

Corollary 5. The volumes of bounded semialgebraic sets are $\mathcal{M}^2$-computable real numbers.

Proof. *Let S be a bounded semialgebraic subset of $\mathbb{R}^n$. By the Analytic Cell Decomposition Theorem (i.e. Theorem 3) let $C_1, \dots, C_p$ be analytic cells partitioning the set S. Since S is definable without parameters, the cells $C_1, \dots, C_p$ can also be chosen to be definable without parameters. Hence each of* *the sets $C_1, \ldots, C_p$ is either open restricted analytic cell or has volume zero. Therefore their volumes are $\mathcal{M}^2$-computable real numbers (by virtue of Corollary 4). Thus the volume of S is an $\mathcal{M}^2$-computable real number (as a finite sum of such numbers).*

10 The real periods are $\mathcal{M}^2$-computable real numbers

Theorem 4. ([Yo08], Lemma 24) The ring $\mathcal{P}$ of all periods is generated by

$$\bigcup_{n\in\mathbb{N}^+} \Big\{vol(S) \mid S \subseteq \mathbb{R}^n \;\&\; \text{“}S \text{ is a bounded open semialgebraic set”}\Big\}.$$

Corollary 6. The real periods are $\mathcal{M}^2$-computable real numbers.

Proof. *The set of $\mathcal{M}^2$-computable real numbers is a field. Therefore by Theorem 4 and Corollary 5 we can conclude that every period is an $\mathcal{M}^2$-computable real number.*

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