Sofia University St Kliment Ohridski
Faculty of Mathematics and Informatics
Department of Mathematical Logic and Application

Master Thesis

Definability by Propositional Formulas with Intuitionistic Semantics: Algorithmic Problems

Grigor Kolev
Logic and Algorithms (Mathematics)
Faculty Number 9MI3100001
Supervisor: Prof. Tinko Tinchev

2023

Contents

Chapter 1

Introduction

There is a natural correspondence between partial orders as first-order models and intuitionistic Kripke frames - we can view such structures as either frames or models. Because of this we can use either language to express properties of partial orders. The difference in the semantics though make the languages incomparable in their expressive power - there are properties definable with sentences which are undefinable through propositional intuitionistic formulas and vice versa.

A natural question arises then: can we algorithmically determine whether a property expressible in one language is expressible in the other? Van Benthem poses the following algorithmic problems about the modal language:

1. Is there an algorithm which given an input sentence A, determines whether there exists a modal formula φ such that A and φ have the same models?

2. Is there an algorithm which given an input modal formula φ, determines whether there exists a sentence A such that A and φ have the same models?

3. Is there an algorithm which given an input sentence A and modal formula φ, determines whether A and φ have the same models?

The same correspondence problems arise naturally when considering the intuitionistic language instead of a modal language. In her dissertation, L. Chagrova showed that all three of the problems are undecidable in the intuitionistic case, reducing undecidable problems about Minsky machines to the problems in consideration.

Despite this, we can consider restricted versions of the correspondence problems with respect to particular classes - such instances of the correspondence problem are sometimes decidable, depending on the complexity of the restricted class.

We will focus our work on the first problem.

Definition 1. *We say that an intuitionistic formula* φ *is a definition of the sentence* A *with respect to the class of structures* $\mathcal{K}$*, if for every* $\mathfrak{F} \in \mathcal{K}$*,* $\mathfrak{F} \models A \iff \mathfrak{F} \models \varphi$*.*

Definition 2. *The problem* $IntDef$ *with respect to the class* $\mathcal{K}$ *is the following task: given an input sentence* A*, determine whether there exists a definition* φ *of* A *with respect to* $\mathcal{K}$*.*

The text is structured in the following manner:

- Chapter 2 is a brief refresher on First-order, Monadic second-order and Intuitionistic logic. The used notation is presented and the relevant definitions and theorems are reminded to the reader.

- Chapter 3 examines a number of classes based on linear orders. Using classical results due to Rabin, decidability of the first-order theories of the classes is shown. By showing that certain properties of the models of sentences can be effectively determined, the problem of definability with respect to those classes is proven to be decidable.

- Chapter 4 is about undecidable instances of the definability problem. The technique of stable classes due to Balbiani and Tinchev is briefly commented on and is used as the main tool to prove undecidability of the definability problem by reducing validity in a class to definability. Undecidability of the first-order theories and stability of some natural classes of structures is proven, showing that the definability problem with respect to those classes is undecidable.

- Chapter 5 gives a brief summary of the results and alludes to a further problem about a class of structures with similar nature to a class we have considered in Chapter 4.

Chapter 2

Preliminaries

2.1 General

The general framework for the present work will be the theory ZFC. Unless otherwise specified, we will use standard terminology and notation with its usual meaning when reasoning about sets and set operations. We will use the standard notation ω for the set of natural numbers and lowercase greek letters for ordinals, sometimes using n or k with indices for natural numbers. A central role will take partially ordered sets and here we will briefly list the most relevant definitions and properties concerning them.

Definition 3. *A partial order is a pair $\langle X, \leq \rangle$, where X is a set and $\leq$ is a reflexive, transitive and antisymmetric binary relation.*

A strict partial order is a pair $\langle X, < \rangle$, where X is a set and $<$ is an irreflexive and transitive binary relation.

With every partial order $\langle X, \leq \rangle$ we can associate in a natural way the corresponding strict partial order $\langle X, (\leq \setminus \{\langle a, a \rangle \mid a \in X\}) \rangle$ and dually with every strict partial order $\langle X, < \rangle$ we can associate the partial order $\langle X, (< \cup \{\langle a, a \rangle \mid a \in X\}) \rangle$.

We will use the following accompanying notions when working with a partial order $\mathfrak{P} = \langle X, \leq \rangle$:

- *The inverse relation is $\geq = \{\langle b, a \rangle \mid a \leq b\}$. The inverse partial order of $\mathfrak{P}$ is $\mathfrak{P}^\star = \langle X, \geq \rangle$.*
- *The elements $a \in X$ and $b \in X$ are comparable if $a \leq b$ or $b \leq a$. If they are not comparable we say that they are incomparable.*
- *The element $a \in X$ is minimal if for every element $b \in X$ comparable with a, $a \leq b$.*

- *The element* $a \in X$ *is the least element of* X *if for every* $b \in X$, $a \leq b$.
- *The set* $Y \subseteq X$ *is a chain if every two elements* $a, b \in Y$ *are comparable.*
- *The set* $Y \subseteq X$ *is an antichain if every two distinct elements* $a, b \in Y$ *are incomparable.*
- *The set* $Y \subseteq X$ *is upward closed (or upper set, or upper cone) if for every element* $a \in Y$ *and every element* $b \in X$, $a \leq b$ *implies* $b \in X$.
- *The upper closure of* $Y \subseteq X$ *is the set* $Y \uparrow = \{x \in X \,|\, (\exists y \in Y)(y \leq x)\}$.
- $\mathfrak{P}$ *is dense if for every two elements* $a, b \in X$ *such that* $a < b$, *there exists an element* $c \in X$ *such that* $a < c < b$.
- $\mathfrak{P}$ *is a linear order if every two elements* $a, b \in X$ *are comparable.*
- *The element* $b \in X$ *is the successor of* $a \in X$ *if* $a < b$ *and there is no element* $c \in X$ *such that* $a < c < b$.
- *The partial orders* $\mathfrak{P}_1 = \langle X, \leq \rangle$ *and* $\mathfrak{P}_2 = \langle Y, \sqsubseteq \rangle$ *are isomorphic if there exists a bijective function* $f : X \to Y$ *such that for every* $a, b \in X$, $a \leq b \iff f(a) \sqsubseteq f(b)$. *We write* $\mathfrak{P}_1 \cong \mathfrak{P}_2$.

Remark 1. *Some relevant properties for any partial order* $\mathfrak{P} = \langle X, \leq \rangle$ *are the following:*

- *For every* $Y \subseteq X$, $\langle Y, \leq \cap (Y \times Y) \rangle$ *is a partial order.*
- *If* $\langle X, \leq \rangle$ *is a linear order and* $Y \subseteq X$, *then* $\langle Y, \leq \cap (Y \times Y) \rangle$ *is a linear order.*
- *If* $\mathfrak{P}$ *is an infinite linear order, then there is an infinite set* $Y \subseteq X$, *such that* $\langle Y, \leq \cap (Y \times Y) \rangle$ *is isomorphic to* ω *or* $\omega^\star$.

We will concern ourselves with computational aspects in the context of a certain notion of definability. Since our work will mostly consist of analysis of the models of theories, for the sake of readability we will use a somewhat high level of abstraction. We will usually present algorithms in the form of natural language description, from which it will be clear how to construct a decision procedure in a formal manner.

Usually the problems we will consider will be of the following form: is there an effective procedure, determining whether a formula A has a property we are interested in.

Remark 2. *Post's theorem.*

The set A is decidable precisely when both A and its complement are decidable.

2.2 First-order languages and logic

The main setting of this thesis will be first-order languages, theories and classes of models. We will work purely with semantic tools and here we will outline briefly the relevant notions. For further reference the reader may consult [3].

Definition 4. *A relational first-order language (abbreviated RFOL) $\mathcal{L}$ with equality consists of the following:*

1. *Logical symbols:*
   - *A countably infinite set VAR of individual variables. Usually we will denote individual variables with lowercase latin letters x, y, z, t, m, sometimes with indices.*
   - *The propositional connectives $\neg, \wedge$.*
   - *The quantifier $\exists$*
   - *The symbol for formal equality $\doteq$*
2. *Nonlogical symbols:*
   - *A set of predicate symbols $Pred(\mathcal{L})$. Usually we will denote predicate symbols with uppercase latin R, S, T, E or $\leq, \sqsubseteq, <$, sometimes with indices. For each symbol $R \in Pred(\mathcal{L})$ there is an associated arity $1 \leq \#(R) < \omega$.*

   We say that $\mathcal{L}$ has cardinality κ if the set $Pred(\mathcal{L})$ has cardinality κ.

The following definitions hold for arbitrary RFOL $\mathcal{L}$:

Definition 5. *An atomic formula of $\mathcal{L}$ is one of the following:*

- $(x \doteq y)$ *for any* $x, y \in VAR$
- $R(x_1, \cdots, x_n)$ *for any* $R \in Pred(\mathcal{L})$, $\#(R) = n$ *and* $x_1, \cdots, x_n \in VAR$

Definition 6. *A $\mathcal{L}$-formula is any atomic formula, and if A and B are formulas and $x \in VAR$, then*

- $(A \wedge B)$ *is a formula*
- $(\neg A)$ *is a formula*
- $\exists x A$ *is a formula*

Usually formulas will be denoted by uppercase latin A, B, C, D, sometimes with indices. The set of all $\mathcal{L}$-formulas will be denoted by $FOR(\mathcal{L})$.

Remark 3. *We will often use the additional propositional connectives $\vee, \rightarrow$ and the quantifier $\forall$:*

- *$A \vee B$ is an abbreviation for $\neg(\neg A \wedge \neg B)$*
- *$A \rightarrow B$ is an abbreviation for $\neg A \vee B$*
- *$\forall x A$ is an abbreviation for $\neg \exists x (\neg A)$*

Definition 7. *Given an $\mathcal{L}$-formula A, we define the variables occurring in A - $vars(A)$, the free variables of A - $fv(A)$, the bound variables of A - $bv(A)$, and the quantifier rank of A - $qr(A)$, as usual:*

- *if $A = x \doteq y$, then $vars(A) = \{x, y\}$, $fv(A) = \{x, y\}$, $bv(A) = \emptyset$, $qr(A) = 0$*
- *if $A = R(x_1, \cdots, x_n)$, then $vars(A) = \{x_1, \cdots, x_n\}$, $fv(A) = \{x_1, \cdots, x_n\}$, $bv(A) = \emptyset$, $qr(A) = 0$*
- *if $A = \neg B$, then $vars(A) = vars(B)$, $fv(A) = fv(B)$, $bv(A) = bv(B)$, $qr(A) = qr(B)$*
- *if $A = B \wedge C$, then $vars(A) = vars(B) \cup vars(C)$, $fv(A) = fv(B) \cup fv(C)$, $bv(A) = bv(B) \cup bv(C)$, $qr(A) = max\{qr(B), qr(C)\}$*
- *if $A = \exists x B$, then $vars(A) = vars(B) \cup \{x\}$, $fv(A) = fv(B) \setminus \{x\}$, $bv(A) = bv(B) \cup \{x\}$, $qr(A) = qr(B) + 1$*

If $fv(A) = \emptyset$ we will say that A is a sentence. The set of all $\mathcal{L}$-sentences will be denoted by $SENT(\mathcal{L})$.

Definition 8. *The $\mathcal{L}$-formula B is a variant of the $\mathcal{L}$-formula A if B is obtained by repeatedly executing the following procedure:*

Suppose $A = \cdots \exists x C \cdots$ and $y \notin vars(C)$ is a variable. Then obtain $A' = \cdots \exists y C[x/y] \cdots$ by replacing each free occurrence of x in C with y, i.e.:

- *$z[x/y] = z$, if $z \neq x$ is a variable*
- *$x[x/y] = y$*
- *$(x_1 \doteq x_2)[x/y] = (x_1[x/y] \doteq x_2[x/y])$*

- $(R(x_1, \cdots, x_n))[x/y] = R(x_1[x/y], \cdots, x_n[x/y])$ *for every* $R \in Pred(\mathcal{L})$
- $(\neg D)[x/y] = \neg(D[x/y])$
- $(D_1 \wedge D_2)[x/y] = D_1[x/y] \wedge D_2[x/y]$
- $(\exists x D)[x/y] = \exists x D$
- $(\exists z D)[x/y] = \exists z(D[x/y])$, *if* $z \neq x$ *is a variable*

Remark 4. *For every* $\mathcal{L}$*-formula* A *there is an* $\mathcal{L}$*-formula* B*, such that* B *is a variant of* A *and* $fv(B) \cap bv(B) = \emptyset$.

Definition 9. *Let* $\mathcal{L}$ *be a RFOL. An* $\mathcal{L}$*-model(or* $\mathcal{L}$*-structure) is any pair* $\mathfrak{A} = \langle \mathcal{A}, I \rangle$*, where:*

- $\mathcal{A}$ *is a nonempty set called the universe of* $\mathfrak{A}$*. For a model* $\mathfrak{A}$ *we will denote the universe of* $\mathfrak{A}$ *with* $|\mathfrak{A}|$.
- I *is an interpretation of the nonlogical symbols of* $\mathcal{L}$*, i.e.* $I : Pred(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{A}^\star)$ *such that for every* $R \in Pred(\mathcal{L})$, $I(R) \subseteq \mathcal{A}^{\#(R)}$*. For a model* $\mathfrak{A}$ *and predicate symbol* $R \in Pred(\mathcal{L})$ *we will denote its interpretation* $I(R)$ *with* $R^{\mathfrak{A}}$.

When it is clear what the language $\mathcal{L}$ *is in the context of our arguments, we may refer to* $\mathcal{L}$*-models(*$\mathcal{L}$*-structures) as just models(structures).*

We will usually denote models with uppercase gothic letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}$*, sometimes with indices.*

Definition 10. *A variable assignment is any function* $V : VAR \rightarrow |\mathfrak{A}|$.

For a model $\mathfrak{A}$*, variable assignment* V*, variable* $x \in VAR$ *and element* $a \in |\mathfrak{A}|$*, the modified assignment* V_x^a *is the variable assignment such that*

- $V_x^a(x) = a$
- $V_x^a(z) = V(z)$*, for* $z \neq x$

Definition 11. *For any* $\mathcal{L}$*-model* $\mathfrak{A}$ *and variable assignment* V*, the satisfaction relation* $\models$ *is defined as follows:*

- $\mathfrak{A}, V \models x \doteq y$*, if* $V(x) = V(y)$
- $\mathfrak{A}, V \models R(x_1, \cdots, x_n)$*, if* $\langle V(x_1), \cdots, V(x_n) \rangle \in R^{\mathfrak{A}}$
- $\mathfrak{A}, V \models \neg A$*, if* $\mathfrak{A}, V \not\models A$

- $\mathfrak{A}, V \models A \wedge B$, *if* $\mathfrak{A}, V \models A$ *and* $\mathfrak{A}, V \models B$
- $\mathfrak{A}, V \models \exists x A$, *if there is an element* $a \in |\mathfrak{A}|$ *such that* $\mathfrak{A}, V_x^a \models A$

We can readily see that:

- $\mathfrak{A}, V \models A \vee B \iff \mathfrak{A}, V \models A$ *or* $\mathfrak{A}, V \models B$
- $\mathfrak{A}, V \models A \rightarrow B \iff \mathfrak{A}, V \models B$ *or* $\mathfrak{A}, V \not\models A$
- $\mathfrak{A}, V \models \forall x A \iff$ *for every element* $a \in |\mathfrak{A}|$, $\mathfrak{A}, V_x^a \models A$

We say that:

- *A is valid in the model* $\mathfrak{A}$ *and write* $\mathfrak{A} \models A$*, if for every variable assignment* V, $\mathfrak{A}, V \models A$*. We also say that* $\mathfrak{A}$ *is a model of* A*.*
- *A set* $\Gamma \subseteq FOR(\mathcal{L})$ *is valid in the model* $\mathfrak{A}$ *and write* $\mathfrak{A} \models \Gamma$*, if for every formula* $A \in \Gamma$, $\mathfrak{A} \models A$*.*

Remark 5. *A direct consequence of the definition of the satisfaction relation is that if* $\mathfrak{A}$ *is a model,* A *is a formula and* V_1 *and* V_2 *are variable assignments such that for every* $x \in fv(A)$ $V_1(x) = V_2(x)$*, then* $\mathfrak{A}, V_1 \models A \iff \mathfrak{A}, V_2 \models A$

Because of that, when A *is a sentence we have that* $\mathfrak{A} \models A \iff$ *for all variable assignments* V, $\mathfrak{A}, V \models A \iff$ *there exists a variable assignment* V *such that* $\mathfrak{A}, V \models A$*, i.e. sentences state inherent properties of the models and their validity does not depend on the variable assignment.*

Remark 6. *By Remark 3, for every formula* A *we can find a variant* B *of* A *such that* $fv(B) \cap bv(B) = \emptyset$*. Moreover, if* B *is a variant of* A*, for every model* $\mathfrak{A}$ *and every variable assignment* V *we have that* $\mathfrak{A}, V \models A \iff \mathfrak{A}, V \models B$*. Therefore in our analysis of formulas we can always assume that the formulas we are working with have the property that* $fv(A) \cap bv(A) = \emptyset$*.*

From the properties of formulas and satisfaction we have stated thus far we can develop the following convenient notation: for a formula A *with* $fv(A) = \{x_1, \cdots, x_n\}$, $\mathfrak{A} \models A[\![\overline{a}]\!]$ *is an abbreviation for "for every assignment* V*, such that* $V(x_i) = a_i$ *for every* $1 \leq i \leq n$, $\mathfrak{A}, V \models A$*".*

Definition 12. *Let* $\mathcal{K}$ *be a class of models. We say that:*

- *The sentence* A *is valid in* $\mathcal{K}$ *and write* $\mathcal{K} \models \mathfrak{A}$ *if for every model* $\mathfrak{A} \in \mathcal{K}$, $\mathfrak{A} \models A$*. If* $\mathcal{K}$ *is the class of all models we say that* A *is valid.*

- *The sentence A is satisfiable in $\mathcal{K}$ if there is a model $\mathfrak{A} \in \mathcal{K}$ such that $\mathfrak{A} \models A$. If $\mathcal{K}$ is the class of all models we say that A is satisfiable. We have that A is satisfiable (in $\mathcal{K}$) precisely when $\neg A$ is not valid (in $\mathcal{K}$).*

- *We will denote the class of all models of A from $\mathcal{K}$ with $Mod_{\mathcal{K}}(A)$.*

- *The set $\Gamma \subseteq SENT$ is satisfiable (in $\mathcal{K}$) if there is a model $\mathfrak{A}(\in \mathcal{K})$, such that $\mathfrak{A} \models \Gamma$.*

- *$\mathcal{K}$ is axiomatizable if there exists a set $\Gamma \subseteq SENT(\mathcal{L})$ such that for every model $\mathfrak{A}$, $\mathfrak{A} \models \Gamma \iff \mathfrak{A} \in \mathcal{K}$. We say that Γ axiomatizes $\mathcal{K}$*

- *$\mathcal{K}$ is finitely axiomatizable if there exists a finite set $\Gamma \subseteq SENT(\mathcal{L})$ such that Γ axiomatizes $\mathcal{K}$. Since the conjunction of a finite number of formulas is a formula, we can equivalently state that $\mathcal{K}$ is finitely axiomatizable if there exists a sentence A such that $\{A\}$ axiomatizes $\mathcal{K}$.*

- *$T = \{A \in SENT(\mathcal{L}) \mid A$ is valid in $\mathcal{K}\}$ is the theory of $\mathcal{K}$ and we will denote it by $th(\mathcal{K})$. When $\mathcal{K} = \{\mathfrak{A}\}$ we denote it simply by $th(\mathfrak{A})$.*

Definition 13. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-models. We say that:*

- *$\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent and write $\mathfrak{A} \equiv \mathfrak{B}$ if $th(\mathfrak{A}) = th(\mathfrak{B})$.*

- *$\mathfrak{A}$ and $\mathfrak{B}$ are k-elementarily equivalent for $k < \omega$ and write $\mathfrak{A} \equiv_k \mathfrak{B}$ if for every $\mathcal{L}$-sentence A with $qr(A) \leq k$ we have that $\mathfrak{A} \models A \iff \mathfrak{B} \models A$.*

Definition 14. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-models. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic and write $\mathfrak{A} \cong \mathfrak{B}$ if there is a bijection $f : |\mathfrak{A}| \to |\mathfrak{B}|$ such that for every symbol $R \in Pred(\mathcal{L})$ of arity $\#(R) = n$ and every $a_1, \cdots, a_n \in |\mathfrak{A}|$, $\mathfrak{A} \models R(x_1, \cdots, x_n)[\![a_1, \cdots, a_n]\!] \iff \mathfrak{B} \models R(x_1, \cdots, x_n)[\![f(a_1), \cdots, f(a_n)]\!]$.*

Remark 7. *If $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$.*

Definition 15. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-models. We say that:*

- *$\mathfrak{A}$ is a submodel of $\mathfrak{B}$ if $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and for every symbol $R \in Pred(\mathcal{L})$, $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap (|\mathfrak{A}| \times |\mathfrak{A}|)$.*

- *$\mathfrak{A}$ is an extension of $\mathfrak{B}$ if $\mathfrak{B}$ is a submodel of $\mathfrak{A}$.*

- $\mathfrak{A}$ *is isomorphically embedded in* $\mathfrak{B}$ *if* $\mathfrak{A}$ *is isomorphic to a submodel of* $\mathfrak{B}$.

Definition 16. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* $\mathcal{L}$*-models. We say that:*

- $\mathfrak{A}$ *is an elementary submodel of* $\mathfrak{B}$ *if* $\mathfrak{A}$ *is a submodel of* $\mathfrak{B}$ *and for every* $\mathcal{L}$*-formula* A *with* $fv(A) = \{x_1, \cdots, x_n\}$ *and every* $a_1, \cdots, a_n \in |\mathfrak{A}|$, $\mathfrak{A} \models A[\![a_1, \cdots, a_n]\!] \iff \mathfrak{B} \models A[\![a_1, \cdots, a_n]\!]$.

- $\mathfrak{A}$ *is an elementary extension of* $\mathfrak{B}$ *if* $\mathfrak{B}$ *is an elementary submodel of* $\mathfrak{A}$.

- $\mathfrak{A}$ *is elementarily embedded in* $\mathfrak{B}$ *if* $\mathfrak{A}$ *is isomorphic to an elementary submodel of* $\mathfrak{B}$.

Remark 8. *If* $\mathfrak{A}$ *is elementarily embedded in* $\mathfrak{B}$*, then* $\mathfrak{A} \equiv \mathfrak{B}$*. In particular, if* $\mathfrak{A}$ *is an elementary submodel of* $\mathfrak{B}$*, then* $\mathfrak{A} \equiv \mathfrak{B}$.

Definition 17. *Let* A, B *be* $\mathcal{L}$*-formulas with* $fv(A) = \{y, x_1, \cdots, x_n\}$ *and* $vars(A) \cap vars(B) = \emptyset$*. We define the relativization of* B *with respect to* A *and* y*, denoted by* $(B)^{A,y}$*, as follows:*

- $(z_1 \doteq z_2)^{A,y} = z_1 \doteq z_2$

- $(R(z_1, \cdots, z_2))^{A,y} = R(z_1, \cdots, z_n)$ *for every symbol* $R \in Pred(\mathcal{L})$

- $(\neg C)^{A,y} = \neg(C)^{A,y}$

- $(C \wedge D)^{A,y} = (C)^{A,y} \wedge (D)^{A,y}$

- $(\exists z C)^{A,y} = \exists z(A[y/z] \wedge (C)^{A,y})$

Observe that if $fv(B) = \{z_1, \cdots, z_k\}$*, then* $fv((B)^{A,y}) = \{x_1, \cdots, x_n, z_1, \cdots, z_k\}$.

Definition 18. *Let* $\mathfrak{A}$ *be an extension of* $\mathfrak{B}$*,* $a_1, \cdots, a_n \in |\mathfrak{A}|$ *and* $A \in FOR(\mathcal{L})$ *such that* $fv(A) = \{y, x_1, \cdots, x_n\}$*. Then* $\mathfrak{B}$ *is called the relativized reduct of* $\mathfrak{A}$ *with respect to* A *and* $\overline{a}$ *if* $|\mathfrak{B}| = \{b \mid \mathfrak{A} \models A[\![b, \overline{a}]\!]\}$.

Theorem 1. *Relativization theorem. (for proof consult [4] theorem 5.1.1).*

Let $\mathfrak{B}$ *be the relativized reduct of* $\mathfrak{A}$ *with respect to* A *and* $\overline{a}$ *, where* $fv(A) = \{y, \overline{x}\}$*. Let* $B \in FOR(\mathcal{L})$ *be such that* $vars(A) \cap vars(B) = \emptyset$ *and* $fv(B) = \{z_1, \cdots, z_k\}$*. Then for every* $b_1, \cdots, b_k$ $\mathfrak{B} \models B[\![\overline{b}]\!] \iff \mathfrak{A} \models (B)^{A,y}[\![\overline{a}, \overline{b}]\!]$.

Theorem 2. *Compactness theorem.*

A set $\Gamma \subseteq SENT(\mathcal{L})$ *of sentences is satisfiable precisely when every finite* $\Delta \subseteq \Gamma$ *is satisfiable.*

Theorem 3. *Downward Löwenheim–Skolem theorem.*

Let $\mathcal{L}$ *be a language of cardinality* κ_1 *and* $\mathfrak{B}$ *be an* $\mathcal{L}$*-model of cardinality* κ_2*. Then for every cardinality* κ *such that* $\aleph_0 + \kappa_1 \leq \kappa \leq \kappa_2$*, and for every set* $S \subseteq |\mathfrak{B}|$ *of cardinality at most* κ*, there exists an elementary submodel* $\mathfrak{A}$ *of* $\mathfrak{B}$ *such that* $S \subseteq |\mathfrak{A}|$ *and* $|\mathfrak{A}|$ *is of cardinality* κ*.*

Theorem 4. *Tarski [9]*

The theory of the class of all lattices is undecidable. The theory of the class of all partial orders is undecidable.

Theorem 5. *Rogers [7]*

The theory of the class G *of all models for the language* $\mathcal{L} = \{E\}$ *satisfying the axioms* $\forall x E(x,x)$ *and* $\forall x \forall y (E(x,y) \rightarrow E(y,x))$ *is undecidable.*

Theorem 6. *Lavrov [5]*

For the class G *as defined in the previous theorem and the class* G^{fin} *consisting of the finite models in* G*,* $th(G)$ *and* $FOR(E) \setminus th(G^{fin})$ *are recursively inseparable, that is they are disjoint and there exists no decidable set* C *such that* $C \cap th(G) = \emptyset$ *and* $(SENT(E) \setminus th(G^{fin}))$*.*

2.3 Intuitionistic propositional logic

The other logic we will mainly work with will be the intuitionistic propositional logic (which we will abbreviate Int). We will consider standard Kripke semantics and again focus on the semantic portion. For further reference, the reader may consult [2].

Definition 19. *The language of Int consists of:*

- *A countably infinite set $PVAR$ of propositional variables. Usually we will denote propositional variables with p, q, r, s, t, sometimes with indices.*
- *The constant symbols $\top, \bot$*
- *The propositional connectives $\wedge, \vee, \rightarrow$*

Definition 20. *Intuitionistic formula (Int-formula):*
$\top$, $\bot$ *and* p *for every* $p \in PVAR$ *are Int-formulas, and if* φ_1 *and* φ_2 *are Int-formulas, then*

- $(\varphi_1 \wedge \varphi_2)$ *is an Int-formula*
- $(\varphi_1 \vee \varphi_2)$ *is an Int-formula*
- $(\varphi_1 \rightarrow \varphi_2)$ *is an Int-formula*

Usually we will denote formulas with φ, ψ *or* χ, *sometimes with indices.*

Remark 9. *We will often use the additional connective* $\neg$. *The formula* $(\neg\varphi)$ *is an abbreviation for* $\varphi \rightarrow \bot$.

Definition 21. *For a formula* φ *we will denote with* $vars(\phi)$ *the set of all propositional variables, occurring in* φ. *Formally:*

- $vars(\top) = vars(\bot) = \emptyset$
- $vars(p) = \{p\}$ *for* $p \in PVAR$
- $vars(\varphi \wedge \psi) = vars(\varphi \vee \psi) = vars(\varphi \rightarrow \psi) = vars(\varphi) \cup vars(\psi)$

Definition 22. *A Kripke frame is any partial order* $\mathfrak{F} = \langle W, \leq \rangle$ *with* $W \neq \emptyset$.

Definition 23. *Let* $\mathfrak{F} = \langle W, \leq \rangle$ *be a Kripke frame. A variable assignment for* $\mathfrak{F}$ *is a function* $V : PVAR \rightarrow \mathcal{P}(W)$ *such that for every* $p \in PVAR$, $V(p)$ *is an upper set.*

Definition 24. *A Kripke model over a Kripke frame $\mathfrak{F}$ is any pair $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, where V is a variable assignment for $\mathfrak{F}$.*

Definition 25. *For any Kripke model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ over $\mathfrak{F} = \langle W, \leq \rangle$ and any point $x \in W$, the satisfaction relation $\models$ is defined as follows:*

- $\mathfrak{M}, x \models \top$
- $\mathfrak{M}, x \not\models \bot$
- $\mathfrak{M}, x \models p$, *if* $x \in V(p)$
- $\mathfrak{M}, x \models \varphi_1 \vee \varphi_2$, *if* $\mathfrak{M}, x \models \varphi_1$ *or* $\mathfrak{M}, x \models \varphi_2$
- $\mathfrak{M}, x \models \varphi_1 \wedge \varphi_2$, *if* $\mathfrak{M}, x \models \varphi_1$ *and* $\mathfrak{M}, x \models \varphi_2$
- $\mathfrak{M}, x \models \varphi_1 \to \varphi_2$, *if for every* $y \in W$ *such that* $x \leq y$, *if* $\mathfrak{M}, y \models \varphi_1$, *then* $\mathfrak{M}, y \models \varphi_2$

We can readily see that:

- $\mathfrak{M}, x \models \neg\varphi \iff$ *for every* $y \in W$ *such that* $x \leq y$, $\mathfrak{M}, y \not\models \varphi$

We say that:

- φ *is true in* $\mathfrak{M}$ *and write* $\mathfrak{M} \models \varphi$ *if for every* $x \in W$, $\mathfrak{M}, x \models \varphi$.
- φ *is true at* $x \in W$ *in* $\mathfrak{F}$ *and write* $\mathfrak{F}, x \models \varphi$ *if for every model* $\mathfrak{M}$ *over* $\mathfrak{F}$, $\mathfrak{M}, x \models \varphi$.
- φ *is valid in* $\mathfrak{F}$ *and write* $\mathfrak{F} \models \varphi$ *if for every model* $\mathfrak{M}$ *over* $\mathfrak{F}$, $\mathfrak{M} \models \varphi$.
- φ *is satisfiable if there is a Kripke model* $\mathfrak{M}$ *and point* $x \in W$ *such that* $\mathfrak{M}, x \models \varphi$
- φ *is valid if for every Kripke model* $\mathfrak{F}$, $\mathfrak{F} \models \varphi$.

Definition 26. *Let $\mathcal{K}$ be a class of Kripke frames. The logic of $\mathcal{K}$ is $Log(\mathcal{K}) = \{\varphi \mid$ for every $\mathfrak{F} \in \mathcal{K}, \mathfrak{F} \models \varphi\}$. We say that φ is valid in $\mathcal{K}$ and write $\mathcal{K} \models \varphi$ if $\varphi \in Log(\mathcal{K})$. If $\mathcal{K} = \{\mathfrak{F}\}$, then we denote the logic of $\mathcal{K}$ simply by $Log(\mathfrak{F})$.*

Definition 27. *Let $\mathfrak{F} = \langle W, \leq \rangle$ be a Kripke frame and $X \subseteq W$, $X \neq \emptyset$. The generated subframe of X is the frame $\mathfrak{F}_X = \{X\uparrow, \leq \cap (X\uparrow \times X\uparrow)\}$.*

If $X = \{y\}$ for some $y \in W$ we will denote the generated subframe simply by $\mathfrak{F}_y$ and say that $\mathfrak{F}_y$ is rooted with root y.

Remark 10. *Generated subframes theorem.*

Let $\mathfrak{F} = \langle W, \leq \rangle$ *be a Kripke frame. The following properties hold for generated subframes:*

- $\mathfrak{F}, x \models \varphi \iff \mathfrak{F}_x \models \varphi$
- $\mathfrak{F} \models \varphi$ *precisely when for every generated subframe* $\mathfrak{F}_X$, $\mathfrak{F}_X \models \varphi$
- $\mathfrak{F} \models \varphi$ *precisely when for every* $x \in W$, $\mathfrak{F}_x \models \varphi$

Definition 28. *Let* $\mathfrak{F} = \langle W, \leq \rangle$ *and* $\mathfrak{G} = \langle U, \sqsubseteq \rangle$ *be Kripke frames. We say that a function* $f : W \to U$ *is a p-morphism if the following conditions are satisfied:*

- f *is surjective*
- *for every* $x, y \in W$, *if* $x \leq y$ *then* $f(x) \sqsubseteq f(y)$
- *for every* $x, y \in W$, *if* $f(x) \sqsubseteq f(y)$ *then there is some* $z \in W$ *such that* $f(y) = f(z)$ *and* $x \leq z$

We say that $\mathfrak{G}$ *is a p-morphic image of* $\mathfrak{F}$ *if such function* f *exists.*

Remark 11. *P-morphism theorem.*

For any Kripke frame $\mathfrak{F}$ *and* $\mathfrak{G}$, *if* $\mathfrak{G}$ *is a p-morphic image of* $\mathfrak{F}$ *and* $\mathfrak{F} \models \varphi$ *then* $\mathfrak{G} \models \varphi$.

Remark 12. *If* $\mathfrak{F}_3$ *is a p-morphic image of* $\mathfrak{F}_2$ *and* $\mathfrak{F}_2$ *is a p-morphic image of* $\mathfrak{F}_1$, *then* $\mathfrak{F}_3$ *is a p-morphic image of* $\mathfrak{F}_1$.

Remark 13. *Consider the formulas* $\varphi_{depth \leq n}$ *for* $1 \leq n < \omega$, *defined as follows:*

- $\varphi_{depth \leq 1} = p_1 \vee \neg p_1$
- $\varphi_{depth \leq n+1} = p_{n+1} \vee (p_{n+1} \to \varphi_{depth \leq n})$

Then the class of frames validating $\varphi_{depth \leq n}$ *is precisely the class of all partial orders, such that every chain of elements in the frame contains no more than n elements.*

2.4 Monadic second-order languages and logic

In order to show that we can effectively determine some properties of formulas, we will investigate certain classes of models in the context of the more expressive monadic second-order language. Here we will very briefly list the most relevant syntactical and semantic notions for second-order languages and theories.

Definition 29. *A relational monadic second-order language (RMSOL) $\mathcal{L}_{II}$ with equality is a first-order language $\mathcal{L}_I$ extended with a countably infinite set of set variables $SVAR$.*

We will denote set variables with uppercase latin letters, sometimes with indices.

Definition 30. *An atomic formula of $\mathcal{L}_{II}$ is one of the following:*

- *An atomic formula of $\mathcal{L}_I$.*
- *$(x \in Y)$, for any $x \in VAR$ and $Y \in SVAR$.*

Definition 31. *An $\mathcal{L}_{II}$-formula is any atomic $\mathcal{L}_{II}$-formula, and if A and B are formulas, $x \in VAR$, $Y \in SVAR$, then:*

- *$(\neg A)$ is a formula*
- *$(A \wedge B)$ is a formula*
- *$\exists x A$ is a formula*
- *$\exists Y A$ is a formula*

The connectives $\vee$ and $\rightarrow$, and universal quantification over individual variables are defined as in the first-order case.

The formula $\forall Y A$ is an abreviation for $\neg\exists Y \neg A$.

The formula $(\exists x \in Y)A$ is an abbreviation for $\exists x(x \in Y \wedge A)$.

The formula $(\exists x \notin Y)A$ is an abbreviation for $\exists x(\neg x \in Y \wedge A)$

The formula $X \subseteq Y$ is an abbreviation for $\forall z(z \in X \rightarrow z \in Y)$.

The formula $X \doteq Y$ is an abbreviation for $X \subseteq Y \wedge Y \subseteq X$.

The formula $(\exists X \subseteq Y)A$ is an abbreviation for $\exists X(X \subseteq Y \wedge A)$.

Definition 32. *The notions of occurring variables, free variables and bound variables in a formula A are defined similarly to the first-order case, taking into account the variables over sets. We will again denote them by $vars(A), fv(A)$ and $bv(A)$, respectively.*

Variants of formulas are again defined similarly to the first-order case, again taking into account the variables over sets.

Remark 14. *For every $\mathcal{L}_{II}$-formula A, there is a variant B of A, such that $fv(B) \cap bv(B) = \emptyset$.*

Definition 33. *Let $\mathcal{L}_{II}$ be a RMSOL. An $\mathcal{L}_{II}$-model is any first-order model $\mathfrak{A}$ for the language $\mathcal{L}_I$.*

Definition 34. *A variable assignment is any function $V : (VAR \cup SVAR) \to (|\mathfrak{A}| \cup \mathcal{P}(|\mathfrak{A}|))$, such that for every $x \in VAR$, $V(x) \in |\mathfrak{A}|$ and for every $Y \in SVAR$, $V(Y) \subseteq |\mathfrak{A}|$.*

The modified assignment V_x^a for $x \in VAR$ is defined as in the first-order case, copying the behaviour of V over set variables as well.

The modified assignment V_Y^T for $Y \in SVAR$ and $T \subseteq |\mathfrak{A}|$ is the following variable assignment:

- $V_Y^T(x) = V(x)$ *for* $x \in VAR$
- $V_Y^T(Y) = T$
- $V_Y^T(Z) = V(Z)$ *for* $Z \in SVAR$, $Z \neq Y$

Definition 35. *For an $\mathcal{L}_{II}$-model $\mathfrak{A}$ and variable assignment V, the satisfaction relation $\models$ is defined as follows:*

- $\mathfrak{A}, V \models x \doteq y$, *if* $V(x) = V(y)$
- $\mathfrak{A}, V \models x \in Y$, *if* $V(x) \in V(Y)$
- $\mathfrak{A}, V \models R(x_1, \cdots, x_n)$, *if* $\langle V(x_1), \cdots, V(x_n)\rangle \in R^{\mathfrak{A}}$
- $\mathfrak{A}, V \models \neg A$, *if* $\mathfrak{A}, V \not\models A$
- $\mathfrak{A}, V \models A \wedge B$, *if* $\mathfrak{A}, V \models A$ *and* $\mathfrak{A}, V \models B$
- $\mathfrak{A}, V \models \exists x A$, *if there is an element* $a \in |\mathfrak{A}|$ *such that* $\mathfrak{A}, V_x^a \models A$
- $\mathfrak{A}, V \models \exists Y A$, *if there is a set* $T \subseteq |\mathfrak{A}|$ *such that* $\mathfrak{A}, V_Y^T \models A$

It directly follows that:

- $\mathfrak{A}, V \models A \vee B$, *if* $\mathfrak{A}, V \models A$ *or* $\mathfrak{A}, V \models B$
- $\mathfrak{A}, V \models A \to B$, *if* $\mathfrak{A}, V \models \neg A$ *or* $\mathfrak{A}, V \models B$
- $\mathfrak{A}, V \models X \subseteq Y$, *if* $V(X) \subseteq V(Y)$
- $\mathfrak{A}, V \models X \doteq Y$, *if* $V(X) = V(Y)$

- $\mathfrak{A}, V \models \forall x A$, *if for every individual* $a \in |\mathfrak{A}|$, $\mathfrak{A}, V_x^a \models A$
- $\mathfrak{A}, V \models \forall X A$, *if for every set* $T \subseteq |\mathfrak{A}|$, $\mathfrak{A}, V_X^T \models A$

Remark 15. *Similarly to the first-order case, if* A *is a variant of* B*, then* $\mathfrak{A}, V \models A \iff \mathfrak{A}, V \models B$*, and if* $V_1(\eta) = V_2(\eta)$ *for every* $\eta \in fv(A)$*, then* $\mathfrak{A}, V_1 \models A \iff \mathfrak{A}, V_2 \models A$.

We can thus adopt similar semantic notation:

For a formula A *with* $fv(A) \subseteq \{x_1, \cdots, x_n, Y_1, \cdots, Y_k\}$, $\mathfrak{A} \models A[\![\overline{a}, \overline{T}]\!]$ *is an abbreviation for "for every assignment* V*, such that* $V(x_i) = a_i$ *and* $V(Y_j) = T_j$ *for* $1 \leq i \leq n$ *and* $1 \leq j \leq k$, $\mathfrak{A}, V \models A$*".*

Definition 36. *The notions of validity(in a class of models), satisfiability(in a class), (finite) axiomatization of a class and the theory of a class, isomorphic models and elementary equivalence are stated in the exact same manner as in the first-order case.*

We denote the theory of a class $\mathcal{K}$ *with* $th^{II}(\mathcal{K})$.

Remark 16. *If* $\mathfrak{A} \cong \mathfrak{B}$*, then* $\mathfrak{A} \equiv^{II} \mathfrak{B}$.

Of key importance will be the following classical theorem due to Rabin:

Theorem 7. *Decidability of S2S [6]*

The monadic second order theory S2S of two successors is decidable. As a direct consequence, the monadic theory of at most countable linear orders is decidable.

Chapter 3

Decidable instances of definability

In this chapter we will consider some well-tamed classes of frames for which the resulting instances of the problem of definability are decidable.

Linear orders will be the main building blocks for these classes. A useful and in a sense restricting property of linear orders is that every generated subframe is also a linear order.

Mainly due to this property, it turns out that with respect to the classes we consider, all possible intuitionistic definitions modulo equivalence are the following:

- $\top$
- $\bot$
- $\varphi_{depth \leq n}$ for $1 \leq n < \omega$

The general approach in this chapter will be to consider a class of models and perform careful analysis of the properties of the models of an arbitrary first-order sentence A.

3.1 Finite linear orders

First, we will consider the class of all finite linear orders. Despite not being axiomatizable, the class has the important properties of having a decidable theory and a very specific form of the class $Mod_{LIN^{fin}}(A)$ for any sentence A.

Definition 37. *Denote with LIN^{fin} the class of all finite linear orders.*

Throughout this chapter, with $\mathfrak{F}_n$ for $1 \leq n < \omega$ we will denote the model $\langle n, \leq_n \rangle$, where $\leq_n$ is the usual ordering of n.

Remark 17. *Clearly, for any model $\mathfrak{F} \in LIN^{fin}$, there exists a unique $1 \leq n < \omega$ such that $\mathfrak{F} \cong \mathfrak{F}_n$. Without loss of generality, we will assume that all models of the class are of this form.*

Now, consider an arbitrary sentence A. The class $Mod(A)$ has the following simple and finitary characterization:

Proposition 1. *Denote $LIN^{fin>n} = \{\mathfrak{F}_k \mid k > n\}$ for every $1 \leq n < \omega$, and for any sentence A, the class $Mod_{\leq n}(A) = \{\mathfrak{F}_k \mid k \leq n, \mathfrak{F}_k \models A\}$. Let $q = qr(A)$.*

Then

- *If $\mathfrak{F}_{2^q} \models A$, then $Mod(A) = Mod_{\leq 2^q}(A) \cup LIN^{fin>2^q}$.*
- *If $\mathfrak{F}_{2^q} \not\models A$, then $Mod(A) = Mod_{\leq 2^q}(A)$.*

Proof. A standard result for finite linear orders is that for every $k < \omega$ and $n_1, n_2 \geq 2^k$, $\mathfrak{F}_{n_1} \equiv_k \mathfrak{F}_{n_2}$.

Consider a natural number $n > 2^q$. Since $q = qr(A)$, and since $\mathfrak{F}_{2^q} \equiv_q \mathfrak{F}_n$, we can conclude that $\mathfrak{F}_n \models A \iff \mathfrak{F}_{2^q} \models A$.

Therefore for any $n > 2^q$, $\mathfrak{F}_n \in Mod(A) \iff \mathfrak{F}_{2^q} \in Mod(A)$. □

This shows that instead of working with the whole class $Mod(A)$, we can instead argue about the finite set $N(A) = \{n \leq 2^{qr(A)} \mid \mathfrak{F}_n \models A\}$. We obtain the following useful corollaries:

Corollary 1.

1. *The set $N(A)$ is decidable.*
2. *The theory $th(LIN^{fin})$ is decidable.*
3. *It is decidable whether given a sentence A, the class $Mod(A)$ is closed under taking subframes.*

Proof. 1. Since we work in a finite RFOL, for every $1 \leq n < \omega$ we can effectively determine whether $\mathfrak{F}_n \models A$ by following the definition of $\models$.

2. $A \in th(LIN^{fin})$ precisely when $Mod(A) = LIN^{fin}$, which in turn happens precisely when $N(A) = \{1, \cdots, 2^{qr(A)}\}$.

3. Since the set $N(A)$ is decidable and bounded by $2^{qr(A)}$, we can effectively check if it is downward closed. This happens precisely when the class $Mod(A)$ has the desired property.

□

Remark 18. *Observe that for any $1 \leq n_1 \leq n_2 < \omega$, the frame $\mathfrak{F}_{n_1}$ is a generated subframe of $\mathfrak{F}_{n_2}$. Therefore, for any formula φ, $\mathfrak{F}_{n_2} \models \varphi$ implies $\mathfrak{F}_{n_1} \models \varphi$, and so the class of all frames in LIN^{fin} validating φ is closed under taking subframes.*

Theorem 8. *The problem $IntDef$ with respect to the class LIN^{fin} is decidable.*

Proof. Suppose we are given a sentence A. The following procedure effectively determines whether A is definable:

Generate the set $N(A)$.

1. If $N(A) = \{1, \cdots, 2^{qr(A)}\}$, then $LIN^{fin} \models A$ and therefore $\top$ is a definition of A.

2. If $N(A) = \emptyset$, then $LIN^{fin} \models \neg A$ and therefore $\bot$ is a definition of A.

3. If $N(A) \neq \{1, \cdots, 2^{qr(A)}\}$ and $N(A) \neq \emptyset$:

   (a) If $N(A)$ is not downward closed, then by Remark 17 A is undefinable.

   (b) Otherwise, if $N(A)$ is downward closed, then $2^{qr(A)} \notin N(A)$ since $N(A) \neq \{1, \cdots, 2^{qr(A)}\}$ and therefore $Mod(A) = \{\mathfrak{F}_n | n \in N(A)\}$. Since $N(A) \neq \emptyset$, there exists a maximal number $n \in N(A)$. Now all models of A are precisely the frames $\mathfrak{F}_1, \cdots, \mathfrak{F}_n$ and therefore the formula $\varphi_{depth \leq n}$ is a definition of A.

□

3.2 Linear orders

Next, we shall consider the class of all linear orders. We already know what to do with the finite models of a sentence A, so the main task will be to handle the infinite models. To do so, we will first make a few observations on the intuitionistic side and use the fact that the theory of all infinite linear orders is decidable.

Definition 38. *Denote with LIN the class of all linear orders, with LIN^{inf} the class of all infinite linear orders, with $LIN^{countable}$ the class of all at most countable linear orders and with LIN^{cinf} the class of all countably infinite linear orders.*

First we will show that the intuitionistic logic $Log(LIN)$ posseses the finite model property. This will in essence reduce the problem of definability to the finite case.

Proposition 2. *For every frame $\mathfrak{F} \in LIN$ and every intuitionistic formula φ such that $\mathfrak{F} \not\models \varphi$, there exists a finite frame $\mathfrak{F}_{fin} \in LIN$ such that $\mathfrak{F}_{fin} \not\models \varphi$.*

Proof. Let $\mathfrak{F} = \langle W, R\rangle \in LIN$ be a frame and φ be an intuitionistic formula such that $vars(\varphi) \subseteq \{p_1, \cdots, p_n\}$ and $\mathfrak{F} \not\models \varphi$. Let $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ be a Kripke model and $x \in W$ such that $\mathfrak{M}, x \not\models \varphi$.

For a subset $P \subseteq \{p_1, \cdots, p_n\}$ we shall say that $x \in W$ realizes P and write $x \models P$ if $P = \{q \in \{p_1, .., p_n\} \mid \mathfrak{M}, x \models q\}$. We shall say that P is realized if some $x \in W$ realizes it. It is clear that every $x \in W$ realizes a unique P and we shall denote it by P_x.

- $W_{fin} = \{P_x \mid x \in W\}$
- $\leq_{fin} = \{\langle P_x, P_y\rangle \mid x, y \in W, x \leq y\}$
- $V_{fin}(p_i) = \{P \in W_{fin} \mid p_i \in P\}$

We can readily see that $\leq_{fin}$ is a well-defined linear ordering of W_{fin} since V is upward closed. Moreover, $P_1 \leq_{fin} P_2 \iff P_1 \subseteq P_2$. Since $W_{fin} \subseteq \mathcal{P}(\{p_1, \cdots, p_n\})$, the set W_{fin} is finite. Denote $\mathfrak{F}_{fin} = \langle W_{fin}, \leq_{fin}\rangle$ and $\mathfrak{M}_{fin} = \langle \mathfrak{F}_{fin}, V_{fin}\rangle$.

We now claim that for every $x \in W$ and every formula ψ with $vars(\psi) \subseteq \{p_1, \cdots, p_n\}$, $\mathfrak{M}, x \models \psi \iff \mathfrak{M}_{fin}, P_x \models \psi$.

Induction on ψ:

- $\psi = \bot$

  $\mathfrak{M}, x \not\models \bot$ and $\mathfrak{M}_{fin}, P_x \not\models \bot$

- $\psi = p_i$

  $\mathfrak{M}, x \models p_i \iff p_i \in P_x \iff P_x \in V_{fin}(p_i) \iff \mathfrak{M}_{fin}, P_x \models p_i$

- $\psi = \psi_1 \vee \psi_2$

  $\mathfrak{M}, x \models \psi \iff \mathfrak{M}, x \models \psi_1$ or $\mathfrak{M}, x \models \psi_2 \overset{(ih)}{\iff} \mathfrak{M}_{fin}, P_x \models \psi_1$ or $\mathfrak{M}_{fin}, P_x \models \psi_2 \iff \mathfrak{M}_{fin}, P_x \models \psi$

- $\psi = \psi_1 \wedge \psi_2$

  $\mathfrak{M}, x \models \psi \iff \mathfrak{M}, x \models \psi_1$ and $\mathfrak{M}, x \models \psi_2 \overset{(ih)}{\iff} \mathfrak{M}_{fin}, P_x \models \psi_1$ and $\mathfrak{M}_{fin}, P_x \models \psi_2 \iff \mathfrak{M}_{fin}, P_x \models \psi$

- $\psi = \psi_1 \to \psi_2$

  1. Suppose $\mathfrak{M}, x \not\models \psi$. Then there exists $y \geq x$ such that $\mathfrak{M}, y \models \psi_1$ and $\mathfrak{M}, y \not\models \psi_2$. Then by (ih) $\mathfrak{M}_{fin}, P_y \models \psi_1$ and $\mathfrak{M}_{fin}, P_y \not\models \psi_2$. Then since $x \leq y$, we have that $P_x \leq_{fin} P_y$ and $\mathfrak{M}_{fin}, P_x \not\models \psi$.
  2. Suppose $\mathfrak{M}_{fin}, P_x \not\models \psi$. Then there is $P_y \geq_{fin} P_x$ such that $\mathfrak{M}_{fin}, P_y \models \psi_1$ and $\mathfrak{M}_{fin}, P_y \not\models \psi_2$.
     (a) If $x \leq y$, then by (ih) we have that $\mathfrak{M}, y \models \psi_1$ and $\mathfrak{M}, y \not\models \psi_2$. Then $\mathfrak{M}, x \not\models \psi$.
     (b) Otherwise, $y < x$ ($\mathfrak{F}$ is a linear order) and then $P_y \leq_{fin} P_x$. Since we know that $P_x \leq_{fin} P_y$, this means that $P_x = P_y$. Therefore $\mathfrak{M}_{fin}, P_x \models \psi_1$ and $\mathfrak{M}_{fin}, P_x \not\models \psi_2$ and by (ih) $\mathfrak{M}, x \models \psi_1$ and $\mathfrak{M}, x \not\models \psi_2$. Therefore $\mathfrak{M}, x \not\models \psi$.

Now since $\mathfrak{M}, x \not\models \varphi$, we can conclude that $\mathfrak{M}_{fin}, P_x \not\models \varphi$ and thus $\mathfrak{F}_{fin} \not\models \varphi$.

□

To finish our analysis on the intuitionistic side we will need the following property, which lets us, in conjunction with the above, to immediately state that an intuitionistic formula φ is valid in LIN if it is valid in any infinite frame.

Proposition 3. *Let $\mathfrak{F} = \langle W, \leq \rangle \in LIN$ be an infinite linear order and $1 \leq n < \omega$. Then $\mathfrak{F}_n$ is a p-morphic image of $\mathfrak{F}$.*

Proof. Let $a_1 < a_2 < \cdots < a_{n-1} \in W$. Since $\mathfrak{F}$ is infinite, we can always choose such elements for any positive n.

Consider the frame $\mathfrak{F}' = \langle W', \leq \cap (W' \times W') \rangle$, where $W' = \{a \in W \mid a \leq a_{n-1}\}$, and the following two functions:

1. $f : W \to W'$, defined as
   - $f(a) = a$, if $a \leq a_{n-1}$
   - $f(a) = a_{n-1}$, if $a > a_{n-1}$
2. $g : W' \to n$, defined as
   - $g(a) = k$, where k is the least number $k \leq n$ such that $a \leq a_k$

Now clearly f is a p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$ and g is a p-morphism from $\mathfrak{F}'$ onto $\mathfrak{F}_n$. Therefore $\mathfrak{F}_n$ is a p-morphic image of $\mathfrak{F}$

□

On the first-order side, we will use a well-known result due to Rabin:

Proposition 4. *$th(LIN)$ and $th(LIN^{inf})$ are decidable.*

Proof. By [6], the monadic second-order theory of the class $LIN^{countable}$ of all at most countable linear orders is decidable. Therefore, $th(LIN^{countable})$ is decidable and directly $th(LIN)$ is decidable since by the Downward Lowenheim-Skolem theorem $th(LIN) = th(LIN^{countable})$.

Consider the following second-order sentence:

- $Inf = \exists Y \forall x \exists y (x < y) \vee \exists Y \forall x \exists y (x > y)$

Then for a countable model $\mathfrak{A} \in LIN^{countable}$, clearly if $\mathfrak{A} \models Inf$ then $\mathfrak{A}$ is an infinite linear order. Conversely, if $\mathfrak{A}$ is a countably infinite linear order, then it contains a submodel isomorphic to either ω or $\omega^\star$. Evaluating Y with the universe of this model shows that $\mathfrak{A} \models Inf$.

Therefore any at most countable linear order $\mathfrak{A}$ is infinite precisely when $\mathfrak{A} \models Inf$, hence $LIN^{cinf} \models A \iff LIN^{countable} \models Inf \to A$ for any second-order sentence A. Hence $th^{II}(LIN^{cinf})$ is decidable and therefore $th(LIN^{cinf})$ is decidable. Again by the Downward Lowenheim-Skolem theorem this means that $th(LIN^{inf}) = th(LIN^{cinf})$ is decidable.

□

Now we have all the preparation needed to state the following:

Proposition 5. *The problem $IntDef$ with respect to the class LIN is decidable.*

Proof. Suppose we are given a sentence A. The following procedure effectively determines whether A is definable:

1. If $LIN \models A$, then $\top$ is a definition of A.

2. If $LIN \models \neg A$, then $\bot$ is a definition of A.

3. If A and $\neg A$ are satisfiable:

   (a) If $LIN^{inf} \models \neg A$, then A has only finitely(modulo isomorphism) many finite models. Otherwise by the Compactness theorem A would have an infinite model which contradicts our assumption. Proceed with finding a definition of A as in the case of LIN^{fin}.

   (b) If $LIN^{inf} \not\models \neg A$, then we claim that A is undefinable:
   Let $\mathfrak{F} \in LIN^{inf}, \mathfrak{F} \models A$ and assume that φ is a definition of A. Then since every $\mathfrak{G} \in LIN^{fin}$ is a p-morphic image of $\mathfrak{F}$, $LIN^{fin} \models \varphi$. Since LIN has the finite model property, $LIN \models \varphi$ and therefore $LIN \models A$ - contradiction.

$\square$

3.3 Disjoint unions of linear orders

The third class of models we will consider is the class of all disjoint unions of linear orders. We will adopt a similar strategy as for the class of all linear orders: we will wish to show that the models of a sentence A have properties similar to those needed in the previous case.

We will need to see that the theory of the class is decidable, that we can effectively determine whether A has a model with an infinite chain(this condition corresponds to the need to rule out infinite models in the case of linear orders), and whether A is valid in all models of bounded depth, i.e. when we manage to reasonably bound the size of the models A should have in order to be definable, we need to be able to determine that in fact A is true in all such models.

Definition 39. *Denote with $DLIN$ the class of all disjoint unions of linear orders. For an index set $I \neq \emptyset$ and an indexed family $(\mathfrak{F}_i)_{i\in I}$ of disjoint linear orders, we will denote its disjoint union model with $\bigsqcup\limits_{i\in I}\mathfrak{F}_i$, defined as follows:*

- $\bigsqcup\limits_{i\in I}\mathfrak{F}_i = \bigcup\limits_{i\in I}|\mathfrak{F}_i|$
- $\leq^{\bigsqcup\limits_{i\in I}\mathfrak{F}_i} = \bigcup\limits_{i\in I} \leq^{\mathfrak{F}_i}$

Proposition 6. *The class $DLIN$ is finitely axiomatized by the following formulas:*

- $P_1 = \forall x(x \leq x)$
- $P_2 = \forall x\forall y\forall z(x \leq y \wedge y \leq z \to x \leq z)$
- $P_3 = \forall x\forall y(x \leq y \wedge y \leq x \to x \doteq y)$
- $D_1 = \forall x\forall y\forall z(x \leq y \wedge x \leq z \to y \leq z \vee z \leq y)$
- $D_2 = \forall x\forall y\forall z(x \geq y \wedge x \geq z \to y \leq z \vee z \leq y)$

The first three are the usual axioms for partial orders. The last two in essence force a model to be a collection of linear orders - there is no branching allowed.

Proof. Suppose first that $\mathfrak{F} \in DLIN$, $\mathfrak{F} = \bigsqcup\limits_{i\in I}\mathfrak{F}_i$ where $(\mathfrak{F}_i)_{i\in I}$ is an indexed family of linear orders.

In particular $(\mathfrak{F}_i)_{i\in I}$ is a family of partial orders and the disjoint union of partial orders is a partial order. Therefore $\mathfrak{F}$ satisfies P_1, P_2, P_3.

Suppose $a, b, c \in |\mathfrak{F}|$ such that $a \leq^{\mathfrak{F}} b$ and $a \leq^{\mathfrak{F}} c$. Then by the definition of $\leq^{\mathfrak{F}}$ and the fact that the union is disjoint, there exists an index $i \in I$ such that $a \leq^{\mathfrak{F}_i} b$ and $a \leq^{\mathfrak{F}_i} c$. In particular, $b, c \in \mathfrak{F}_i$. Since $\mathfrak{F}_i$ is a linear order, this means that $b \leq^{\mathfrak{F}_i} c$ or $c \leq^{\mathfrak{F}_i} b$. Therefore $b \leq^{\mathfrak{F}} c$ or $c \leq^{\mathfrak{F}} b$. Thus $\mathfrak{F}$ satisfies D_1. The argument about D_2 is symmetric.

Now suppose that $\mathfrak{F}$ satisfies the axioms P_1, P_2, P_3, D_1, D_2. Because of P_1, P_2 and P_3, $\mathfrak{F}$ is a partial order. Consider the following family of models $(\mathfrak{F}_i)_{i\in I}$, where:

- I is an arbitrary maximal antichain in $\mathfrak{F}$. Such exists as a consequence of the Axiom of choice.
- $|\mathfrak{F}_i| = \{a \in |\mathfrak{F}| \mid a \leq^{\mathfrak{F}} i \vee i \leq^{\mathfrak{F}} a\}$ for $i \in I$.
- $\leq^{\mathfrak{F}_i} = \leq^{\mathfrak{F}} \cap (|\mathfrak{F}_i| \times |\mathfrak{F}_i|)$ for $i \in I$.

Immediately, $\mathfrak{F}_i$ is a partial order for every $i \in I$. We claim that $\mathfrak{F}_i$ is a linear order: let $a, b \in |\mathfrak{F}_i|$, then by the definition of $|\mathfrak{F}_i|$, ($a \leq^{\mathfrak{F}} i$ or $i \leq^{\mathfrak{F}} a$) and ($b \leq^{\mathfrak{F}} i$ or $i \leq^{\mathfrak{F}} b$).

We examine the possible cases:

- $a \leq^{\mathfrak{F}} i$ and $i \leq^{\mathfrak{F}} b$

  Then since $\leq^{\mathfrak{F}}$ is a partial order, $a \leq^{\mathfrak{F}} b$ and therefore $a \leq^{\mathfrak{F}_i} b$.
- $b \leq^{\mathfrak{F}} i$ and $i \leq^{\mathfrak{F}} a$

  Symmetric to the previous case.
- $i \leq^{\mathfrak{F}} a$ and $i \leq^{\mathfrak{F}} b$

  Then by axiom D_1 we have that $a \leq^{\mathfrak{F}} b$ or $b \leq^{\mathfrak{F}} a$. Therefore $a \leq^{\mathfrak{F}_i} b$ or $b \leq^{\mathfrak{F}_i} a$.
- $a \leq^{\mathfrak{F}} i$ and $b \leq^{\mathfrak{F}} i$

  Symmetric to the previous case taking axiom D_2 instead of D_1.

Therefore any two elements of $|\mathfrak{F}_i|$ are comparable and $\mathfrak{F}_i$ is a linear order.

Let $i, j \in I$ and choose $k \in |\mathfrak{F}_i| \cap |\mathfrak{F}_j| \neq \emptyset$. Then ($i \leq^{\mathfrak{F}} k$ or $k \leq^{\mathfrak{F}} i$) and ($j \leq^{\mathfrak{F}} k$ or $k \leq^{\mathfrak{F}} i$). Arguing as above, we see that then i and j must be comparable. Since I is an antichain, this is only possible if $i = j$. Therefore the family is disjoint.

Let $a \in |\mathfrak{F}|$. Since I is a maximal antichain, $I \cup \{a\} = I$ or $I \cup \{a\}$ is not an antichain. In the first case, $a \in I$ and therefore $a \in |\mathfrak{F}_a|$. In the second, a is comparable to some $i \in I$ and therefore $a \in |\mathfrak{F}_i|$. Therefore $|\mathfrak{F}| \subseteq |\bigsqcup_{i \in I} \mathfrak{F}_i|$, so $|\mathfrak{F}| = |\bigsqcup_{i \in I} \mathfrak{F}_i|$. Since $\leq^{\mathfrak{F}} = \underset{i \in I}{\leq}^{\mathfrak{F}_i}$, $\mathfrak{F} = \bigsqcup_{i \in I} \mathfrak{F}_i$.

With this we have shown that $\mathfrak{F} \in DLIN$. □

To show that the theory of the class is decidable and that we can effectively determine whether a sentence A has a model with an infinite chain, we will show that we can embed the at most countable models of $DLIN$ into models of LIN. In order to extract the desired reduction we will need to consider the second-order theory of LIN.

An important consideration is that the second-order theory of the full class LIN is undecidable ([8]) but if we limit ourselves to only countable models, the theory is decidable ([6]) and this will suffice.

Definition 40. *Denote with $DLIN^{countable}$ the class of all at most countable disjoint unions of linear orders.*

Proposition 7.

- *By application of the Downward Löwenheim–Skolem theorem, $th(DLIN^{countable}) = th(DLIN)$.*

  Indeed, since $DLIN^{countable} \subseteq DLIN$, we have that $th(DLIN) \subseteq th(DLIN^{countable})$.

  In the other direction, suppose that $A \notin th(DLIN)$. Then since $DLIN$ is axiomatizable, there is a model $\mathfrak{F} \in DLIN$ such that $\mathfrak{F} \not\models A$, i.e. $\mathfrak{F} \models \neg A$. By the Downward Löwenheim–Skolem theorem, there is a countable elementary submodel $\mathfrak{F}'$ of $\mathfrak{F}$. Then $\mathfrak{F}' \equiv \mathfrak{F}$ and therefore $\mathfrak{F}' \models \neg A$, i.e. $\mathfrak{F}' \not\models A$. Since $\mathfrak{F}' \in DLIN$, $A \notin th(DLIN^{countable})$. In conclusion, $th(DLIN^{countable}) \subseteq th(DLIN)$.

- *The property of a model having an infinite chain is not expressible with a first-order sentence. Nevertheless, by using the full strength of the Downward Löwenheim–Skolem we can show that a sentence A has a model in $DLIN$ with this property precisely when A has a model in $DLIN^{countable}$ with the property:*

  If A has a model $\mathfrak{F} \in DLIN^{countable}$ with an infinite chain, then immediately $\mathfrak{F} \in DLIN$.

  Suppose $\mathfrak{F} \in DLIN$, $\mathfrak{F} \models A$ and $C \subseteq |\mathfrak{F}|$ is an infinite chain. Pick a countably infinite subset $C' \subseteq C$. By the Downward Löwenheim–Skolem *theorem there exists an elementary submodel $\mathfrak{F}'$ of $\mathfrak{F}$ such that $C \subseteq |\mathfrak{F}'|$. Then $\mathfrak{F}' \equiv \mathfrak{F}$ and therefore $\mathfrak{F}' \models A$ and $\mathfrak{F}' \in DLIN^{countable}$.*

The above remarks show that it is enough to consider only at most countable models when checking for those properties. Now we will proceed with the promised embedding:

Definition 41. *Let $\mathfrak{A} \in DLIN^{countable}$ and without loss of generality, assume that $\mathfrak{A} = \bigsqcup_{i<\alpha} \mathfrak{A}_i$, where α is a countable ordinal and for every index i, $\mathfrak{A}_i \in LIN^{countable}$. Without loss of generality we will assume that $|\mathfrak{A}| \cap \alpha = \emptyset$.*

The linearization of $\mathfrak{A}$ with respect to the index set α is the model $\mathfrak{B} \in LIN^{countable}$, defined as follows:

- $|\mathfrak{B}| = |\mathfrak{A}| \cup (\bigcup \alpha)$
- $\leq^{\mathfrak{B}} = \leq^{\mathfrak{A}} \cup \{\langle \beta, y\rangle \mid y \in |\mathfrak{A}_\gamma|, \beta < \gamma < \alpha\} \cup \{\langle x, \beta\rangle \mid x \in |\mathfrak{A}_\gamma|, \gamma \leq \beta < \bigcup \alpha\} \cup \{\langle \beta, \gamma\rangle \mid \beta \leq \gamma < \bigcup \alpha\} \cup \{\langle x, y\rangle \mid x \in |\mathfrak{A}_\beta|, y \in |\mathfrak{A}_\gamma|, \beta < \gamma < \alpha\}$

In essence, the linearization of a model $\mathfrak{A}$ is the result of glueing together all the linear orders one after the other in a fashion dependent on the index set α. Between the linear orders we have inserted special separators which will be our guidemarks where one chain ends and starts another.

The following proposition presents the translation and establishes the connection between satisfaction of a sentence in a model $\mathfrak{A} \in DLIN^{countable}$ and in its linearization with respect to an ordering of the index set.

Proposition 8. *For a monadic predicate M consider the following translation $tr_M(A)$ of first-order formulas A in $\mathcal{L} = \{\leq\}$:*

- $tr_M(x \doteq y) = x \doteq y$
- $tr_M(x \leq y) = x \leq y \wedge \neg\exists m(m \in M \wedge x < m < y)$
- $tr_M(\neg A) = \neg tr_M(A)$
- $tr_M(A \wedge B) = tr_M(A) \wedge tr_M(B)$
- $tr_M(\exists x A) = \exists x(x \notin M \wedge tr_M(A))$

Let $\mathfrak{A} \in DLIN^{countable}$, $\mathfrak{A} = \bigsqcup_{i\in I} \mathfrak{A}_i$ and $\mathfrak{B} \in LIN^{countable}$ be the linearization of $\mathfrak{A}$ with respect to I. Then for any sentence A, $\mathfrak{A} \models A \iff \mathfrak{B} \models tr_M(A)[\![I]\!]$.

Proof. We will prove by induction that for all variables $\overline{x}$ and all $\overline{a} \in |\mathfrak{A}|$ and for all formulas A with free variables among $\overline{x}$, $\mathfrak{A} \models A[\![\overline{a}]\!] \iff \mathfrak{B} \models tr_M(A)[\![I, \overline{a}]\!]$

- $A = x_i \doteq x_j$

  Then $\mathfrak{A} \models x_i \doteq x_j[\![\overline{a}]\!] \iff a_i = a_j \iff \mathfrak{B} \models x_i \doteq x_j[\![I, \overline{a}]\!] \iff \mathfrak{B} \models tr_M(A)[\![I, \overline{a}]\!]$

- $A = x_i \leq x_j$

  Suppose first that $\mathfrak{A} \models x_i \leq x_j[\![\overline{a}]\!]$. Then $a_i \leq^{\mathfrak{A}} a_j$ and since $\mathfrak{B}$ is the linearization of $\mathfrak{A}$, this means that $a_i \leq^{\mathfrak{B}} a_j$ and there is no $k \in I$ such that $a_i <^{\mathfrak{B}} k <^{\mathfrak{B}} a_j$. Therefore $\mathfrak{B} \models x_i \leq x_j \wedge \neg\exists m(m \in M \wedge x_i < m < x_j)[\![I, \overline{a}]\!]$, i.e. $\mathfrak{B} \models tr_M(A)[\![I, \overline{a}]\!]$.

  Now suppose that $\mathfrak{B} \models tr_M(A)[\![I, \overline{a}]\!]$, i.e. $\mathfrak{B} \models x_i \leq x_j \wedge \neg\exists m(m \in M \wedge x_i < m < x_j)[\![I, \overline{a}]\!]$. Therefore $a_i \leq^{\mathfrak{B}} a_j$ and there is no $k \in I$ such that $a_i <^{\mathfrak{B}} k <^{\mathfrak{B}} a_j$. By the definition of $\leq^{\mathfrak{B}}$ this is only possible if $a_i \leq^{\mathfrak{A}} a_j$. So $\mathfrak{A} \models A[\![\overline{a}]\!]$.

- $A = \neg C$

  Then $\mathfrak{A} \models \neg C[\![\overline{a}]\!] \iff \mathfrak{A} \not\models C[\![\overline{a}]\!] \overset{(ih)}{\iff} \mathfrak{B} \not\models tr_M(C)[\![I, \overline{a}]\!] \iff \mathfrak{B} \models \neg tr_M(C)[\![I, \overline{a}]\!] \iff \mathfrak{B} \models tr_M(A)[\![I, \overline{a}]\!]$

- $A = C \wedge D$

  Then $\mathfrak{A} \models C \wedge D[\![\overline{a}]\!] \iff \mathfrak{A} \models C[\![\overline{a}]\!]$ and $\mathfrak{A} \models D[\![\overline{a}]\!] \overset{(ih)}{\iff} \mathfrak{B} \models tr_M(C)[\![I, \overline{a}]\!]$ and $\mathfrak{B} \models tr_M(D)[\![I, \overline{a}]\!] \iff \mathfrak{B} \models tr_M(C) \wedge tr_M(D)[\![I, \overline{a}]\!] \iff \mathfrak{B} \models tr_M(C \wedge D)[\![I, \overline{a}]\!] \iff \mathfrak{B} \models tr_M(A)[\![I, \overline{a}]\!]$

- $A = \exists x_i C$

  Then $\mathfrak{A} \models \exists x_i C[\![\overline{a}]\!] \iff$ there exists $a \in |\mathfrak{A}| : \mathfrak{A} \models C[\![\overline{a}, a]\!] \overset{(ih)}{\iff}$ there exists $a \in |\mathfrak{A}| : \mathfrak{B} \models C[\![I, \overline{a}, a]\!] \iff$ there exists $a \in |\mathfrak{B}| : \mathfrak{B} \models x_i \notin M \wedge C[\![I, \overline{a}, a]\!] \iff \mathfrak{B} \models \exists x_i((x_i \notin M) \wedge C)[\![I, \overline{a}]\!] \iff \mathfrak{B} \models tr_M(A)[\![I, \overline{a}]\!]$

□

Remark 19. *For every $\mathfrak{B} \in LIN^{countable}$ and every set $M \subset |\mathfrak{B}|$ with the properties:*

1. *M is of order type $\alpha \leq \omega$.*

2. *For every* $m \in M$ *there are* $x, y \in |\mathfrak{B}| \setminus M$ *such that* $x < m < y$.

3. *If* $m_1 \in M$ *and* $m_2 \in M$ *is the successor of* m_1 *then there is some* $x \in |\mathfrak{B}| \setminus M$ *such that* $m_1 \leq x \leq m_2$.

we can find a structure $\mathfrak{A} \in DLIN^{countable}$ *such that* $\mathfrak{B}$ *is isomorphic to the linearization of* $\mathfrak{A}$ *with respect to* M.

Let $M = \{m_0, m_1, \cdots\}$ *be an enumeration of* M.

If $M = \emptyset$ *take* $I = \{0\}, \mathfrak{A}_0 = \mathfrak{B}$.

If $M \neq \emptyset$ *and there is some* $x \in |\mathfrak{B}| \setminus M$ *which is an upper bound for* M *take:*

- $I = \alpha + 1$
- $\mathfrak{A}_0 = \{x \in |\mathfrak{B}| \mid (\forall m \in M)(x < m)\}$
- $\mathfrak{A}_{n+1} = \{x \in |\mathfrak{B}| \mid m_n < x < m_{n+1}\}$ *for* $n < \alpha$
- $\mathfrak{A}_\alpha = \{x \in |\mathfrak{B}| \mid (\forall m \in M)(x > m)\}$

If $M \neq \emptyset$ *and* M *is unbounded by* $\mathfrak{B} \setminus M$ *take:*

- $I = 1 + \alpha$
- $\mathfrak{A}_0 = \{x \in |\mathfrak{B}| \mid x < m_0\}$
- $\mathfrak{A}_{n+1} = \{x \in |\mathfrak{B}| \mid m_n < x < m_{n+1}\}$ *for* $n < \alpha$

Now if $\mathfrak{B}'$ *is the linearization of* $\mathfrak{A}$ *with respect to* M, *the function* $f : |\mathfrak{B}| \to |\mathfrak{B}'|$ *defined as follows is an isomorphism:*

- $f(a) = a$ *for* $a \in |\mathfrak{A}|$.
- $f(m_n) = n$ *for* $n < \alpha$

Corollary 2. *For any sentence* A, $DLIN^{countable} \models A \iff LIN^{countable} \models \forall M(B \to tr_M(A))$ *where* B *is the conjunction of the following sentences:*

- $B_1 = (\forall I \subseteq M)((\exists m \in M)(\forall x \in I)(x < m) \to fin(I))$
- $B_2 = (\forall m \in M)(\exists x \notin M)(\exists y \notin M)(x < m < y)$
- $B_3 = (\forall m_1 \in M)(\forall m_2 \in M)(succ_M(m_1, m_2) \to (\exists x \notin M)(m_1 < x < m_2))$

where we use the following abbreviations:

- $succ_M(m_1, m_2) = m_1 < m_2 \wedge (\forall m \in M)(m < m_1 \vee m_2 < m)$
- $fin(X) = \neg inf(X)$
- $inf(X) = (\exists Y \subseteq X)((\forall y \in Y)(\exists z \in Y)(y < z) \vee (\forall y \in Y)(\exists z \in Y)(y > z))$

Proof. First note that the sentences B_1, B_2 and B_3 correspond to the conditions we gave in order to be able to represent a model in $LIN^{countable}$ as an isomorphic copy of the linearization of a model in $DLIN^{countable}$.

Suppose that $DLIN^{countable} \not\models A$. Then there is some $\mathfrak{A} \in DLIN^{countable}$ such that $\mathfrak{A} \not\models A$. Let $\mathfrak{A} = \bigsqcup_{i<\alpha} \mathfrak{A}_i$, $\alpha \leq \omega$ and $\mathfrak{B} \in LIN^{countable}$ be the linearization of $\mathfrak{A}$ with respect to α. Then by Proposition 8 $\mathfrak{B} \not\models tr_M(A)[\![I]\!]$. But $\mathfrak{B} \models B[\![I]\!]$. Therefore $\mathfrak{B} \models \exists m(B \wedge \neg tr_M(A))$, i.e. $\mathfrak{B} \not\models \forall M(B \to tr_M(A))$.

Now suppose that $LIN^{countable} \not\models \forall M(B \to tr_M(A))$. Then there is some $\mathfrak{B} \in LIN^{countable}$ such that $\mathfrak{B} \not\models \forall M(B \to tr_M(A))$. Therefore there is $I \subseteq |\mathfrak{B}|$ such that $\mathfrak{B} \models B \wedge \neg tr_M(A)[\![I]\!]$. But since $\mathfrak{B} \models B[\![I]\!]$ and B forces that I fulfills all conditions of Remark 20, there is $\mathfrak{A} \in DLIN^{countable}$ such that $\mathfrak{B}$ is isomorphic to the linearization of $\mathfrak{A}$ with respect to I. Now since $\mathfrak{B} \not\models tr_M(A)[\![I]\!]$, we can conclude that $\mathfrak{A} \not\models A$. □

Corollary 3. *The theory $th(DLIN)$ is decidable.*

Proof. We established that $DLIN^{countable} \models A \iff LIN^{countable} \models \forall M(B \to tr_M(A))$. Since the theory $th^{II}(LIN^{countable})$ is decidable ([6]) and the translation tr_M is effective, this gives us a decision procedure for $th(DLIN^{countable}) = th(DLIN)$. □

Corollary 4. *It is decidable whether a sentence A has a model with an infinite chain in $DLIN$.*

Proof. Arguing as in the previous Corollary, we can prove that A has no countable model with an infinite chain precisely when $LIN^{countable} \models \forall M(B \wedge tr_M(A) \to C)$ where $C = \forall X(\neg(\exists m \in M)(\exists x_1 \in X)(\exists x_2 \in X)(x_1 < m < x_2) \to fin(X))$.

The formula C says that every subset of the linear order such that it is not partitioned by an element in the interpretation of M is finite. Since such segments correspond exactly to the chains in the delinearized model, the property is assured.

Now by the decidability of the second-order theory of $LIN^{countable}$, we can effectively check if A has a model with an infinite chain in $DLIN^{countable}$, but by Proposition 7, this happens precisely when A has a model with an infinite chain in $DLIN$. □

Corollary 5. *For any sentence A and any $0 < n < \omega$ it is decidable whether for every $\mathfrak{A} \in DLIN$ with chain sizes at most n, $\mathfrak{A} \models A$.*

Proof. The sentence A has the desired property precisely when $DLIN \models C \to A$, where

$$C = \forall x_1 \cdots \forall x_{n+1}(\bigwedge_{1 \le i < j \le n+1} (x_i \le x_j) \to \bigvee_{1 \le i < j \le n+1} (x_i \doteq x_j)).$$

The sentence C says that among any $n+1$ elements in the model which form a chain, at least two are equal. □

Proposition 9. *The problem $IntDef$ with respect to the class $DLIN$ is decidable.*

Proof. Suppose we are given a sentence A. The following procedure effectively determines whether A is definable:

1. If $DLIN \models A$, then $\top$ is a definition of A
2. If $DLIN \models \neg A$, then $\bot$ is a definition of A
3. If A and $\neg A$ are satisfiable in $DLIN$:
   (a) If A has a model in $DLIN$ with an infinite chain, then arguing as in the case of LIN, A is undefinable.
   (b) If all models of A contain only finite chains, then by a Compactness argument there exists $n < \omega$ such that all models of A have chains with at most n elements.

       If $LIN^{fin} \models \neg A$, then A is undefinable: A is satisfiable so it has at least one model $\mathfrak{F}$. Assuming that φ is a definition of A, $\mathfrak{F} \models \varphi$. But taking the generated subframe $\mathfrak{F}_x$ by any point $x \in |\mathfrak{F}|$ results in a linear order. Moreso, $\mathfrak{F}_x \models \varphi$ and by the assumption $\mathfrak{F}_x \models A$. But $LIN^{fin} \models \neg A$ - contradiction.

       If $LIN^{fin} \not\models \neg A$, search for the maximal $n \le 2^{qr(A)}$ such that $\mathfrak{F}_n \models A$. Such must exist by the characterization of $Mod_{LIN^{fin}}(A)$ established earlier.

       If $\mathfrak{A} \models A$ for every $\mathfrak{A} \in DLIN$ with chains with at most n elements, then $\varphi_{depth \le n}$ is a definition of A.

Otherwise, A is undefinable: assume that φ is a definition of A. Choose a model $\mathfrak{F} \in DLIN$ such that every chain contains at most n elements and $\mathfrak{F} \not\models A$. Then every rooted subframe of $\mathfrak{F}$ is isomorphic to $\mathfrak{F}_k$ for some $1 \leq k \leq n$. Since $\mathfrak{F}_n \models A$, $\mathfrak{F}_n \models \varphi$ and so $\mathfrak{F}_k \models A$ for any $1 \leq k \leq n$. Therefore $\mathfrak{F} \models \varphi$. But φ is a definition of A, so $\mathfrak{F} \models A$ - contradiction.

□

Remark 20. *The theory of the class $DLIN^{fin}$ of all finite disjoint unions of finite linear orders and the problem of definability with respect to it are decidable.*

Recall the notation $\mathfrak{F}_n$ for the linear order $\langle n, \leq_n \rangle$.

Using Ehrenfeucht-Fraisse games, we can see that if $n \geq 1$, then every model $\mathfrak{F} = \bigsqcup_{i \in I} \mathfrak{A}_i \in DLIN^{fin}$ is n-elementarily equivalent to a reduced model $\mathfrak{G} \in DLIN^{fin}$, obtained in the following way:

1. *Replace any $\mathfrak{A}_i$ with more than 2^n elements with a fresh isomorphic copy of $\mathfrak{F}_{2^n}$. Denote the resulting family $(\mathfrak{B}_i)_{i \in I}$. The strategy to show that the resulting disjoint union is n-elementarily equivalent to the original model is the classical strategy for finite linear orders.*

2. *Obtain a maximal $J \subseteq I$ with the property that for any $i \in I$, the model $\mathfrak{A}_i$ contains at most n isomorphic copies in the family $(\mathfrak{B}_j)_{j \in J}$. The strategy is straightforward: when Spoiler picks an element from any linear order that has not been used yet, Spoiler picks the corresponding element from one of its isomorphic copies in the other model.*

Denote the class of such reduced models $DLIN_n^{fin}$. This subclass is finite modulo isomorphism, therefore given a sentence A we need only consider the frames reduced by $n = qr(A)$ in order to determine whether A is valid in the class. Hence, the theory of the class is decidable.

A decision procedure for the problem $IntDef$ with respect to the class is similar to the one for the full class $DLIN$:

1. *If $DLIN^{fin} \models A$ or $DLIN^{fin} \models \neg A$, then respectively $\top$ or $\bot$ is a definition of A.*

2. *If A and $\neg A$ are satisfiable in the class, then:*

    (a) *Find the maximal size k of a chain in a model of A in $DLIN_n^{fin}$ where $n = qr(A)$*

(b) *If* $k = 2^n$, *then the sentence* A *is undefinable:*

Suppose that A *is definable and* φ *is a definition of* A. *Since there exists a model* $\mathfrak{A} \in DLIN_n^{fin}$, *such that* $\mathfrak{A} \models A$ *and the maximal size of a chain in* $\mathfrak{A}$ *is* 2^n, *then the finite linear order* $\mathfrak{F}_{2^n}$ *is a generated subframe of* $\mathfrak{A}$.

Since φ *is a definition of* A, *this means that* $\mathfrak{A} \models \varphi$, *hence* $\mathfrak{F}_{2^n} \models \varphi$. *Since* $\mathfrak{F}_t$ *is a generated subframe of* $\mathfrak{F}_{2^n}$ *for* $1 \leq t \leq 2^n$, $\mathfrak{F}_t \models \varphi$. *Therefore,* $\mathfrak{F}_t \models A$ *for any* $1 \leq t \leq 2^n$.

By our previous results, this means that $LIN^{fin} \models A$. *Therefore,* $LIN^{fin} \models \varphi$. *But then any rooted generated subframe of a frame* $\mathfrak{B} \in DLIN^{fin}$ *validates* φ, *thus* $DLIN^{fin} \models \varphi$. *But then this means that* $DLIN^{fin} \models A$ *- contradiction.*

(c) *If* $k < 2^n$, *this means that every* $\mathfrak{F} \in DLIN^{fin}$ *such that* $\mathfrak{F} \models A$ *is the disjoint union of families of linear orders with at most* k *elements. The rest is similar to the the case of the full* $DLIN$:

   i. *If for every model* $\mathfrak{A} \in DLIN_n^{fin}$ *such that every chain in* $\mathfrak{A}$ *contains at most* k *elements,* $\mathfrak{A} \models A$, *then* $\varphi_{depth \leq k}$ *is a definition of* A.

   ii. *Otherwise,* A *is undefinable.*

Chapter 4

Undecidable instances of definability

In this chapter we will survey a few classes of models with undecidable instances of the definability problem.

Our main tool will be a general method developed in [1] by Tinchev and Balbiani consisting of a reduction of the problem of deciding the validity of sentences with respect to the class to the problem of definability with respect to the class.

We will first present the method in the case of well known classes with undecidable theories and then consider some natural classes in the context of partial orders which happen to have undecidable theories.

The main tool to prove undecidability will be the deduction theorem and a variation of the method we used when considering disjoint unions of linear orders: we will embed models of a class known to have an undecidable theory into the class in consideration.

4.1 Outline of the method for reduction

First we will outline the general method developed by Tinchev and Balbiani in its full form.

Consider a sentence C and a class of models $\mathcal{K}$. The main idea is to conceive a translation of C into a sentence $tr(C)$ for the same language such that:

1. C is valid in $\mathcal{K}$ precisely when its translation $tr(C)$ is unsatisfiable in $\mathcal{K}$.

2. If $tr(C)$ is satisfiable in $\mathcal{K}$, then the sentence $tr(C)$ is forced to be undefinable with respect to $\mathcal{K}$.

If $tr(C)$ is unsatisfiable in $\mathcal{K}$, then $tr(C)$ is definable with respect to $\mathcal{K}$ with definition $\bot$. Then clearly the translation tr is a reduction of the problem of deciding the validity of sentences in the class to the problem of definability: C is valid in $\mathcal{K}$ precisely when $tr(C)$ is definable.

We will first explore the details of the construction in the class PO of all partial orders.

In order to force the translation tr to have the first property, we will exploit the notion of relativized reducts.

Recall that for models $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{B}$ is the relativized reduct of $\mathfrak{A}$ with respect to the formula A and parameters $\overline{a} \in |\mathfrak{A}|$ if $|\mathfrak{B}|$ is a submodel of $\mathfrak{A}$ and $|\mathfrak{B}| = \{a \in |\mathfrak{A}| \mid \mathfrak{A} \models A[\![a, \overline{a},]\!]\}$. In particular, $\mathfrak{A}$ possesses a relativized reduct with respect to A and $\overline{a}$ precisely when $\mathfrak{A} \models \exists y A[\![\overline{a}]\!]$. If this is the case, the relativized reduct is unique.

We will usually seek a formula A with $fv(A) = \{y, x_1, \cdots, x_n\}$. The Relativization theorem gives us a connection between truth in a model and truth in its reduct with respect to A and $\overline{a}$, namely if $\mathfrak{B}$ is the relativized reduct of $\mathfrak{A}$ with respect to A and $\overline{a}$, and B is a sentence, then $\mathfrak{B} \models B \iff \mathfrak{A} \models (B)^{A,y}[\![\overline{a}]\!]$.

Consider an arbitrary sentence D and the sentence $R_D = \exists x_1 \cdots \exists x_n(\exists y A \wedge (\neg D)^{A,y})$. What R_D expresses is the property that a model has a relativized reduct in which D is not true.

Consider the translation $tr(D) = R_D$. If every partial order $\mathfrak{B}$ is the relativized reduct of some partial order $\mathfrak{A}$ with respect to A and parameters $a_1, \cdots, a_n \in |\mathfrak{A}|$, then tr satisfies the first property:

- If $PO \models D$ and we assume that there is a model $\mathfrak{F} \in PO$ such that $\mathfrak{F} \models R_D$ then there is a relativized reduct of $\mathfrak{F}$ such that $\mathfrak{F} \models \neg D$. But the reduct is a partial order, therefore $PO \not\models D$ - contradiction.

- If $PO \models \neg R_D$ and we assume that there is a model $\mathfrak{B} \in PO$ such that $\mathfrak{B} \models \neg D$ then since there exists a partial order $\mathfrak{A}$ and parameters $a_1, \cdots, a_n \in |\mathfrak{A}|$ such that $\mathfrak{B}$ is the relativized reduct of $\mathfrak{A}$, then $\mathfrak{A} \models R_D$ - contradiction.

For now we have restricted ourselves to the class PO since this class has the useful property that it is closed with respect to submodels and therefore a relativized reduct of a partial order is guaranteed to be a partial order. When the class is not axiomatizable with universal formulas we will have to be more careful in the translation in order to force the reducts to belong to the class.

So far we have seen how we can achieve only the first property by carefully choosing a formula A - this does not yet grant us the desired reduction. The strategy for the second property will be to extend the translation, adding in a sentence which expresses a property that is intuitionistically undefinable and separates the models in the class. We will also need to strengthen the property we desire of A.

Consider the translation $tr(D) = \exists x_1 \cdots \exists x_n (\exists y A \wedge (\neg D)) \wedge B$, where the sentences A and B have the property that for every model $\mathfrak{F}$ there exist models $\mathfrak{F}_1$ and $\mathfrak{F}_2$, such that:

- $\mathfrak{F}_1 \models B$ and there exist parameters $b_1, \cdots, b_n \in |\mathfrak{F}_1|$ such that $\mathfrak{F}$ is the relativized reduct of $\mathfrak{F}_1$ with respect to A and $b_1, \cdots, b_n$.
- $\mathfrak{F}_2 \not\models B$ and $Log(\mathfrak{F}_1) \subseteq Log(\mathfrak{F}_2)$.

This forces that if $tr(D)$ is satisfiable, then arguing as before D must not be valid in the class. Therefore we can find a model $\mathfrak{F} \models \neg D$. Now take the models $\mathfrak{F}_1$ and $\mathfrak{F}_2$ as above. $\mathfrak{F}_1 \models tr(D)$ and assuming we can find a definition φ of $tr(D)$, the model $\mathfrak{F}_2$ would have to validate φ, hence validate $tr(D)$. But $\mathfrak{F}_2 \not\models B$ - contradiction.

Now we will produce such a reduction for the class PO.

Proposition 10. *Validity in the class PO reduces to definability with respect to PO.*

Proof. Consider the following formulas:

- $A = x < y$
- $B = \neg \exists x \forall y (x \leq y)$
- $tr(C) = \exists x (\exists y A \wedge (\neg C)^{A,y}) \wedge B$

The sentence B expresses the property that a model does not have a least element.

We will show that for any sentence C, $PO \models C \iff tr(C)$ is definable.

Suppose first that $PO \models C$ and assume that $PO \not\models \neg tr(C)$. Then there is a model $\mathfrak{F} \in PO$ such that $\mathfrak{F} \models tr(C)$. Therefore $\mathfrak{F} \models \exists x(\exists y A \wedge (\neg C)^{A,y})$.

Take $a \in \mathfrak{F}$ such that $\mathfrak{F} \models \exists y A \wedge (\neg C)^{A,y}[\![a]\!]$.

Since $\mathfrak{F} \models \exists y A[\![a]\!]$, there exists a relativized reduct $\mathfrak{G}$ of $\mathfrak{F}$ with respect to A and a. Since $\mathfrak{F} \models (\neg C)^{A,y}[\![a]\!]$, by the relativization theorem we have that $\mathfrak{G} \models \neg C$. But every submodel of a partial order is a partial order, therefore $\mathfrak{G} \in PO$ and $\mathfrak{G} \models \neg C$. But $PO \models C$ - contradiction.

Since $PO \models \neg tr(C)$, $\bot$ is a definition of $tr(C)$ and so $tr(C)$ is definable.

Now assume that $tr(C)$ is definable with definition φ, $PO \not\models C$ and take $\mathfrak{F} \in PO$ such that $\mathfrak{F} \models \neg C$.

Take two elements $a, b \notin |\mathfrak{F}|$ and consider the following models:

- $\mathfrak{F}_1$ with universe $|\mathfrak{F}| \cup \{a\}$ and $\leq^{\mathfrak{F}_1} = \leq^{\mathfrak{F}} \cup \{\langle a, c\rangle \mid c \in |\mathfrak{F}_1|\}$
- $\mathfrak{F}_2$ with universe $|\mathfrak{F}| \cup \{a, b\}$ and $\leq^{\mathfrak{F}_1} = \leq^{\mathfrak{F}_1} \cup \{\langle b, b\rangle\}$

We can readily see that $\mathfrak{F}_1 \not\models B$ since a is the least element of $\mathfrak{F}_1$ and therefore $\mathfrak{F}_1 \not\models tr(C)$. Since φ is a definition of $tr(C)$, $\mathfrak{F}_1 \not\models \varphi$. But $\mathfrak{F}_1$ is a generated subframe of $\mathfrak{F}_2$ and therefore $\mathfrak{F}_2 \not\models \varphi$, i.e. $\mathfrak{F}_2 \not\models tr(C)$.

Since $\mathfrak{F}_2 \models B$ (a and b are incomparable), we can conclude that $\mathfrak{F}_2 \not\models \exists x(\exists y A \wedge (\neg C)^{A,y})$. Now since $\mathfrak{F}$ is the relativized reduct of $\mathfrak{F}_2$ with respect to A and a and $\mathfrak{F} \models \neg C$ we have that $\mathfrak{F}_2 \models (\neg C)^{A,y}[\![a]\!]$. Since $\mathfrak{F}_2 \models \exists y A[\![a]\!]$ we can conclude that $\mathfrak{F}_2 \models \exists x(\exists y A \wedge (\neg C)^{A,y})$ - contradiction.

□

Corollary 6. *The problem $IntDef$ with respect to the class PO is undecidable.*

Proof. By [9], the theory of lattices is undecidable. Since the class of all latices is finitely axiomatizable, by the Deduction theorem a sentence A is valid in the class of lattices precisely when $PO \models Lattice \to A$, where $Lattice$ is the axiom for the class.

Therefore the theory $th(PO)$ is undecidable and by the above reduction, the problem or definability with respect to the class PO is undecidable. □

4.2 Stable classes

We will now consider the notion of stable classes, introduced in [1] by Tinchev and Balbiani in the case of definability with modal formulas. It abstracts away the piecing of the reduction, given suitable formulas A and B which possess the properties we sketched in the previous section.

Definition 42. *Consider a class $\mathcal{K}$ of models. We say that $\mathcal{K}$ is stable if there exist a sentence B and a formula A with $fv(A) = \{y, x_1, \cdots, x_n\}$, such that:*

- *If $\mathfrak{F} \in \mathcal{K}$ and $a_1, \cdots, a_n \in |\mathfrak{F}|$ and $\mathfrak{F}_1$ is the relativized reduct of $\mathfrak{F}$ with respect to A and $\overline{a}$, then $\mathfrak{F}_1 \in \mathcal{K}$.*
- *If $\mathfrak{F} \in \mathcal{K}$, then there exist models $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{K}$ such that $Log(\mathfrak{F}_2) \subseteq Log(\mathfrak{F}_1)$, $\mathfrak{F}_1 \not\models B$, $\mathfrak{F}_2 \models B$ and $\mathfrak{F}$ is the relativized reduct of $\mathfrak{F}_2$ with respect to A and some parameters $\overline{a}$ from $|\mathfrak{F}_2|$.*

Theorem 9. *If the class $\mathcal{K}$ is stable, then the problem of deciding the validity of sentences in $\mathcal{K}$ is reducible to the problem of definability with respect to $\mathcal{K}$.*

Proof. The proof of Theorem 1 in [1] for the modal case taken verbatim constitutes a proof for the intuitionistic case. □

4.3 Examples of stable classes

Here we will outline two examples: the classes of dense partial orders and of lattices. A technical detail which did not appear in the proof for the class of all partial orders but is necessary to accomodate to here, is that the formula A should force the relativized reducts we consider to be members of the respective class of models in consideration. Since the two classes are finitely axiomatizable extensions of PO, this can be easily achieved by just plugging in the relativized axiom.

Proposition 11. *Consider the class DPO of all dense partial orders(i.e. the partial orders $\mathfrak{F}$ such that $\mathfrak{F} \models Dense$, where $Dense = \forall x \forall y(x < y \rightarrow \exists z(x < z < y))$). The class DPO is stable.*

Proof. Consider the following formulas:

- $D = x_1 < y \wedge x_2 < y$
- $A = D \wedge (Dense)^{D,y}$

- $B = \neg\exists x \forall y (x \leq y)$

We will see that the formulas A and B are witnesses of the stability of DPO.

- Every reduct of a model $\mathfrak{F} \in DPO$ is in DPO:

Suppose that $\mathfrak{F}_1$ is the relativized reduct of $\mathfrak{F}$ with respect to A and a, b. Since $\mathfrak{F}_1$ is a submodel of $\mathfrak{F}$, $\mathfrak{F}_1$ is a partial order, hence we must only see to the density axiom.

Take an arbitrary element $c \in |\mathfrak{F}_1|$. Then $\mathfrak{F} \models A[\![c, a, b]\!]$. In particular, $\mathfrak{F} \models (Dense)^{D,y}[\![c, a, b]\!]$. Since $y \notin fv((Dense)^{D,y})$, then $\mathfrak{F} \models (Dense)^{D,y}[\![a, b]\!]$. Since $\mathfrak{F}_1$ is the relativized reduct of $\mathfrak{F}$ with respect to A and a, b, then by the relativization theorem, $\mathfrak{F}_1 \models Dense$.

- For every model $\mathfrak{F}$, there exist models $\mathfrak{F}_1$ and $\mathfrak{F}_2$ with the desired properties for stability:

Let $\mathfrak{F} \in DPO$ and without loss of generality assume that $|\mathfrak{F}| \cap (\mathbb{Q} \times \{0, 1\}) = \emptyset$. Consider the following models:

- $\mathfrak{F}_1$ with universe $|\mathfrak{F}| \cup (\mathbb{Q}^{\geq 0} \times \{0\})$ and $\leq^{\mathfrak{F}_1} = \leq^{\mathfrak{F}} \cup \{\langle\langle q_1, 0\rangle, \langle q_2, 0\rangle\rangle | q_1 \leq^{\mathbb{Q}} q_2\} \cup \{\langle p, c\rangle \mid c \in |\mathfrak{F}|, p \in (\mathbb{Q}^{\geq 0} \times \{0\})\}$
- $\mathfrak{F}_2$ with universe $|\mathfrak{F}| \cup (\mathbb{Q}^{\geq 0} \times \{0, 1\})$ and $\leq^{\mathfrak{F}_1} = \leq^{\mathfrak{F}} \cup \{\langle\langle q_1, i\rangle, \langle q_2, i\rangle\rangle | q_1 \leq^{\mathbb{Q}} q_2, i \in \{0, 1\}\} \cup \{\langle q, c\rangle \mid c \in |\mathfrak{F}|, q \in (\mathbb{Q}^{\geq 0} \times \{0, 1\})\}$

It is immediate that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are partial orders.

If $q_1, q_2 \in (\mathbb{Q}^{\geq 0} \times \{0, 1\})$ and $q_1 < q_2$, then $q_1, q_2 \in (\mathbb{Q}^{\geq 0} \times \{0\})$ or $q_1, q_2 \in (\mathbb{Q}^{\geq 0} \times \{1\})$. Since $\mathbb{Q}^{\geq 0}$ is dense, there is an element $q \in (\mathbb{Q}^{\geq 0} \times \{0, 1\})$ such that $q_1 < q < q_2$. If $\langle q, i\rangle < a$, then $\langle q, i\rangle < \langle q + 1, i\rangle < a$. Therefore $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are dense.

$\mathfrak{F}_1$ is a generated subframe of $\mathfrak{F}_2$, so $Log(\mathfrak{F}_2) \subseteq Log(\mathfrak{F}_1)$.

$\mathfrak{F}_1$ has a least element $\langle 0, 0\rangle$ and therefore $\mathfrak{F}_1 \not\models B$.

$\langle 0, 0\rangle$ and $\langle 0, 1\rangle$ are incomparable, therefore $\mathfrak{F}_2$ does not have a least element and hence $\mathfrak{F}_2 \models B$.

$\mathfrak{F}$ is the relativized reduct of $\mathfrak{F}_2$ with respect to A and $\langle 0, 0\rangle$, $\langle 0, 1\rangle$: for every element $a \in |\mathfrak{F}|$, $\langle 0, 0\rangle <^{\mathfrak{F}_2} a$ and $\langle 0, 1\rangle <^{\mathfrak{F}_2} a$. Any element $b \in (\mathbb{Q}^{\geq 0} \times \{i\})$ is incomparable with $\langle 0, 1 - i\rangle$.

□

Corollary 7. *The problem IntDef with respect to the class DPO is undecidable.*

Proof. As we will see later in Corollary 11, $th(DPO)$ is undecidable. By the stability of the class, this means that the problem of definability with respect to the class is undecidable. □

Proposition 12. *Consider the class LAT of all lattices.*
The class LAT is stable.

Proof. Consider the following formulas:

- $D = x < y$
- $atom(x) = \exists z(z < x \wedge \forall y(y < x \rightarrow y \doteq z))$ is a formula stating that the interpretation of x is an atom in the lattice
- $B = \exists x \exists y(\neg x \doteq y \wedge atom(x) \wedge atom(y))$ is a sentence saying that a lattice has at least two distinct atoms.
- *Lattice* is the axiom for lattices.
- $A = D \wedge (Lattice)^{D,y}$

We will see that the formulas A and B are witnesses of the stability of LAT.

- Every reduct of a model $\mathfrak{F} \in LAT$ is in LAT - similar to the proof for DPO.
- For every model $F \in LAT$, there exist models $\mathfrak{F}_1 \in LAT$ and $\mathfrak{F}_2 \in LAT$ with the desired properties for stability:

Take the elements $0, a_1, a_2, b \notin |\mathfrak{F}|$ and consider the following models:

- $\mathfrak{F}_1$ with universe $|\mathfrak{F}| \cup \{a_1, b\}$ and $\leq^{\mathfrak{F}_1} = \leq^{\mathfrak{F}} \cup \{\langle a_1, b\rangle, \langle a_1, a_1\rangle, \langle b, b\rangle\} \cup \{\langle p, c\rangle \mid c \in |\mathfrak{F}|, p \in \{a_1, b\}\}$
- $\mathfrak{F}_2$ with universe $|\mathfrak{F}| \cup \{0, a_1, a_2, b\}$ and $\leq^{\mathfrak{F}_2} = \leq^{\mathfrak{F}} \cup$ $\{\langle 0, a_1\rangle\ \langle 0, a_2\rangle, \langle a_1, b\rangle, \langle a_2, b\rangle\}^{\star} \cup \{\langle p, c\rangle \mid c \in |\mathfrak{F}|, p \in \{0, a_1, a_2, b\}\}$ where $X^{\star}$ is the transitive closure of X

We can immediately see that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are lattices.
$\mathfrak{F}_1$ is a generated subframe of $\mathfrak{F}_2$, so $Log(\mathfrak{F}_2) \subseteq Log(\mathfrak{F}_1)$
$\mathfrak{F}_1$ has one atom, namely b. Therefore $\mathfrak{F}_1 \not\models B$.
$\mathfrak{F}_2$ has two atoms, namely a_1, a_2. Therefore $\mathfrak{F}_2 \models B$.
$\mathfrak{F}$ is the relativized reduct of $\mathfrak{F}_2$ with respect to A and b.

□

Corollary 8. *The problem $IntDef$ with respect to the class LAT is undecidable.*

Proof. By [9], the theory of the class of all lattices is undecidable. Therefore by the stability of the class, the problem of definability with respect to the class LAT is undecidable. □

4.4 Some classes with undecidable theories

A natural class of partial orders to consider is the class of all partial orders with bounded depth, where depth is interpreted as the maximal size of a chain in the set.

Definition 43. *For every $1 \leq n < \omega$ denote with $PO_{depth \leq n}$ the class of all partial orders such that every chain contains no more than n elements.*

We will show that for $n \geq 2$, the theory of the class $PO_{depth \leq n}$ and the problem $IntDef$ with respect to the class are undecidable.

Proposition 13. *For every $1 \leq n < \omega$, the class $PO_{depth \leq n}$ is axiomatizable.*

The respective axiom is the conjunction of the axiom P for partial orders and the axiom D_n, where

$D_n = \forall x_1 \cdots \forall x_{n+1} (\bigwedge_{1 \leq i \leq n} (x_i \leq x_{i+1}) \rightarrow \bigvee_{i<j} (x_i \doteq x_j))$

Proof. Suppose $\mathfrak{F} \in PO_{depth \leq n}$. Then $\mathfrak{F}$ is a partial order and every chain contains at most n elements.

Let $a_1, \cdots, a_{n+1} \in |\mathfrak{F}|$ and suppose that $a_i \leq^{\mathfrak{F}} a_{i+1}$ for $1 \leq i \leq n$. Then the set $\{a_1, \cdots, a_{n+1}\}$ is a chain and therefore contains at most n elements. Therefore there are some indices $1 \leq i < j \leq n+1$ such that $a_i = a_j$.

In the other direction, suppose $\mathfrak{F} \models P \wedge D_n$. Then $\mathfrak{F}$ is a partial order. Let $C \subseteq |\mathfrak{F}|$ be a chain. Assume that C contains more than n elements and pick $a_1, \cdots, a_{n+1} \in C$ such that $a_i <^{\mathfrak{F}} a_{i+1}$ for $1 \leq i \leq n$. Then $\mathfrak{F} \models \bigwedge_{1 \leq i \leq n} (x_i \leq x_{i+1})[\![\overline{a}]\!]$ but $\mathfrak{F} \not\models \bigvee_{i<j} (x_i \doteq x_j)[\![\overline{a}]\!]$. Therefore $\mathfrak{F} \not\models D_n$ - contradiction.

□

Remark 21. *The theory of the class $PO_{depth \leq 1}$ is decidable. Indeed, the class is axiomatizable and contains a unique model of any cardinality. Therefore the theory is α-categorical for any cardinal α and therefore complete. Since the class is finitely axiomatizable, we can conclude that the theory is decidable.*

As for the problem $IntDef$ with respect to the class, if an intuitionistic formula φ is true in any frame in the class, it must be true in all frames in the class since all frames are disjoint unions of copies of the single-point frame. In fact, the logic of the class is exactly the Classical propositional logic. Therefore the only definable formulas are the valid formulas in the class. Since the theory of the class is decidable, we can conclude that the problem $IntDef$ with respect to $PO_{depth \leq 1}$ is decidable.

While bounded in depth, the models remain unbounded in width and when $n \geq 2$ they can take shapes of all kind. It turns out that this is enough for the theory to be complex enough to be undecidable. In order to prove it, we will consider the class of models of a symmetric and reflexive binary relation - such models we will call graphs. The theory of this class is undecidable as shown by Rogers in [7] and we will use this in order to prove that the theories of the classes of consideration are also undecidable.

Definition 44. *Let G be the class of all graphs, i.e. the class of all models for the language $\mathcal{L} = \{E\}$ satisfying the axiom $\forall x E(x,x) \wedge \forall x \forall y (E(x,y) \rightarrow E(y,x))$.*

We will show that any graph can be encoded in a model in the following subclass of $PO_{depth \leq n}$ for $n \geq 2$:

Definition 45. *The class CON is the class of all models $\mathfrak{F} \in PO_{depth \leq 2}$ such that each element has exactly 0 or exactly 2 elements strictly above it and for any two elements, if there exists a common element strictly below them, then it is unique.*

Proposition 14. *The class CON is axiomatized by the following set of axioms:*

- *The axiom $P_{depth \leq 2}$ for the class $PO_{depth \leq 2}$.*
- $P_{0,2} = \forall x(\neg \exists y (x < y) \vee \exists y_1 \exists y_2 (x < y_1 \wedge x < y_2 \wedge \neg y_1 \doteq y_2 \wedge \forall z (x < z \rightarrow z \doteq y_1 \vee z \doteq y_2)))$, *stating that strictly above each element there are exactly 0 or exactly 2 elements.*
- $L = \forall x \forall y \forall z_1 \forall z_2 (\neg x \doteq y \wedge z_1 < x \wedge z_1 < y \wedge z_2 < x \wedge z_2 < y \rightarrow z_1 \doteq z_2)$ *stating that existence of a common element strictly below two elements implies its uniqueness.*

The following formulas will be useful to us:

- $vertex(x) = \neg \exists y (x < y)$, *stating that an individual has no elements above it.*
- $edge(x) = \exists y (x < y)$, *stating that an individual has an element above it. With the other properties the class posseses, this means that there are exactly 2 elements above it.*
- $connected(x,y) = \exists z (z \leq x \wedge z \leq y)$, *stating that there exists a common element below the interpretations of x and y.*

Definition 46. *Consider a graph $\mathfrak{G} \in G$, which without loss of generality is such that $|\mathfrak{G}| \cap \{\{x, y\} \mid x, y \in |\mathfrak{G}|\} = \emptyset$. The connectivity map of $\mathfrak{G}$ is the model $\mathfrak{C} \in CON$ defined as follows:*

- $|\mathfrak{C}| = |\mathfrak{G}| \cup \{\{x, y\} \mid \langle x, y\rangle \in E^{\mathfrak{G}}\}$
- $\leq^{\mathfrak{C}} = \{\langle\{x, y\}, x\rangle \mid x \neq y, \langle x, y\rangle \in E^{\mathfrak{C}}\} \cup \{\langle x, x\rangle \ \mid x \in |\mathfrak{C}|\}$

An easy way to visualize the connectivity map is to imagine that the graph is made up of points on a grid and the edges are represented by some loose rope connecting points. If to every edge is attached a weight, it will sink below the points the rope connects. The resulting diagram of a partial order, in which the individuals are the original points in the grid and the weights, is the connectivity map we have now defined.

Remark 22. *For every $\mathfrak{C} \in CON$ we can find a model $\mathfrak{G} \in G$ such that $\mathfrak{C}$ is isomorphic to the connectivity map of $\mathfrak{G}$.*

Indeed, for arbitrary $\mathfrak{C} \in CON$, consider the model $\mathfrak{G}$, defined as follows:

- $|\mathfrak{G}| = \{a \in |\mathfrak{C}| \mid \mathfrak{C} \models vertex(x)[\![a]\!]\}$
- $E^{\mathfrak{G}} = \{\langle a, b\rangle \mid \mathfrak{C} \models connected(x, y)[\![a, b]\!]\}$

Now if $\mathfrak{C}'$ is the connectivity map of $\mathfrak{G}$, then the function $f : |\mathfrak{C}| \to |\mathfrak{C}'|$ is clearly an isomorphism:

- *$f(a) = a$, if $a \in |\mathfrak{G}|$*
- *$f(a) = \{b, c\}$, if $\mathfrak{C} \models edge(x)[\![a]\!]$ and b and c are the two elements above a in $\mathfrak{C}$.*

Proposition 15. *Consider the following translation from $\mathcal{L}_1 = \{E\}$ to $\mathcal{L}_2 = \{\leq\}$:*

- $tr(x \doteq y) = x \doteq y$
- $tr(E(x, y)) = connected(x, y)$
- $tr(\neg A) = \neg tr(A)$
- $tr(A \wedge B) = tr(A) \wedge tr(B)$
- $tr(\exists x A) = \exists x(vertex(x) \wedge tr(A))$

Let $\mathfrak{G} \in G$ and $\mathfrak{C} \in CON$ be the connectivity map of $\mathfrak{G}$. Then for all sentences A, $\mathfrak{G} \models A \iff \mathfrak{C} \models tr(A)$.

Proof. We will prove by induction that for all variables $\overline{x}$ and for all parameters $\overline{a} \in |\mathfrak{G}|$ and for all formulas A with free variables among $\overline{x}$, $\mathfrak{G} \models A[\![\overline{a}]\!] \iff \mathfrak{C} \models tr(A)[\![\overline{a}]\!]$:

- $A = x_i \doteq x_j$

  Then $\mathfrak{G} \models x_i \doteq x_j[\![\overline{a}]\!] \iff a_i = a_j \iff \mathfrak{C} \models x_i \doteq x_j[\![\overline{a}]\!] \iff \mathfrak{C} \models tr(A)[\![\overline{a}]\!]$

- $A = E(x, y)$

  Suppose first that $\mathfrak{G} \models A[\![\overline{a}]\!]$, i.e. $\mathfrak{G} \models E(x_i, x_j)[\![\overline{a}]\!]$. First, if $a_i = a_j$, then $a_i \leq^{\mathfrak{C}} a_j$ and $a_i \leq^{\mathfrak{C}} a_j$, therefore $\mathfrak{C} \models \mathit{connected}(x_i, x_j)[\![\overline{a}]\!]$, i.e. $\mathfrak{C} \models tr(A)[\![\overline{a}]\!]$. Now, if $a_i \neq a_j$, and $\langle a_i, a_j\rangle \in E^{\mathfrak{G}}$, by the definition of $\leq^{\mathfrak{C}}$ this means that $\{a_i, a_j\} \leq^{\mathfrak{C}} a_i$ and $\{a_i, a_j\} \leq^{\mathfrak{C}} a_j$. Therefore $\mathfrak{C} \models z \leq x_i \wedge z \leq x_j[\![\{a_i, a_j\}, \overline{a}]\!]$ and so $\mathfrak{C} \models \exists z(z \leq x_i \wedge z \leq x_j)[\![\overline{a}]\!]$. In conclusion, $\mathfrak{C} \models \mathit{connected}(x_i, x_j)[\![\overline{a}]\!]$, i.e. $\mathfrak{C} \models tr(A)[\![\overline{a}]\!]$.

  Now suppose that $\mathfrak{C} \models tr(A)[\![\overline{a}]\!]$, i.e. $\mathfrak{C} \models \mathit{connected}(x_i, x_j)[\![\overline{a}]\!]$, i.e. $\mathfrak{C} \models \exists z(z \leq x_i \wedge z \leq x_j)[\![\overline{a}]\!]$. Then if $a_i = a_j$, $\langle a_i, a_j \in E^{\mathfrak{G}}$ and so $\mathfrak{G} \models E(x_i, x_j)[\![\overline{a}]\!]$, i.e. $\mathfrak{G} \models A[\![\overline{a}]\!]$. Otherwise, there is some $a \in |\mathfrak{C}|$ such that $\mathfrak{C} \models z \leq x_i \wedge z \leq x_j[\![a, \overline{a}]\!]$, i.e. $a \leq^{\mathfrak{C}} a_i$ and $a \leq^{\mathfrak{C}} a_j$. By the definition of $\leq^{\mathfrak{C}}$ this is only possible when $a = \{a_i, a_j\}$. But since $a \in |\mathfrak{C}|$, this can only mean that $\langle a_i, a_j\rangle \in E^{\mathfrak{G}}$. Therefore $\mathfrak{G} \models E(x_i, x_j)[\![\overline{a}]\!]$, i.e. $\mathfrak{G} \models A[\![\overline{a}]\!]$.

- $A = \neg B$

  Then $\mathfrak{G} \models A[\![\overline{a}]\!] \iff \mathfrak{G} \models \neg B[\![\overline{a}]\!] \iff \mathfrak{G} \not\models B[\![\overline{a}]\!] \overset{ih}{\iff} \mathfrak{C} \not\models tr(B)[\![\overline{a}]\!] \iff \mathfrak{C} \models \neg tr(B)[\![\overline{a}]\!] \iff \mathfrak{C} \models tr(A)[\![\overline{a}]\!]$

- $A = B \wedge C$

  Then $\mathfrak{G} \models A[\![\overline{a}]\!] \iff \mathfrak{G} \models B \wedge C[\![\overline{a}]\!] \iff \mathfrak{G} \models B[\![\overline{a}]\!]$ and $\mathfrak{G} \models C[\![\overline{a}]\!] \overset{ih}{\iff} \mathfrak{C} \models tr(B)[\![\overline{a}]\!]$ and $\mathfrak{C} \models tr(C)[\![\overline{a}]\!] \iff \mathfrak{C} \models tr(B) \wedge tr(C)[\![\overline{a}]\!] \iff \mathfrak{C} \models tr(A)[\![\overline{a}]\!]$

- $A = \exists x B$

  First suppose that $\mathfrak{G} \models A[\![\overline{a}]\!]$, i.e. $\mathfrak{G} \models \exists x B$. Then there is some $a \in |\mathfrak{G}|$ such that $\mathfrak{G} \models B[\![a, \overline{a}]\!]$. By the induction hypothesis, $\mathfrak{C} \models tr(B)[\![a, \overline{a}]\!]$. But by the definition of $\leq^{\mathfrak{C}}$ we have $\mathfrak{C} \models \mathit{vertex}(x)[\![a]\!]$, since $a \in |\mathfrak{G}|$. Therefore $\mathfrak{C} \models vertex(x) \wedge tr(B)[\![a, \overline{a}]\!]$. In conclusion, $\mathfrak{C} \models \exists x(vertex(x) \wedge tr(B))[\![\overline{a}]\!]$, i.e. $\mathfrak{C} \models tr(A)[\![\overline{a}]\!]$.

Now suppose that $\mathfrak{C} \models tr(A)[\![\overline{a}]\!]$, i.e. $\mathfrak{C} \models \exists x(vertex(x) \wedge tr(B))[\![\overline{a}]\!]$. Then there is some $a \in |\mathfrak{C}|$ such that $\mathfrak{C} \models vertex(x) \wedge tr(B)[\![a, \overline{a}]\!]$. Since $\mathfrak{C} \models vertex(x)[\![a]\!]$, we must have that $a \in |\mathfrak{G}|$ by the definition of $\leq^{\mathfrak{G}}$. Now $\mathfrak{C} \models tr(B)[\![a, \overline{a}]\!]$ and by the induction hypothesis $\mathfrak{G} \models B[\![a, \overline{a}]\!]$. In conclusion, $\mathfrak{G} \models \exists x B[\![\overline{a}]\!]$, i.e. $\mathfrak{G} \models A[\![\overline{a}]\!]$.

□

Corollary 9. *The theory $th(CON)$ is undecidable.*

Proof. We will show that the translation tr as defined above is a reduction between the problems of deciding the validity of sentences in the theories of the classes G and CON, i.e. $G \models A \iff CON \models tr(A)$:

Suppose first that $G \not\models A$. Then there is a model $\mathfrak{G} \in G$ such that $\mathfrak{G} \not\models A$. Consider the connectivity map $\mathfrak{C} \in CON$ of $\mathfrak{G}$. Then by Proposition 15, $\mathfrak{C} \not\models tr(A)$. Therefore $CON \not\models tr(A)$.

Now suppose that $CON \not\models tr(A)$. Then there is a model $\mathfrak{C} \in CON$ such that $\mathfrak{C} \not\models tr(A)$. By our above remark, $\mathfrak{C}$ is isomorphic to the connectivity map of some model $\mathfrak{G} \in G$. Therefore by Proposition 15, $\mathfrak{G} \not\models A$ and hence $G \not\models A$.

Thus we have shown that tr is a reduction and since $th(G)$ is undecidable by [7], the theory $th(CON)$ must also be undecidable. □

Corollary 10. *The theory $th(PO_{depth \leq n})$ is undecidable for every $2 \leq n < \omega$.*

Proof. Since the class CON is a subclass of $PO_{depth \leq n}$ and is finitely axiomatizable, a reduction of the theory $th(CON)$ to the theory $th(PO_{depth \leq n})$ is given by the deduction theorem:

$CON \models A \iff PO_{depth \leq n} \models C \rightarrow A$,

where C is the axiom for CON.

□

We will consider a variant of the class CON, consisting of dense partial orders.

The models in this class are obtained by replacing every non-reflexive arrow in a CON model by a dense linear order with first and last elements the original ends of the arrow.

Definition 47. *The class $DCON$ is the class of all models of the following axioms:*

- $\forall x(mid(x) \to \exists! y(min(y) \wedge y \leq x))$
- $\forall x(mid(x) \to \exists! y(max(y) \wedge x \leq y))$
- $\forall x(min(x) \wedge \neg max(x) \to \exists! y_1 \exists! y_2(max(y_1) \wedge max(y_2) \wedge x < y_1 \wedge x < y_2 \wedge \neg y_1 \doteq y_2))$
- $\forall x_1 \forall x_2 \forall y_1 \forall y_2(\neg y_1 \doteq y_2 \wedge min(x_1) \wedge min(x_2) \wedge max(y_1) \wedge max(y_2) \wedge x_1 \leq y_1 \wedge x_2 \leq y_1 \wedge x_1 \leq y_2 \wedge x_2 \leq y_2 \to x_1 \doteq x_2)$
- $Dense = \forall x \forall y(x < y \to \exists z(x < z < y))$

where we use the following abbreviations:

- $min(x) = \neg \exists y(y < x)$, *i.e. the interpretation of* x *is a minimal element*
- $max(x) = \neg \exists y(x < y)$, *i.e. the interpretation of* x *is a maximal element*
- $mid(x) = \neg min(x) \wedge \neg max(x)$

Remark 23. *If* $\mathfrak{C} \in CON$*, the model* $\mathfrak{D} \in DCON$*, defined as follows is the densification of* $\mathfrak{C}$*:*

- $|\mathfrak{D}| = |\mathfrak{C}| \cup (\mathbb{Q} \times \{\{a, b\} \mid \mathfrak{C} \models min(x) \wedge max(y) \wedge x < y[\![a, b]\!]\})$
- $\leq^{\mathfrak{D}} = \leq^{C} \cup \{\langle a, \langle q, \{a, b\}\rangle\rangle, \langle\langle q, \{a, b\}\rangle, b\rangle \mid q \in \mathbb{Q}, a <^{\mathfrak{C}} b\}$

Conversely, if $\mathfrak{D} \in DCON$*,* $\mathfrak{D}$ *is isomorphic to the densification of the following model* $\mathfrak{C} \in CON$*:*

- $|\mathfrak{C}| = \{a \in |\mathfrak{D}| \mid \mathfrak{D} \models min(x) \vee max(x)[\![a]\!]\}$
- $\leq^{\mathfrak{C}} = \leq^{\mathfrak{D}} \cap (|\mathfrak{C}| \times |\mathfrak{C}|)$

Observe that the model $\mathfrak{C}$ *is the relativized reduct of* $\mathfrak{D}$ *with respect to the formula* $min(y) \vee max(y)$*.*

Proposition 16. *The theory of the class DCON is undecidable.*

Proof. Consider the formula $A = min(y) \vee max(y)$. We will show that the translation $tr(B) = (B)^{A,y}$ is a reduction of the problem of validity in CON to the problem of validity in $DCON$.

First, suppose that $CON \not\models B$. Then there exists a model $\mathfrak{C} \in CON$ such that $\mathfrak{C} \not\models B$. Take the densification $\mathfrak{D} \in DCON$ of $\mathfrak{C}$. Then $\mathfrak{C}$ is the relativized reduct of D with respect to the formula A (with no parameters since A has only one free variable y). Since $\mathfrak{C} \not\models B$, by the relativization theorem we have that $\mathfrak{D} \not\models (B)^{A,y}$, i.e. $\mathfrak{D} \not\models tr(B)$. Therefore, $DCON \not\models tr(B)$.

Now, suppose that $DCON \not\models tr(B)$ and take a model $\mathfrak{D} \in DCON$ such that $\mathfrak{D} \not\models tr(B)$. By the axiomatization of $DCON$, there exists at least one minimal element, therefore $\mathfrak{D} \models \exists y A$ and a relativized reduct of $\mathfrak{D}$ with respect to A exists. Taking this reduct produces a model $\mathfrak{C} \in CON$ and since $tr(B) = (B)^{A,y}$, $\mathfrak{D} \not\models (B)^{A,y}$, therefore by the relativization theorem $\mathfrak{C} \not\models B$ and thus $CON \not\models B$.

Since the theory of the class CON is undecidable and tr is a reduction, the theory of the class $DCON$ is also undecidable. □

Corollary 11. *The theory of the class DPO of all dense partial orders is undecidable.*

Proof. Since $DCON \subseteq DPO$ and $DCON$ is finitely axiomatizable, for every sentence A it is true that $DCON \models A \iff DPO \models C \to A$ where C is the axiom for $DCON$. Therefore $th(DPO)$ is undecidable. □

Definition 48. *Denote with $PO_{isuccessors \leq n}$ for each $n < \omega$ the class of all partial orders such that every element has at most n immediate successors, i.e. $PO_{isuccessors \leq n}$ is the class of all models of the following axioms:*

- *The axiom P for partial orders.*
- $S_n = \forall x \forall y_1 \cdots \forall y_{n+1} (\bigwedge_{1 \leq i \leq n+1} (succ(x, y_i)) \to \bigvee_{1 \leq i < j \leq n+1} (y_i \doteq y_j))$

  where $succ(x, y) = x < y \wedge \neg \exists z (x < z < y)$.

Proposition 17. *For every $0 \leq n < \omega$, the theory $PO_{isuccessors \leq n}$ is undecidable.*

Proof. We can readily see that the class $PO_{isuccessors \leq 0}$ is in fact the class DPO of all dense partial orders:

On one side, for each model $\mathfrak{F} \in DPO$ and each element $a \in |\mathfrak{F}|$, a cannot have any immediate successors, otherwise it would not fulfill the density condition. Therefore $\mathfrak{F} \in PO_{isuccessors \leq 0}$

Now, if $\mathfrak{F} \in PO_{isuccessors \leq 0}$, then consider any two elements $a, b \in |\mathfrak{F}|$ such that $a <^{\mathfrak{F}} b$. Then since a has no successors, b is not a successor of a and therefore there is an element $c \in |\mathfrak{F}|$ such that $a <^{\mathfrak{F}} c <^{\mathfrak{F}} b$. Therefore $\mathfrak{F}$ is a dense partial order.

Therefore $PO_{isuccessors \leq 0} = DPO$ and therefore has undecidable theory.

Now by an argument similar as above, since $PO_{isuccessors\leq 0}$ is finitely axiomatized and a subclass of $PO_{successors\leq n}$ for every $n < \omega$, the theory of the class $PO_{isuccessors\leq n}$ is undecidable for every $n < \omega$.

□

4.5 Stability of the considered classes

We will show that all the classes considered in the previous section are stable.

Proposition 18. *The classes*

- $PO_{depth \leq n}$ *for* $2 \leq n < \omega$
- CON
- $PO_{isuccessors \leq n}$ *for* $n < \omega$

are stable.

Proof. Let $\mathcal{K}$ be any of the listed classes.

Consider the following formulas:

- Ax is the axiom for $\mathcal{K}$.
- $isolated(x) = \forall y(x \leq y \vee y \leq x \rightarrow x \doteq y)$, stating that the interprata-tion of x is incomparable with any other point.
- $D = \neg x_1 \doteq y \wedge \neg x_2 \doteq y$
- $A = D \wedge (Ax)^{D,y}$
- $B = \exists x \exists y(\neg x \doteq y \wedge isolated(x) \wedge isolated(y))$, stating that a model contains at least two distinct points, incomparable with any other.

We will show that the formulas A and B are witnesses of the stability of $\mathcal{K}$:

Clearly, any relativized reduct of a model $\mathfrak{F} \in \mathcal{K}$ with respect to A and parameter $a \in |\mathfrak{F}|$ is in $\mathcal{K}$.

Let $\mathfrak{F} \in \mathcal{K}$ and take the elements $a, b \notin |\mathfrak{F}|$. We will show that there exist models $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{K}$ with the desired properties for stability:

- $\mathfrak{F}_1$ with universe $\{a\}$ and $\leq^{\mathfrak{F}_1} = \{\langle a, a\rangle\}$.
- $\mathfrak{F}_2$ with universe $|\mathfrak{F}| \cup \{a, b\}$ and $\leq^{\mathfrak{F}_2} = \leq^{\mathfrak{F}} \cup \{\langle a, a\rangle, \langle b, b\rangle\}$.

We can readily see that if $\mathcal{K}$ is any of the considered classes, $\mathfrak{F}_1 \in \mathcal{K}$ and $\mathfrak{F}_2 \in \mathcal{K}$.

$\mathfrak{F}_1$ is a generated subframe of $\mathfrak{F}_2$, therefore $Log(\mathfrak{F}_2) \subseteq Log(\mathfrak{F}_1)$.

$\mathfrak{F}_1 \not\models B$, since $|\mathfrak{F}_2|$ contains a single point.

$\mathfrak{F}_2 \models B$, since a and b are isolated.

$\mathfrak{F}$ is the relativized reduct of $\mathfrak{F}_2$ with respect to A and a, b. □

Corollary 12. *The problem* $IntDef$ *with respect to any of the considered in this section classes is undecidable.*

4.6 Finite restrictions of the classes

For a class of models $\mathcal{K}$ denote with $\mathcal{K}^{fin}$ the class of all finite models in $\mathcal{K}$. We will see that most of the results in the previous two sections hold for the finite restrictions of the classes.

Proposition 19. *The theory of the class CON^{fin} is undecidable (not even semidecidable).*

Proof. Given a finite graph $\mathfrak{G} \in G^{fin}$, the connectivity map $\mathfrak{C}$ of $\mathfrak{G}$ is also finite. Therefore, the same reduction as in the the case of the full classes works in the finite case without modification, i.e. for every sentence A, $G^{fin} \models A \iff CON^{fin} \models tr(A)$.

By [5], $th(G)$ and $FOR(E) \setminus th(G^{fin})$ are recursively inseparable, i.e. they are disjoint and there exists no decidable set C such that $C \cap th(\mathfrak{G}) = \emptyset$ and $(SENT(E) \setminus th(G^{fin})) \subseteq C$. In particular, this means that the set $SENT(E) \setminus th(G^{fin})$ is not decidable.

Since the language $\mathcal{L} = \{E\}$ is finite and all models in G^{fin} are finite, an exhaustive search for countermodels semidecides the set $SENT(E) \setminus th(G^{fin})$. By Post's theorem, since this set is semidecidable and is not decidable, its complement is not semidecidable, i.e. $th(G^{fin})$ is not semidecidable.

Since tr reduces the problem of validity in G^{fin} to the problem of validity in CON^{fin}, the theory $th(CON^{fin})$ is not semidecidable. □

Corollary 13. *The theory of the class $PO^{fin}_{depth \le n}$ is not semidecidable for every $2 \le n < \omega$.*

Proof. Since $CON^{fin} \subseteq PO^{fin}_{depth \le n}$ for $n \ge 2$, the deduction theorem gives us the reduction $CON^{fin} \models A \iff PO^{fin}_{depth \le n} \models C \to A$ where C is the axiom for CON. □

Corollary 14. *The theory of the class $PO^{fin}_{isuccessors \le n}$ is not semidecidable for $2 \le n < \omega$.*

Proof. Since the class $CON^{fin} \subseteq PO^{fin}_{isuccessors \le n}$ for $2 \le n < \omega$, again the deduction theorem yields a reduction. □

Proposition 20. *The classes*

- CON^{fin}
- $PO^{fin}_{depth \le n}$ *for* $n \ge 2$
- $PO^{fin}_{isuccessors \le n}$ *for* $n \ge 2$

are stable.

Proof. In the proof of stability of the full classes, if the model $\mathfrak{F}$ is finite, then so are the models $\mathfrak{F}_1$ and $\mathfrak{F}_2$ we constructed. Therefore, the same formulas A and B as in the previous section are witnesses of the stability of the classes. $\square$

Corollary 15. *The problem $IntDef$ with respect to any of the above classes of finite models is not semidecidable.*

Remark 24. *The theory of the class $PO^{fin}_{isuccessors\leq 0}$ is decidable.*

The models in the class are finite disjoint unions of copies of the single-point frame. Using Ehrenfeucht-Fraisse games, we can directly see that if $\mathfrak{A}, \mathfrak{B} \in PO^{fin}_{isuccessors\leq 0}$ with respectively $k_1 \geq n$ and $k_2 \geq n$ elements, then $\mathfrak{A} \equiv_n \mathfrak{B}$. Therefore, given a sentence A with $qr(A) = n$, it suffices to check if A is valid in all models with at most n elements, which modulo isomorphism are finitely many.

Now arguing as in the the remark for $PO_{depth\leq 1}$, a sentence A is definable precisely when $PO^{fin}_{isuccessors\leq 0} \models A$.

Chapter 5

Conclusion

In the present work we have examined the algorithmic problem of definability of first-order sentences with intuitionistic formulas with respect to several classes of models.

We have seen that the following classes have decidable theories and decidable instances of the definability problem:

- LIN^{fin}, the class of all finite linear orders
- LIN, the class of all linear orders
- $DLIN$, the class of all disjoint unions of linear orders
- $DLIN^{fin}$
- $PO_{depth \leq 1}$, the class of all disjoint unions of single-point frames
- $PO^{fin}_{depth \leq 1}$

The following classes have undecidable theories and undecidable instances of the definability problem (not even semidecidable in the case of the classes of finite models):

- CON, the class of all connectivity maps
- $PO_{depth \leq n}$ for $2 \leq n < \omega$, the classes of partial orders bounded in chain size
- $PO^{fin}_{depth \leq n}$ for $2 \leq n < \omega$
- $PO_{isuccessors \leq n}$ for $n < \omega$, the classes of partial orders bounded in the number of immediate successors

- $PO^{fin}_{isuccessors \leq n}$ for $2 \leq n < \omega$

Another natural class to consider is the class $PO_{width \leq n}$ of all partial orders with bounded size of antichains. In a certain sense, the class is dual to the class $PO_{depth \leq n}$, but the transitivity axiom plays a much more significant role. The author could not determine whether the theory of the class is decidable or not, but is slightly more inclined to believe that it is decidable. The models essentially consist of a finite number of chains with arrows inbetween them and transitivity tames the possible configurations of arrows, but we were unable to determine whether this makes them tame enough for the theory to be decidable.

Bibliography

[1] Philippe Balbiani and Tinko Tinchev. "Undecidable problems for modal definability". In: *Journal of Logic and Computation* 27.3 (2017), pp. 901–920.

[2] A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford Logic Guides. Clarendon Press, 1997.

[3] Chen Chung Chang and H. Jerome Keisler. *Model theory. [By] C. C. Chang and H. J. Keisler*. North-Holland Pub. Co.; American Elsevier Amsterdam, New York, 1973, xii, 550 p.

[4] Wilfrid Hodges. *Model Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1993.

[5] I. Lavrov. "The effective non-separability of the set of identically true formulae and the set of finitely refutable formulae for certain elementary theories". In: *Algebra i Logika. Sem* 2.1 (1963), pp. 5–18.

[6] Michael O. Rabin. "Decidability of Second-Order Theories and Automata on Infinite Trees". In: *Transactions of the American Mathematical Society* 141 (1969), pp. 1–35.

[7] H. Rogers. "Certain Logical Reduction and Decision Problems". In: *Annals of Mathematics, Second Series* 64.2 (1956), pp. 264–284.

[8] Saharon Shelah. "The Monadic Theory of Order". In: *Annals of Mathematics, Second Series* 102.3 (1975), pp. 379–419.

[9] A. Tarski. "Undecidability of the theories of lattices and projective geometry". In: *The Journal of Symbolic Logic* 14.1 (1949), pp. 77–78.