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Subrecursive Computability in Analysis

Master Thesis

by

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Contents

Chapter 1

Introduction.

Throughout the thesis the set of non-negative integers is denoted with $\mathbb{N}$, the set of rational numbers is denoted with $\mathbb{Q}$ and the set of real numbers is denoted with $\mathbb{R}$.

A real number α is *computable*, if there exists an effective method, which generates arbitrarily close rational approximations to α.
A generalization of this concept is derived when we relativize the requirement for the existence of "effective method".

Following [6], let $\mathcal{F}$ be a class of total functions in $\mathbb{N}$.

A real number α is called $\mathcal{F}$-*computable*, if there exist three one-argument functions $f, g, h \in \mathcal{F}$, such that

$$\left| \frac{f(n) - g(n)}{h(n) + 1} - \alpha \right| \leq \frac{1}{n+1}$$

for all $n \in \mathbb{N}$.

When $\mathcal{F}$ is the class of all recursive functions we obtain the extensively studied notion of *recursive* real number, which constitutes (according to Church-Turing thesis) a mathematical counterpart of the notion of computable real number. Of course, when $\mathcal{F}$ is the class of all total functions in $\mathbb{N}$, we obtain the whole set of real numbers.

In our considerations, the role of $\mathcal{F}$ will be played by the subrecursive classes $\mathcal{E}^2$, $\mathcal{L}^2$ and $\mathcal{M}^2$, which are proper subclasses of the class of primitive recursive functions. The role of α will be taken by famous constants, arisen from the study of various branches of mathematics. The main goal is to show that many of these constants possess the property of being $\mathcal{F}$-computable (for $\mathcal{F}$ – one of the three classes). Such a property is evidence for their low computing complexity.

Chapter 2

General definitions and properties. Basic methods.

2.1. The classes $\mathcal{E}^2$, $\mathcal{L}^2$ and $\mathcal{M}^2$.

We say that a class $\mathcal{F}$ of total functions in the natural numbers is closed under bounded primitive recursion, if whenever the functions $g : \mathbb{N}^n \to \mathbb{N}$, $h : \mathbb{N}^{n+2} \to \mathbb{N}$, $b : \mathbb{N}^{n+1} \to \mathbb{N}$ ($n \in \mathbb{N}$) belong to the class $\mathcal{F}$, the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$, satisfying the conditions
$f(x_1, \ldots, x_n, 0) = g(x_1, \ldots, x_n)$,
$f(x_1, \ldots, x_n, y+1) = h(x_1, \ldots, x_n, y, f(x_1, \ldots, x_n, y))$,
$f(x_1, \ldots, x_n, y) \leq b(x_1, \ldots, x_n, y)$, for all $x_1 \in \mathbb{N}, \ldots, x_n \in \mathbb{N}$, $y \in \mathbb{N}$,
also belongs to the class $\mathcal{F}$.

Definition 2.1.1. $\mathcal{E}^2$ is the smallest class of total functions in the natural numbers, which contains the constant 0, the successor function $\lambda x.x+1$, the projections $\lambda x_1, \ldots, x_n.x_i$, $n \geq 1$, $i \in [1, n]$, i, n – natural numbers, the addition and the multiplication and is closed under substitution and bounded primitive recursion.

The class $\mathcal{E}^2$ is known as the second class in the Grzegorczyk's hierarchy.

We say that a class $\mathcal{F}$ of total functions in the natural numbers is closed under bounded summation, if whenever the function $g : \mathbb{N}^{n+1} \to \mathbb{N}$ ($n \in \mathbb{N}$) belongs to the class $\mathcal{F}$, the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$, defined by $f(x_1, \ldots, x_n, y) = \sum_{z=0}^{y} g(x_1, \ldots, x_n, z)$ for all natural $x_1, \ldots, x_n, y$, also belongs to the class $\mathcal{F}$.

Definition 2.1.2. $\mathcal{L}^2$ is the smallest class of total functions in the natural numbers, which contains the constant 0, the successor function, the projections, the addition and the Kronecker's delta δ: $\delta(x, y) = 1$, if $x = y$ and $\delta(x, y) = 0$, if $x \neq y$, which is closed under substitution and bounded summation.

The class $\mathcal{L}^2$ is known as the Skolem's class of lower elementary functions.
It is well known that $\mathcal{L}^2 \subseteq \mathcal{E}^2$, because the bounded summation can be expressed by means of bounded primitive recursion.
It is an open question if the last inclusion is proper.

We say that a class $\mathcal{F}$ of total functions in the natural numbers is closed under the bounded least number operator, if whenever the function $g : \mathbb{N}^{n+1} \to \mathbb{N}$ ($n \in \mathbb{N}$) belongs to the class $\mathcal{F}$, the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$, defined by
$f(x_1, \ldots, x_n, y)$ = the least $z \leq y$, such that $g(x_1, \ldots, x_n, z) = 0$, if such z exist,
$f(x_1, \ldots, x_n, y) = y+1$, if for every $z \leq y$, $g(x_1, \ldots, x_n, z) > 0$, for all $x_1, \ldots, x_n, y$ – natural numbers, also belongs to the class $\mathcal{F}$. For the function f we use the notation
$f(x_1, \ldots, x_n, y) = \mu z \leq y\, [g(x_1, \ldots, x_n, z) = 0]$.

Definition 2.1.3. $\mathcal{M}^2$ is the smallest class of total functions in the natural numbers, which contains the constant 0, the successor function, the projections, the modified subtraction $\lambda x, y.\ x \dot{-} y$, the multiplication[1] and is closed under substitution and the bounded least number operator.

The class $\mathcal{M}^2$ seems to be the most natural subrecursive class, due to the following characterization: a total function f in the natural numbers belongs to the class $\mathcal{M}^2$ if and only if f is bounded by a polynomial and the graph of f is a Δ_0-definable relation. Here, Δ_0-definable relation means that it is defined by a formula from Peano's arithmetic with the sole use of bounded quantifiers.

It is well known that $\mathcal{M}^2 \subseteq \mathcal{L}^2$, because of the fact that the bounded least number operator can be expressed by means of bounded summation. It is an open question if the last inclusion is proper.

From now on, we denote by $\mathcal{F}$ any of the three classes $\mathcal{E}^2$, $\mathcal{L}^2$, $\mathcal{M}^2$.
We know that $\mathcal{F}$ is closed under substitution. It can be shown that $\mathcal{F}$ is closed under the bounded least number operator (the check is separate for the three cases).
Moreover, all constants, the addition, the multiplication and the functions
$\lambda x,y.\ x \dot{-} y$, $\lambda x,y.\min(x, y)$, $\lambda x,y.\max(x, y)$, $\lambda x,y.[x/(y+1)]$, $\lambda x,y.(x \bmod (y+1))$, $\lambda x,y.|x-y|$ belong to the class $\mathcal{F}$.

Let R be an n-ary relation in the natural numbers. We regard R as an $\mathcal{F}$-relation if and only if the characteristic function χ_R of R belongs to the class $\mathcal{F}$
($\chi_R(x_1, \ldots, x_n) = 0$, if $(x_1, \ldots, x_n) \in R$, $\chi_R(x_1, \ldots, x_n) = 1$, if $(x_1, \ldots, x_n) \notin R$).
Using the aforementioned characterization of $\mathcal{M}^2$, it is easily seen that
R is an $\mathcal{M}^2$-relation if and only if R is Δ_0-definable.
The $\mathcal{L}^2$-relations will also be called lower elementary relations.
It can be proven that the class of $\mathcal{F}$-relations is closed under the boolean operations and under the use of bounded quantifiers. Moreover, the class of $\mathcal{F}$-relations contains the relations $=, \neq, \leq, \geq, <, >$.

Property 2.1.4. Let $k \in \mathbb{N}$, $k \geq 1$ and $f : \mathbb{N}^k \to \mathbb{N}$ is a total function. If the graph of f is an $\mathcal{F}$-relation, then the function $\varphi : \mathbb{N}^{k+1} \to \mathbb{N}$, defined by

$$\varphi(t, n_1, \ldots, n_k) = \min(t, f(n_1, \ldots, n_k))$$

belongs to the class $\mathcal{F}$.
Proof. We have $\varphi(t, n_1, \ldots, n_k) = \mu y \leq t\, [y = t \vee y = f(n_1, \ldots, n_k)]$.

Now we will investigate some properties of the class $\mathcal{L}^2$.
The following relation is lower elementary (cf. [5, p. 67]): prime $(p) \leftrightarrow p$ is a prime number.
The following function belongs to the class $\mathcal{L}^2$ (in [5, p. 68] we substitute first p for p_n and then n for p in the expressions for Q and e) $\lambda m, p.\exp(m, p)$ – the exponent of p in the factorization of m.
The following function belongs to the class $\mathcal{L}^2$ (cf [5, p. 67]): $\lambda k.\mathrm{p}_k$ – $(k+1)$-th prime number.

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[1] Here, originally, instead of the multiplication function, the function $\lambda x.x^2$ was used, but as prof. Skordev noticed, the corresponding class is much narrower than needed, for example, it does not contain the addition and the multiplication. The new definition of $\mathcal{M}^2$ is now correct.

Further, we cite a property from [9], which will be useful in chapter 4.

Property 2.1.5. Let k be a natural number, $f : \mathbb{N}^{k+1} \to \mathbb{N}$ and $f \in \mathcal{L}^2$. Then the graph of the function $\varphi : \mathbb{N}^{k+1} \to \mathbb{N}$, defined by $\varphi(n_1, \ldots, n_k, n) = \prod_{i=0}^{n} f(n_1,\ldots,n_k,i)$ is lower elementary.

Proof. If $l = 0$, then $\varphi(n_1, \ldots, n_k, n) = l \leftrightarrow \exists i \le n\, f(n_1, \ldots, n_k, i) = 0$.
If $l > 0$, then $\varphi(n_1, \ldots, n_k, n) = l \leftrightarrow \forall i \le n\, (f(n_1, \ldots, n_k, i) > 0 \wedge f(n_1, \ldots, n_k, i) \le l) \wedge$
$\forall p \le l\, (\text{prime}(p) \Rightarrow \exp(l, p) = \sum_{i=0}^{n} \exp(f(n_1,\ldots,n_k,i), p))$. Here we use the fact that $\mathcal{L}^2$ is closed under bounded summation.

In the end we quote two highly non-trivial properties of the class $\mathcal{M}^2$, which we will use in section 2.5 and in chapter 5.

Property 2.1.6. Let k be a natural number, $f : \mathbb{N}^{k+1} \to \mathbb{N}$ and the graph of f is a Δ_0-definable relation. Then the graph of the function $\varphi : \mathbb{N}^{k+1} \to \mathbb{N}$, defined by

$\varphi(n_1, \ldots, n_k, n) = \sum_{i=0}^{[\log_2(n+1)]} f(n_1,\ldots,n_k,i)$ is also a Δ_0-definable relation.

In particular, if $f \in \mathcal{M}^2$, then $\varphi \in \mathcal{M}^2$.
The proof of property 2.1.6 is essentially made in [4].

Property 2.1.7. Let k be a natural number, $f : \mathbb{N}^{k+1} \to \mathbb{N}$ and the graph of f is a Δ_0-definable relation. Then the graph of the function $\varphi : \mathbb{N}^{k+1} \to \mathbb{N}$, defined by

$\varphi(n_1, \ldots, n_k, n) = \prod_{i=0}^{n} f(n_1,\ldots,n_k,i)$ is also a Δ_0-definable relation.

A proof of property 2.1.7 can be found in [1].

2.2. $\mathcal{F}$-expressibility and $\mathcal{F}$-computability.

Throughout this section $\mathcal{F}$ is one of the classes $\mathcal{E}^2$, $\mathcal{L}^2$, $\mathcal{M}^2$. Here we will exhibit the definitions from [6] of $\mathcal{F}$-expressibility and $\mathcal{F}$-computability of functions and we will prove some properties.

Definition 2.2.1. Let $k \ge 1$ and $A : \mathbb{N}^k \to \mathbb{Q}$ is a function. We say that A is $\mathcal{F}$-*expressible*, if there exist three k-argument functions f, g, h from the class $\mathcal{F}$, such that

$$A(n_1, \ldots, n_k) = \frac{f(n_1,\ldots,n_k) - g(n_1,\ldots,n_k)}{h(n_1,\ldots,n_k)+1}$$

for all natural $n_1, \ldots., n_k$.

Simple calculations yield the following facts:
1. If $A : \mathbb{N}^k \to \mathbb{Q}$, $B : \mathbb{N}^k \to \mathbb{Q}$ are $\mathcal{F}$-expressible, then the sum, the difference and the product of the functions A and B are $\mathcal{F}$-expressible functions.
2. If $A : \mathbb{N}^k \to \mathbb{Q}$ is $\mathcal{F}$-expressible and $A(n_1, \ldots, n_k) \ne 0$ for all natural $n_1, \ldots, n_k$, then the quotient $\lambda n_1, \ldots, n_k.\dfrac{1}{A(n_1,\ldots,n_k)}$ is an $\mathcal{F}$-expressible function.

3. If $A : \mathbb{N}^k \to \mathbb{Q}$ is $\mathcal{F}$-expressible and we substitute variables for functions from $\mathcal{F}$ in it, we obtain again an $\mathcal{F}$-expressible function.

Definition 2.2.2. Let $k \geq 1$ and $\theta : \mathbb{N}^k \to \mathbb{R}$ be a function. We say that θ is $\mathcal{F}$-*computable*, if there exists an $\mathcal{F}$-expressible function $A : \mathbb{N}^{k+1} \to \mathbb{Q}$, such that

$$|A(t, n_1, \ldots, n_k) - \theta(n_1, \ldots, n_k)| \leq \frac{1}{t+1}$$

for all natural $t, n_1, \ldots, n_k$.

Clearly, if $\theta : \mathbb{N}^k \to \mathbb{R}$ is $\mathcal{F}$-computable and we substitute variables for functions from $\mathcal{F}$ in it, we obtain again an $\mathcal{F}$-computable function.
It is also clear that an arbitrary $\mathcal{F}$-expressible function $A : \mathbb{N}^k \to \mathbb{Q}$, regarded as a function with range a subset of $\mathbb{R}$, is $\mathcal{F}$-computable.

Property 2.2.3. Let $k \geq 1$, $f : \mathbb{N}^k \to \mathbb{R}$ is a function, $A : \mathbb{N}^{k+1} \to \mathbb{Q}$ is an $\mathcal{F}$-expressible function and let $t|A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)|$ be bounded (as a function of $t, n_1, \ldots, n_k$).
Then f is an $\mathcal{F}$-computable function.
Proof. Let c be a non-zero natural number, such that
$t|A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)| \leq c$ for all natural $t, n_1, \ldots, n_k$.
Let us define $B : \mathbb{N}^{k+1} \to \mathbb{Q}$ by $B(t, n_1, \ldots, n_k) = A(ct + c, n_1, \ldots, n_k)$. Then B is $\mathcal{F}$-expressible and $|B(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)| = |A(ct + c, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)| \leq \dfrac{c}{ct+c} = \dfrac{1}{t+1}$.
Therefore, f is an $\mathcal{F}$-computable function.

Now we will show two properties, which will be useful for proving $\mathcal{F}$-computability.

Property 2.2.4. Let $k \geq 1$ and $f : \mathbb{N}^k \to \mathbb{N}$ be a function, such that $f(n_1, \ldots, n_k) \neq 0$ for all natural $n_1, \ldots, n_k$. Let $\lambda t, n_1, \ldots, n_k.\min(t+1, f(n_1, \ldots, n_k)) \in \mathcal{F}$. Then $\lambda n_1, \ldots, n_k.\dfrac{1}{f(n_1,\ldots,n_k)}$ is an $\mathcal{F}$-computable function.

Proof. The following inequality holds: $\left|\dfrac{1}{\min(t+1, f(n_1,\ldots,n_k))} - \dfrac{1}{f(n_1,\ldots,n_k)}\right| \leq \dfrac{1}{t+1}$.
To prove it, let us fix $t, n_1, \ldots, n_k$.
First case: $f(n_1, \ldots, n_k) \leq t+1$. Then $\min(t+1, f(n_1, \ldots, n_k)) = f(n_1, \ldots, n_k)$ and the inequality is obviously true: $0 \leq \dfrac{1}{t+1}$.

Second case: $f(n_1, \ldots, n_k) > t+1$. Then $\min(t+1, f(n_1, \ldots, n_k)) = t+1$ and $\dfrac{1}{t+1} > \dfrac{1}{f(n_1,\ldots,n_k)}$.

Therefore,

$$\left|\frac{1}{\min(t+1, f(n_1,\ldots,n_k))} - \frac{1}{f(n_1,\ldots,n_k)}\right| = \left|\frac{1}{t+1} - \frac{1}{f(n_1,\ldots,n_k)}\right| = \frac{1}{t+1} - \frac{1}{f(n_1,\ldots,n_k)} \leq \frac{1}{t+1}.$$

It remains to use that $\lambda t, n_1, \ldots, n_k.\dfrac{1}{\min(t+1, f(n_1,\ldots,n_k))}$ is an $\mathcal{F}$-expressible function.

Property 2.2.5. Let $k \geq 1$ and $f : \mathbb{N}^k \to \mathbb{N}$ be a function, such that $f(n_1, \ldots, n_k) \neq 0$ for all natural $n_1, \ldots, n_k$. Let the graph of f be an $\mathcal{F}$-relation. Then

$\lambda n_1, \ldots, n_k . \dfrac{1}{f(n_1, \ldots, n_k)}$ is an $\mathcal{F}$-computable function.

Proof. Using property 2.1.4 we obtain that the function $\lambda t, n_1, \ldots, n_k.\min(t, f(n_1, \ldots, n_k))$ belongs to the class $\mathcal{F}$. Now we apply substitution and obtain

$\lambda t, n_1, \ldots, n_k.\min(t+1, f(n_1, \ldots, n_k)) \in \mathcal{F}$. It remains to use property 2.2.4.

2.3. Closure of $\mathcal{F}$-computable functions under the arithmetical operations.

Throughout this section $\mathcal{F}$ is one of the classes $\mathcal{E}^2$, $\mathcal{L}^2$, $\mathcal{M}^2$.

In this section we will investigate the matter how the arithmetical operations interact with the property $\mathcal{F}$-computability of functions. The proofs essentially follow [6], where they are made for $\mathcal{F}$-computable real numbers.

Proposition 2.3.1. Let $k \geq 1$ and $f : \mathbb{N}^k \to \mathbb{R}$, $g : \mathbb{N}^k \to \mathbb{R}$ are $\mathcal{F}$-computable functions. Then the sum $f+g$ is also an $\mathcal{F}$-computable function.

Proof. We have that f is $\mathcal{F}$-computable, so there exists an $\mathcal{F}$-expressible function

$A : \mathbb{N}^{k+1} \to \mathbb{Q}$, such that $|A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)| \leq \dfrac{1}{t+1}$ for all natural $t, n_1, \ldots, n_k$.

Because of the fact that g is $\mathcal{F}$-computable, there exists an $\mathcal{F}$-computable function

$B : \mathbb{N}^{k+1} \to \mathbb{Q}$, such that $|B(t, n_1, \ldots, n_k) - g(n_1, \ldots, n_k)| \leq \dfrac{1}{t+1}$ for all natural $t, n_1, \ldots, n_k$.

Further, we obtain

$|(A(t, n_1, \ldots, n_k) + B(t, n_1, \ldots, n_k)) - (f(n_1, \ldots, n_k) + g(n_1, \ldots, n_k))| =$
$|(A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)) + (B(t, n_1, \ldots, n_k) - g(n_1, \ldots, n_k))| \leq$
$|A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)| + |B(t, n_1, \ldots, n_k) - g(n_1, \ldots, n_k)| \leq \dfrac{1}{t+1} + \dfrac{1}{t+1} = \dfrac{2}{t+1}$.

Since $A + B$ is $\mathcal{F}$-expressible and the function $\lambda t.\dfrac{2t}{t+1}$ is bounded, we can apply property 2.2.3 and thus we obtain that $f + g$ is $\mathcal{F}$-computable.

Proposition 2.3.2. Let $k \geq 1$ and $f : \mathbb{N}^k \to \mathbb{R}$ be an $\mathcal{F}$-computable function. Then so is the function $-f$.

Доказателство. Since f is $\mathcal{F}$-computable, there exists an $\mathcal{F}$-expressible function

$A : \mathbb{N}^{k+1} \to \mathbb{Q}$, such that $|A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)| \leq \dfrac{1}{t+1}$ for all natural $t, n_1, \ldots, n_k$.

In this situation,

$|((-A(t, n_1, \ldots, n_k)) - (-f(n_1, \ldots, n_k))| = |A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)| \leq \dfrac{1}{t+1}$.

Since $-A$ is $\mathcal{F}$-expressible, we obtain that $-f$ is $\mathcal{F}$-computable.

As a corollary from propositions 2.3.1 and 2.3.2 we deduce that difference of $\mathcal{F}$-computable functions is also an $\mathcal{F}$-computable function.

Proposition 2.3.3. Let $k \geq 1$ and $f : \mathbb{N}^k \to \mathbb{R}$, $g : \mathbb{N}^k \to \mathbb{R}$ be $\mathcal{F}$-computable bounded functions. Then, their product $f.g$ is also an $\mathcal{F}$-computable function.

Proof. Let $|f(n_1, \ldots, n_k)| \leq C$ and $|g(n_1, \ldots, n_k)| \leq D$ for all natural $n_1, \ldots, n_k$, where C and D are natural numbers.

Since f is $\mathcal{F}$-computable, there exists an $\mathcal{F}$-expressible function $A : \mathbb{N}^{k+1} \to \mathbb{Q}$, such that

$|A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)| \leq \dfrac{1}{t+1}$ for all natural $t, n_1, \ldots, n_k$.

Using this inequality, we easily obtain $|A(t, n_1, \ldots, n_k)| \leq |f(n_1, \ldots, n_k)| + \dfrac{1}{t+1} \leq C+1$ for all natural $t, n_1, \ldots, n_k$. Since g is $\mathcal{F}$-computable, there exists an $\mathcal{F}$-expressible function $B : \mathbb{N}^{k+1} \to \mathbb{Q}$, such that $|B(t, n_1, \ldots, n_k) - g(n_1, \ldots, n_k)| \leq \dfrac{1}{t+1}$ for all natural $t, n_1, \ldots, n_k$.

In this situation,

$|A(t, n_1, \ldots, n_k).B(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k).g(n_1, \ldots, n_k)| =$
$|A(t, n_1, \ldots, n_k).B(t, n_1, \ldots, n_k) - A(t, n_1, \ldots, n_k).g(n_1, \ldots, n_k) +$
$A(t, n_1, \ldots, n_k).g(n_1, \ldots, n_k) - f(n_1, \ldots, n_k).g(n_1, \ldots, n_k)| =$
$|A(t, n_1, \ldots, n_k).(B(t, n_1, \ldots, n_k) - g(n_1, \ldots, n_k)) + (A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)).g(n_1, \ldots, n_k)| \leq$
$|A(t, n_1, \ldots, n_k)|.|B(t, n_1, \ldots, n_k) - g(n_1, \ldots, n_k)| +$
$|A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)|.|g(n_1, \ldots, n_k)| \leq \dfrac{C+1}{t+1} + \dfrac{D}{t+1} = \dfrac{C+D+1}{t+1}$.

Since $A.B$ is $\mathcal{F}$-expressible and the function $\lambda t.\dfrac{(C+D+1)t}{t+1}$ is bounded, we can apply property 2.2.3 and thus we obtain that $f.g$ is $\mathcal{F}$-computable.

Proposition 2.3.4. Let $k \geq 1$ and $f : \mathbb{N}^k \to \mathbb{R}$ be an $\mathcal{F}$-computable function. Let $f(n_1, \ldots, n_k) \neq 0$ for all $n_1, \ldots, n_k$ and the quotient $\dfrac{1}{f(n_1,\ldots,n_k)}$ be bounded as a function of $n_1, \ldots, n_k$. Then $\lambda n_1, \ldots, n_k.\dfrac{1}{f(n_1,\ldots,n_k)}$ is an $\mathcal{F}$-computable function.

Доказателство. Let $\left|\dfrac{1}{f(n_1,\ldots,n_k)}\right| \leq D$ for all natural $n_1, \ldots, n_k$, where D is a positive real number. We choose a non-zero natural B, such that $B \geq 2D - 1$.

For this choice of B, we obtain $\left|\dfrac{1}{f(n_1,\ldots,n_k)}\right| \leq D \leq \dfrac{B+1}{2}$, therefore $|f(n_1, \ldots, n_k)| \geq \dfrac{2}{B+1}$ for all natural $n_1, \ldots, n_k$. Since f is $\mathcal{F}$-computable, there exists an $\mathcal{F}$-expressible function $A : \mathbb{N}^{k+1} \to \mathbb{Q}$, such that $|A(t, n_1, \ldots, n_k) - f(n_1, \ldots, n_k)| \leq \dfrac{1}{t+1}$ for all natural $t, n_1, \ldots, n_k$.

Let $t \geq B$. Then $|A(t, n_1, \ldots, n_k)| \geq |f(n_1, \ldots, n_k)| - |f(n_1, \ldots, n_k) - A(t, n_1, \ldots, n_k)| \geq$
$\dfrac{2}{B+1} - \dfrac{1}{t+1} \geq \dfrac{1}{B+1}$ for all natural $n_1, \ldots, n_k$. It follows that for $t \geq B$ and arbitrary $n_1, \ldots, n_k$ we have $A(t, n_1, \ldots, n_k) \neq 0$ and

$$\left|\frac{1}{A(t,n_1,\ldots,n_k)} - \frac{1}{f(n_1,\ldots,n_k)}\right| = \frac{|A(t,n_1,\ldots,n_k) - f(n_1,\ldots,n_k)|}{|A(t,n_1,\ldots,n_k)||f(n_1,\ldots,n_k)|} \leq \frac{h}{t+1},$$

where $h = \dfrac{(B+1)^2}{2}$. We define $C : \mathbb{N}^{k+1} \to \mathbb{Q}$ by $C(m, n_1, \ldots, n_k) = \dfrac{1}{A(m+B, n_1,\ldots,n_k)}$.

Then C is $\mathcal{F}$-expressible and $\left| C(m, n_1, \ldots, n_k) - \frac{1}{f(n_1, \ldots, n_k)} \right| \leq \frac{h}{m+B+1}$.

It remains to use that the function $\lambda m. \frac{mh}{m+B+1}$ is bounded and apply property 2.2.3.

So, in the end, $\lambda n_1, \ldots, n_k. \frac{1}{f(n_1, \ldots, n_k)}$ is an $\mathcal{F}$-computable function.

2.4. Method for proving $\mathcal{E}^2$-computability and $\mathcal{L}^2$-computability.

The method, which we will use for proving $\mathcal{E}^2$-computability and $\mathcal{L}^2$-computability of real-valued functions with natural arguments and real numbers is contained in the next theorem and the corollary after it.
In this section $\mathcal{F}$ is one of the classes $\mathcal{E}^2$ and $\mathcal{L}^2$.

Theorem 2.4.1. Let k be a natural number and $\theta : \mathbb{N}^{k+1} \to \mathbb{R}$ be an $\mathcal{F}$-computable function.

Let the series $\sum_{s=0}^{\infty} \theta(n_1, \ldots, n_k, s)$ be convergent and $\sigma(n_1, \ldots, n_k)$ be its sum

for all natural $n_1, \ldots, n_k$. Let there exist a function $p : \mathbb{N}^{k+1} \to \mathbb{N}$ from the class $\mathcal{F}$, such that

$$\left| \sum_{s=t+1}^{\infty} \theta(n_1, \ldots, n_k, s) \right| \leq \frac{1}{n+1}$$

for arbitrary natural numbers $n_1, \ldots, n_k$, n and $t = p(n_1, \ldots, n_k, n)$.
Then the function σ is also $\mathcal{F}$-computable.

As a special case of theorem 2.4.1 we obtain (when $k = 0$)

Corollary 2.4.2. Let $\theta : \mathbb{N} \to \mathbb{R}$ be an $\mathcal{F}$-computable function. Let the series $\sum_{s=0}^{\infty} \theta(s)$ be

convergent and α be its sum. Let there exist a function $p : \mathbb{N} \to \mathbb{N}$ from the class $\mathcal{F}$, such that

$$\left| \sum_{s=t+1}^{\infty} \theta(s) \right| \leq \frac{1}{n+1}$$

for arbitrary natural number n and $t = p(n)$.
Then the real number α is also $\mathcal{F}$-computable.

The proof of the theorem can be found in [8, section 2]. It is based essentially on the fact that the class $\mathcal{F}$ is closed under bounded summation.

Thus, to prove $\mathcal{F}$-computability of the sum of a series the following two tasks must be solved:

1. Prove $\mathcal{F}$-computability of the general term of the series.
2. Find a proper assessment for the speed of convergence.

2.5. Method for proving $\mathcal{M}^2$-computability.

The method, which we will use to prove $\mathcal{M}^2$-computability is similar to the method from section 2.4. We explicitly point out that the method from section 2.4 is not applicable for the class $\mathcal{M}^2$, because it is not known whether this class is closed under bounded summation. The modified method is based on property 2.1.6 and is the essence of the following theorem

Theorem 2.5.1. Let $\theta : \mathbb{N} \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function, such that the series

$$\sum_{s=0}^{\infty} \theta(s)$$

is convergent and let α be its sum. Let there exist a one-argument function p from $\mathcal{M}^2$, such that

$$\left| \sum_{s=[\log_2(t+1)]+1}^{\infty} \theta(s) \right| \leq \frac{1}{n+1}$$

for all natural n and $t = p(n)$. Then the number α is also $\mathcal{M}^2$-computable.

The proof of theorem is obtained with slight modification of the proof, made in [8, section 2] – propositions 2, 3 and the theorem.

Lemma 2.5.2. Let $\theta : \mathbb{N} \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function. There exist functions $f : \mathbb{N}^2 \to \mathbb{N}$ and $g : \mathbb{N}^2 \to \mathbb{N}$ from the class $\mathcal{M}^2$, such that $\left| \frac{f(s,n) - g(s,n)}{n+1} - \theta(s) \right| \leq \frac{1}{n+1}$ for all natural n and s.

Proof. Using that θ is $\mathcal{M}^2$-computable, we obtain functions $f_1 : \mathbb{N}^2 \to \mathbb{N}$, $g_1 : \mathbb{N}^2 \to \mathbb{N}$ and $h_1 : \mathbb{N}^2 \to \mathbb{N}$ from $\mathcal{M}^2$, such that $\left| \frac{f_1(s,n) - g_1(s,n)}{h_1(s,n)+1} - \theta(s) \right| \leq \frac{1}{n+1}$ for all natural n and s. Let us define $f_0(s, n) = f_1(s, 2n+1)$, $g_0(s, n) = g_1(s, 2n+1)$, $h_0(s, n) = h_1(s, 2n+1)$ for natural n, s.

Then f_0, g_0, h_0 are functions from $\mathcal{M}^2$ and $\left| \frac{f_0(s,n) - g_0(s,n)}{h_0(s,n)+1} - \theta(s) \right| =$

$$\left| \frac{f_1(s, 2n+1) - g_1(s, 2n+1)}{h_1(s, 2n+1)+1} - \theta(s) \right| \leq \frac{1}{2n+1+1} = \frac{1}{2(n+1)} \quad (*).$$

Let $A(i, j) = \left[\frac{i}{j+1} + \frac{1}{2} \right] = \left[\frac{2i+j+1}{2(j+1)} \right]$ for natural i, j. Clearly, $A \in \mathcal{M}^2$.

It is easily seen, that for all natural i, j the following inequality holds:

$$\left| A(i,j) - \frac{i}{j+1} \right| \leq \frac{1}{2} \quad (**).$$

Now, let us define

$f(s, n) = A((n+1)(f_0(s, n) \dot{-} g_0(s, n)), h_0(s, n))$,

$g(s, n) = A((n+1)(g_0(s, n) \dot{-} f_0(s, n)), h_0(s, n))$ for $s, n \in \mathbb{N}$.

It is clear that $f \in \mathcal{M}^2$ and $g \in \mathcal{M}^2$.

We will prove the following inequality:

$\left| f(s, n) - g(s, n) - (n+1) \frac{f_0(s,n) - g_0(s,n)}{h_0(s,n)+1} \right| \leq \frac{1}{2}$. Let us fix n and s.

First case: $f_0(s, n) \leq g_0(s, n)$. Then $f(s, n) = A(0, h_0(s, n)) = 0$ and the inequality transforms into $\left| -g(s, n) - (n+1) \frac{f_0(s,n) - g_0(s,n)}{h_0(s,n)+1} \right| \leq \frac{1}{2}$,

which is equivalent to $\left| g(s, n) - (n+1) \frac{g_0(s,n) - f_0(s,n)}{h_0(s,n)+1} \right| \leq \frac{1}{2}$.

The last inequality is true because of (**).

Second case: $f_0(s, n) > g_0(s, n)$. Then $g(s, n) = A(0, h_0(s, n)) = 0$ and the inequality becomes
$|f(s, n) - (n+1)\dfrac{f_0(s,n)-g_0(s,n)}{h_0(s,n)+1}| \le \dfrac{1}{2}$.

The last inequality is true because of (**).

We multiply the proven inequality with $\dfrac{1}{n+1}$ and we obtain

$|\dfrac{f(s,n)-g(s,n)}{n+1} - \dfrac{f_0(s,n)-g_0(s,n)}{h_0(s,n)+1}| \le \dfrac{1}{2(n+1)}$. This inequality, together with (*) and the triangle inequality yield

$$\left|\frac{f(s,n)-g(s,n)}{n+1} - \theta(s)\right| \le \frac{1}{2(n+1)} + \frac{1}{2(n+1)} = \frac{1}{n+1}$$ for all natural n, s.

Lemma 2.5.3. Let $\theta : \mathbb{N} \to \mathbb{R}$ be an $\mathcal{M}^2$-computable function. Let $\theta^{\Sigma} : \mathbb{N} \to \mathbb{R}$ be defined by $\theta^{\Sigma}(t) = \sum_{s=0}^{[\log_2(t+1)]} \theta(s)$ for all natural t. Then θ^{Σ} is $\mathcal{M}^2$-computable.

Доказателство. Using lemma 2.5.2 we obtain functions $f : \mathbb{N}^2 \to \mathbb{N}$ and $g : \mathbb{N}^2 \to \mathbb{N}$ from the class $\mathcal{M}^2$, such that $\left|\dfrac{f(s,n)-g(s,n)}{n+1} - \theta(s)\right| \le \dfrac{1}{n+1}$ for $n \in \mathbb{N}$, $s \in \mathbb{N}$.

Let $b(t) = [\log_2(t+1)]$ for natural t. An easy application of property 2.1.7 yields the fact that the graph of the function $\lambda k.2^k$ is Δ_0-definable. From the equivalence $y = b(t) \leftrightarrow \exists z \le t+1\ (z = 2^y \wedge 2z > t+1)$ we obtain that the graph of b is Δ_0-definable. Moreover, it is clear that b is bounded by a polynomial of t. Therefore, $b \in \mathcal{M}^2$.

Now let us examine the functions $f^{\Sigma}: \mathbb{N}^2 \to \mathbb{N}$ and $g^{\Sigma}: \mathbb{N}^2 \to \mathbb{N}$, defined by

$f^{\Sigma}(t, n) = \sum_{s=0}^{b(t)} f(s, nb(t)+n+b(t))$, $g^{\Sigma}(t, n) = \sum_{s=0}^{b(t)} g(s, nb(t)+n+b(t))$.

These functions belong to the class $\mathcal{M}^2$, because this class is closed under logarithmically bounded summation (property 2.1.6). In the above inequality we substitute n for $nb(t)+n+b(t)$ and we obtain

$$\left|\frac{f(s,nb(t)+n+b(t)) - g(s,nb(t)+n+b(t))}{nb(t)+n+b(t)+1} - \theta(s)\right| \le \frac{1}{nb(t)+n+b(t)+1} = \frac{1}{(b(t)+1)(n+1)}$$

for all s, n, t – natural numbers. Finally, we sum these inequalities for $s = 0, 1, \ldots, b(t)$ and using the triangle inequality we obtain that

$$\left|\frac{f^{\Sigma}(t,n) - g^{\Sigma}(t,n)}{nb(t)+n+b(t)+1} - \theta^{\Sigma}(t)\right| \le (b(t)+1)\frac{1}{nb(t)+n+b(t)+1} = \frac{1}{n+1}$$ for all natural n, t.

Thus, θ^{Σ} is $\mathcal{M}^2$-computable.

Now we are ready to prove the theorem.

Proof of theorem 2.5.1. Let θ^{Σ} be the function from lemma 2.5.3. From the same lemma we obtain that θ^{Σ} is $\mathcal{M}^2$-computable. Therefore, there exist functions $f_1: \mathbb{N}^2 \to \mathbb{N}$, $g_1 : \mathbb{N}^2 \to \mathbb{N}$ and $h_1 : \mathbb{N}^2 \to \mathbb{N}$ from the class $\mathcal{M}^2$, such that

$$\left|\frac{f_1(t,n) - g_1(t,n)}{h_1(t,n)+1} - \theta^{\Sigma}(t)\right| \le \frac{1}{n+1}$$ for n, $t \in \mathbb{N}$.

Now we define $f(n) = f_1(p(2n+1), 2n+1)$, $g(n) = g_1(p(2n+1), 2n+1)$,
$h(n) = h_1(p(2n+1), 2n+1)$. Naturally, f, g and h belong to $\mathcal{M}^2$ (because $p \in \mathcal{M}^2$). Now we have

$$\left|\frac{f(n)-g(n)}{h(n)+1}-\alpha\right| = \left|\frac{f(n)-g(n)}{h(n)+1}-\sum_{s=0}^{\infty}\theta(s)\right| =$$

$$\left|\frac{f(n)-g(n)}{h(n)+1}-\left(\sum_{s=0}^{[\log_2(p(2n+1)+1)]}\theta(s)+\sum_{s=[\log_2(p(2n+1)+1)]+1}^{\infty}\theta(s)\right)\right| \le$$

$$\left|\frac{f_1(p(2n+1), 2n+1)-g_1(p(2n+1), 2n+1)}{h_1(p(2n+1), 2n+1)+1}-\theta^{\Sigma}(p(2n+1))\right| + \left|\sum_{s=[\log_2(p(2n+1)+1)]+1}^{\infty}\theta(s)\right| \le$$

$\frac{1}{2n+2}+\frac{1}{2n+2} = \frac{1}{n+1}$ for all natural n. So, in the end, α is $\mathcal{M}^2$-computable.

Thus to prove $\mathcal{M}^2$-computability of the sum of a series the following two tasks must be solved:

1. Prove $\mathcal{M}^2$-computability of the general term of the series.
2. Find a proper assessment for the speed of convergence.

2.6. Some inequalities.

In this section we expose some well known or easily proved inequalities, which will be useful when we assess the speed of convergence of series. For the sake of completeness, we give proofs for some of them.

Proposition 2.6.1. Let $a : \mathbb{N} \to \mathbb{R}$ be a monotonically decreasing sequence with limit 0.
Let $N \in \mathbb{N}$. Then the series $\sum_{s=N}^{\infty}(-1)^s a_s$ is convergent and $\left|\sum_{s=N}^{\infty}(-1)^s a_s\right| \le a_N$.
A proof can be found in any detailed book on analysis
(criterion of Leibnitz for alternating series).

Proposition 2.6.2. Let N be a natural number and $f : [N, \infty) \to \mathbb{R}$ be a non-negative monotonically decreasing function. Then for any natural number $M \ge N$ the following inequality holds:

$$\sum_{n=N}^{M} f(n) \le f(N) + \int_{N}^{M} f(x)dx.$$

In particular, if the integral $\int_{N}^{\infty} f(x)dx$ is convergent, then the series $\sum_{n=N}^{\infty} f(n)$ is convergent and the following assessment is true: $\sum_{n=N}^{\infty} f(n) \le f(N) + \int_{N}^{\infty} f(x)dx$.
A proof can be found in any detailed book on analysis
(integral criterion for convergence of series).

Proposition 2.6.3. Let $k : \mathbb{N} \to \mathbb{N}$ be a strongly monotonically increasing sequence.
Then the inequality $k(n) \ge n$ holds for every $n \in \mathbb{N}$.
Proof. Trivial induction on n.

Proposition 2.6.4. Let $a \geq 2$ be a real number. Then for any $k \in \mathbb{N}$ the following inequality holds: $a^k \geq k+1$.
Proof. Induction on k. For $k = 0$, the inequality is true, because $a^0 = 1 \geq 1 = 0+1$.
Let $k \geq 0$ and $a^k \geq k+1$. Then $a^{k+1} = a.a^k \geq 2.(k+1) = 2k+2 \geq k+2$.

Proposition 2.6.5. For every natural k, $2^k > 3k - 3$.
Proof. Induction on k. When $k = 0$, the inequality is true, because $2^0 = 1 > -3 = 3.0 - 3$.
When $k = 1$, the inequality is also true, because $2^1 = 2 > 0 = 3.1 - 3$.
When $k = 2$, the inequality is again true, because $2^2 = 4 > 3 = 3.2 - 3$.
Let $k \geq 2$ and $2^k > 3k - 3$. Then $2^{k+1} = 2.2^k > 2.(3k - 3) = 6k - 6 \geq 3k = 3(k+1) - 3$.

Proposition 2.6.6. For every natural k, $16^{2^k} - 1 \geq 8^{2^k + k - 1}$.
Proof. We have $16^{2^k} = 2^{2^k}.8^{2^k}$ and $8^{2^k+k-1} = 8^{2^k}.2^{3k-3}$. Using proposition 2.6.5 we obtain $2^k > 3k - 3$. It follows that $2^{2^k} > 2^{3k-3}$. So, $16^{2^k} = 2^{2^k}.8^{2^k} > 8^{2^k}.2^{3k-3} = 8^{2^k+k-1}$. Both sides of the last inequality are natural numbers, therefore $16^{2^k} - 1 \geq 8^{2^k+k-1}$.

Proposition 2.6.7. For every $n \in \mathbb{N}$, $2^n \leq (n+1)!$.
Proof. When $n = 0$, the inequality is obviously true: $2^0 = 1 \leq 1 = (0+1)!$. When $n > 0$,
$2^n = \underbrace{2.2.\ldots.2.2}_{n \text{ times}} \leq 2.3.\ \ldots\ n.(n+1) = (n+1)!$.

Proposition 2.6.8. For every $n \in \mathbb{N}$, $2^n \leq \binom{2n}{n}$.

Proof. When $n = 0$, the inequality is true, because $2^0 = 1 = \binom{2.0}{0}$.

When $n > 0$, $\binom{2n}{n} = \prod_{k=0}^{n-1} \frac{2n-k}{n-k} = \prod_{k=0}^{n-1} \frac{2n-2k+k}{n-k} = \prod_{k=0}^{n-1} \left(2 + \frac{k}{n-k}\right) \geq \prod_{k=0}^{n-1} 2 = 2^n$.

Proposition 2.6.9. For every natural k, $p_k \geq k+2$, where p_k is the $(k+1)$-th prime.
Proof. We use proposition 2.6.3, applied to the sequence $\lambda k.p_k - 2$.

Proposition 2.6.10. There exists a natural number g, such that $p_k < (k+1)^2$ for all natural $k > g$.

The proof of the proposition uses the Chebyshev's theorem for the distribution of prime numbers and can be found in [5, p. 67].

Proposition 2.6.11. For every real $x \in [0, 1)$ the following inequalities hold:
$-\ln(1 - x) \leq x.\ \frac{1}{1-x}$ and $\ln(1+x) \leq x$.
Proof. We use that for every $x \in (-1, 1]$ the following expansion holds:
(*) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$. When $x \in [0, 1)$, $-x \in (-1, 0]$, therefore
$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$. So, $-\ln(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots \leq \sum_{s=1}^{\infty} x^s = x.\ \frac{1}{1-x}$.
From (*) and proposition 2.6.1 we obtain $|\ln(1+x)| = \ln(1+x) \leq x$ for $x \in [0, 1)$.

Chapter 3

$\mathcal{E}^2$-computability of continued fractions.

3.1. General statements about continued fractions.

For every finite non-empty sequence $s_0, s_1, \ldots, s_k$ of non-zero real numbers we define $[s_0, s_1, \ldots, s_k]$ – *finite continued fraction*, generated by the sequence:

$$[s_0, s_1, \ldots, s_k] = \cfrac{1}{s_0 + \cfrac{1}{s_1 + \cfrac{1}{s_2 + \cfrac{1}{\ddots + \cfrac{1}{s_{k-1} + \cfrac{1}{s_k}}}}}}.$$

Let $a : \mathbb{N} \to \mathbb{N}\backslash\{ 0 \}$ be a sequence of non-zero natural numbers.

We construct the sequence $b : \mathbb{N} \to \mathbb{R}$ in this way: $b_n = [a_0, a_1, \ldots, a_n]$.

Now let us define two sequences $h : \mathbb{N} \to \mathbb{N}$ and $k : \mathbb{N} \to \mathbb{N}$ by the following recurrence relations:
$h_0 = 1$, $h_1 = a_1$, $h_{n+2} = a_{n+2}.h_{n+1} + h_n$ for all natural n,
$k_0 = a_0$, $k_1 = a_1.a_0 + 1$, $k_{n+2} = a_{n+2}.k_{n+1} + k_n$ for all natural n.

The following properties hold [10, chapter 1]:
1. $k_n < k_{n+1}$ for all natural n.
2. $b_n = \dfrac{h_n}{k_n}$ for all natural n.
3. $b_n = \dfrac{1}{a_0} + \sum_{s=1}^{n} \dfrac{(-1)^s}{k_s.k_{s-1}}$ for all natural n.
4. $\lim_{n \to \infty} b_n$ exists.

Definition 3.1.1. We denote $\overline{a} = \lim_{n \to \infty} b_n$. $\overline{a}$ is called *infinite continued fraction*, generated by the sequence a. The following equality is illustrative (but informal):

$$\overline{a} = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}}.$$

It can be shown that for every sequence $a : \mathbb{N} \to \mathbb{N}\backslash\{ 0 \}$ of non-zero natural numbers, the infinite continued fraction $\overline{a}$ is an irrational number. Conversely, every irrational number from the interval (0, 1) can be represented by an infinite continued fraction, generated by a suitable sequence of non-zero numbers $a : \mathbb{N} \to \mathbb{N}$. The correspondence between the irrational numbers from (0, 1) and the infinite continued fractions, generated by sequence of non-zero natural numbers is bijective. Proofs of these facts can be found in [10, chapter 2].

3.2. Basic result.

Our goal is to prove the following

Theorem 3.2.1. Let $a : \mathbb{N} \to \mathbb{N}\setminus\{0\}$ be a sequence of non-zero numbers, which belongs to the class $\mathcal{E}^2$. Then the number $\overline{a}$ is $\mathcal{E}^2$-computable.

Proof. We use the following equalities from section 3.1:

$$\overline{a} = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{a_0} + \sum_{s=1}^{n} \frac{(-1)^s}{k_s.k_{s-1}} = \frac{1}{a_0} + \sum_{s=1}^{\infty} \frac{(-1)^s}{k_s.k_{s-1}} = \frac{1}{a_0} + \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{k_{s+1}.k_s}.$$

Since the rational numbers are $\mathcal{E}^2$-computable and the $\mathcal{E}^2$-computable numbers constitute a field (cf. [6]), it is sufficient to show $\mathcal{E}^2$-computability of the sum of the series.

To obtain this we use the basic method from section 2.4. For the matter of convergence we use proposition 2.6.1: $\left|\sum_{s=t+1}^{\infty} \frac{(-1)^{s+1}}{k_{s+1}.k_s}\right| \le \frac{1}{k_{t+2}.k_{t+1}} \le \frac{1}{t+1} = \frac{1}{n+1}$ for $t = n$,

because $k_t \ge t$ for any natural t, which follows from proposition 2.6.3.

It remains to show that the sequence $\lambda s.\frac{(-1)^{s+1}}{k_{s+1}.k_s}$ is $\mathcal{E}^2$-computable.

It is sufficient to show $\mathcal{E}^2$-computability of the sequences $\lambda s.(-1)^{s+1}$, $\lambda s.\frac{1}{k_s}$, $\lambda s.\frac{1}{k_{s+1}}$ and to use proposition 2.3.3, which gives that the product of bounded $\mathcal{E}^2$-computable sequences is again an $\mathcal{E}^2$-computable sequence.

We have $(-1)^{s+1} = s \bmod 2 - (s+1) \bmod 2$ for every $s \in \mathbb{N}$, so the first sequence is even $\mathcal{E}^2$-expressible. The $\mathcal{E}^2$-computability of the third sequence follows from the $\mathcal{E}^2$-computability of the second one, because it is obtained by substitution with the function $\lambda s.s+1 \in \mathcal{E}^2$.

For the $\mathcal{E}^2$-computability of the second sequence we define the function $p : \mathbb{N}^2 \to \mathbb{N}$ with $p(s, t) = \min(k_s, t+1)$. We will show that p belongs to the class $\mathcal{E}^2$.

Lemma 3.2.2. Let $f : \mathbb{N}^2 \to \mathbb{N}$, $g_0 : \mathbb{N} \to \mathbb{N}$, $g_1 : \mathbb{N} \to \mathbb{N}$, $h : \mathbb{N}^4 \to \mathbb{N}$, $b : \mathbb{N}^2 \to \mathbb{N}$ and

$f(0, t) = g_0(t)$, $f(1, t) = g_1(t)$,

$f(s+2, t) = h(s, t, f(s+1, t), f(s, t))$,

$f(s, t) \le b(s, t)$,

for all natural s, t. If g_0, g_1, h, b are from the class $\mathcal{E}^2$, then so is f.

Proof. Let $\Pi : \mathbb{N}^2 \to \mathbb{N}$, $L : \mathbb{N} \to \mathbb{N}$ and $R : \mathbb{N} \to \mathbb{N}$ be defined by

$\Pi(x, y) = (x+y)^2 + x$, $L(z) = z \dot{-} [\sqrt{z}]^2$, $R(z) = [\sqrt{z}] \dot{-} L(z)$.

The function $\lambda z.[\sqrt{z}]$ belongs to $\mathcal{E}^2$: $[\sqrt{z}] = \mu y \le z\, [y^2 \le z \wedge z < (y+1)^2]$ for all $z \in \mathbb{N}$.

From this fact it follows that the three functions Π, L, R belong to $\mathcal{E}^2$. Moreover, it is easily seen that the following equalities hold: $L(\Pi(x, y)) = x$, $R(\Pi(x, y)) = y$ for all natural x, y. The map Π can be used to code the ordered pairs of natural numbers (the functions L and R are the decoding maps).

Let us define $F : \mathbb{N}^2 \to \mathbb{N}$ by $F(s, t) = \Pi(f(s+1, t), f(s, t))$ for all natural s, t.

We will show that $F \in \mathcal{E}^2$. For $s, t \in \mathbb{N}$ we have $F(0, t) = \Pi(f(1, t), f(0, t)) = \Pi(g_1(t), g_0(t))$,

$F(s+1, t) = \Pi(f(s+2, t), f(s+1, t)) = \Pi(h(s, t, f(s+1, t), f(s, t)), f(s+1, t)) =$
$\Pi(h(s, t, L(F(s, t)), R(F(s, t))), L(F(s, t)))$.

Moreover, $F(s, t) = \Pi(f(s+1, t), f(s, t)) \le \Pi(b(s+1, t), b(s, t))$ and the last function belongs to $\mathcal{E}^2$ (here we use the obvious fact that Π is monotonically increasing over both its arguments).

Therefore, F is obtained with bounded primitive recursion from functions from $\mathcal{E}^2$ and thus $F \in \mathcal{E}^2$. We have $f(s, t) = R(F(s, t))$ for $s, t \in \mathbb{N}$, so f also belongs to $\mathcal{E}^2$.

For the above function p we have the following representation:
$p(0, t) = \min(k_0, t+1)$, $p(1, t) = \min(k_1, t+1)$,
$p(s+2, t) = \min(k_{s+2}, t+1) = \min(a_{s+2}k_{s+1} + k_s, t+1) = \min(a_{s+2}p(s+1, t) + p(s, t), t+1)$.
The last equality is the sole, which is not obvious. Let us fix s, t.
First case: $k_{s+1} \leq t+1$. Then $k_s < t+1$. Therefore, $p(s+1, t) = k_{s+1}$ and $p(s, t) = k_s$, so the equality is true.
Second case: $k_{s+1} > t+1$. Then $\min(a_{s+2}k_{s+1} + k_s, t+1) = t+1$ (here we use that $a_{s+2} > 0$). Also, $p(s+1, t) = t+1$ and $\min(a_{s+2}p(s+1, t) + p(s, t), t+1) = \min(a_{s+2}(t+1) + p(s, t), t+1) = t+1$, because $a_{s+2} > 0$. Again we have an equality.
From the fact that $a \in \mathcal{E}^2$ and the inequality $p(s, t) \leq t+1$, using lemma 3.2.2, we conclude that $p \in \mathcal{E}^2$. It remains to use property 2.2.4 and thus we obtain that the sequence $\lambda s.\dfrac{1}{k_s}$ is $\mathcal{E}^2$-computable, which gives the end to the proof.

As a corollary of the theorem we obtain another proof for a fact, proven in [7].

Corollary 3.2.3. The number e is $\mathcal{E}^2$-computable.

Proof. It is known that $e = 2 + \overline{a}$, where a is the sequence
1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, ... , which has the following representation:
$a_n = 1$, if ($n \bmod 3 = 0$ or $n \bmod 3 = 2$),
$a_n = 2([n/3]+1)$, if $n \bmod 3 = 1$,
and so $a \in \mathcal{E}^2$.

By the way, in section 5.1 we will see that the number e is even $\mathcal{M}^2$-computable.

Chapter 4

$\mathcal{L}^2$-computability of famous constants.

In this chapter we will apply the method from section 2.4 to prove $\mathcal{L}^2$-computability of different constants.

4.1. Logarithm of positive integer.

In this section, we will show that the numbers $\ln N$ for $N \in \mathbb{N}\setminus\{0\}$ are $\mathcal{L}^2$-computable.

For this purpose we use the following expansion: $\ln(1+\frac{1}{N}) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(s+1)N^{s+1}}$ for $N \in \mathbb{N}\setminus\{0\}$.

From this fact, it follows that $\ln(N+1) = \ln N + \sum_{s=0}^{\infty} \frac{(-1)^s}{(s+1)N^{s+1}}$ for $N \in \mathbb{N}\setminus\{0\}$.

Now we apply induction on $N \geq 1$. When $N = 1$, $\ln N = 0$, which is obviously $\mathcal{L}^2$-computable. Let $N \geq 1$ and $\ln N$ be $\mathcal{L}^2$-computable. In order to prove that $\ln(N+1)$ is $\mathcal{L}^2$-computable, we use the above expansion and the fact, that $\mathcal{L}^2$-computable numbers constitute a field (cf. [6]). It is sufficient to show that the sum of the series is an $\mathcal{L}^2$-computable real number.

We use the basic method. The functions $\lambda s.(-1)^s$ and $\lambda s.\frac{1}{s+1}$ are $\mathcal{L}^2$-expressible, so they are $\mathcal{L}^2$-computable. The function $\lambda s.N^{s+1}$ has a lower elementary graph, because it is obtained by applying bounded product to a constant (property 2.1.5).

Therefore, $\lambda s.\frac{1}{N^{s+1}}$ is $\mathcal{L}^2$-computable from property 2.2.5. Thus the general term of the series is a product of three bounded $\mathcal{L}^2$-computable functions, therefore this general term is itself $\mathcal{L}^2$-computable (proposition 2.3.3). For the matter of convergence we use inequality 2.6.1: $\left|\sum_{s=t+1}^{\infty} \frac{(-1)^s}{(s+1)N^{s+1}}\right| \leq \frac{1}{(t+2)N^{t+2}} \leq \frac{1}{t+1} = \frac{1}{n+1}$ for $t = n$.

Finally, $\ln(N+1)$ is an $\mathcal{L}^2$-computable real number.

4.2. Catalan's constant.

Catalan's constant G is defined by [3, p. 53]:

$$G = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)^2}.$$

We will show that G is $\mathcal{L}^2$-computable. To do this we use the basic method.

Clearly, the general term is $\mathcal{L}^2$-expressible (and hence $\mathcal{L}^2$-computable) function of s:

$\frac{(-1)^s}{(2s+1)^2} = \frac{(s+1) \bmod 2 - s \bmod 2}{(4s^2+4s)+1}$ for all $s \in \mathbb{N}$. For the matter of convergence we use inequality 2.6.1: $\left|\sum_{s=t+1}^{\infty} \frac{(-1)^s}{(2s+1)^2}\right| \leq \frac{1}{(2t+3)^2} \leq \frac{1}{2t+3} \leq \frac{1}{t+1} = \frac{1}{n+1}$ for $t = n$.

So, G is $\mathcal{L}^2$-computable.

4.3. Euler's constant.

In fact, $\mathcal{L}^2$-computability of the Euler's constant γ is proved in [9], but a different representation is used there.

Here we use the following representation [3, p. 30]:

$$\gamma = \sum_{s=0}^{\infty}\left(\frac{1}{s+1} - \ln(1+\frac{1}{s+1})\right) = \sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+2).(s+1)^{j+2}}.$$

In order to show $\mathcal{L}^2$-computability of γ we use the basic method twice.

At first, we will show that the sum of the inner series is an $\mathcal{L}^2$-computable function of s.

After that we will give a suitable assessment for the speed of convergence of the outer series.
The function $\lambda s, j.(s+1)^{j+2}$ has a lower elementary graph (property 2.1.5).

Using property 2.2.5 we obtain that $\lambda s, j.\dfrac{1}{(s+1)^{j+2}}$ is an $\mathcal{L}^2$-computable function.

The functions $\lambda s, j.(-1)^j$ and $\lambda s, j.\dfrac{1}{(j+2)}$ ca $\mathcal{L}^2$-expressible, hence $\mathcal{L}^2$-computable.

Thus the general term of the inner series is a product of three bounded $\mathcal{L}^2$-computable functions, so it is itself $\mathcal{L}^2$-computable (proposition 2.3.3).
For the matter of convergence we use inequality 2.6.1:

$$\left|\sum_{j=t+1}^{\infty}\frac{(-1)^j}{(j+2).(s+1)^{j+2}}\right| \le \frac{1}{(t+3).(s+1)^{t+3}} \le \frac{1}{t+3} \le \frac{1}{t+1} = \frac{1}{n+1} \text{ for } t = n.$$

Thus, the general term of the outer series $\lambda s.\sum_{j=0}^{\infty}\dfrac{(-1)^j}{(j+2).(s+1)^{j+2}}$ is $\mathcal{L}^2$-computable.

For the matter of convergence of the outer series we again use inequality 2.6.1 (for $N = 0$):

$$\left|\sum_{s=t+1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+2).(s+1)^{j+2}}\right| \le \sum_{s=t+1}^{\infty}\left|\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+2).(s+1)^{j+2}}\right| \le \sum_{s=t+1}^{\infty}\frac{1}{2.(s+1)^2} = \frac{1}{2}.\sum_{s=t+1}^{\infty}\frac{1}{(s+1)^2} \le$$

$$\frac{1}{2}.\sum_{s=t+1}^{\infty}\frac{1}{s(s+1)} = \frac{1}{2}.\sum_{s=t+1}^{\infty}\left(\frac{1}{s} - \frac{1}{s+1}\right) = \frac{1}{2}.\frac{1}{t+1} \le \frac{1}{t+1} = \frac{1}{n+1} \text{ for } t = n.$$

4.4. Merten's constant.

The constant of Mertens B_1 has the following representation [3, p. 94]:

$B_1 = \gamma - \sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\dfrac{1}{(j+2).\mathrm{p}_s^{\,j+2}}$, where γ is Euler's constant, p_s is $(s+1)$-th prime number.

Due to the fact that γ is $\mathcal{L}^2$-computable (section 4.3) and that difference of $\mathcal{L}^2$-computable is an $\mathcal{L}^2$-computable number, it is sufficient to focus on the $\mathcal{L}^2$-computability of the double series. We again apply twice the basic method, as in section 4.3.

The function $\lambda s, j.\dfrac{1}{j+2}$ is obviously $\mathcal{L}^2$-computable (it is even $\mathcal{L}^2$-expressible).

We have $\lambda s.\mathrm{p}_s \in \mathcal{L}^2$ and $\mathrm{p}_s^{\,j+2} = \prod_{i=0}^{j+1}\mathrm{p}_s$ is formed by applying bounded product, so using property 2.1.5 we obtain that $\lambda s, j.\mathrm{p}_s^{\,j+2}$ has a lower elementary graph.

From property 2.2.5, $\lambda s, j.\dfrac{1}{\mathrm{p}_s^{\,j+2}}$ is $\mathcal{L}^2$-computable. Now we apply proposition 2.3.3 and thus the general term of the inner series is $\mathcal{L}^2$-computable.

Further, let us investigate the matter of speed of convergence. We have

$$\sum_{j=t+1}^{\infty}\frac{1}{(j+2).\mathrm{p}_s^{\,j+2}} \le \sum_{j=t+1}^{\infty}\frac{1}{\mathrm{p}_s^{\,j+2}} = \frac{1}{\mathrm{p}_s^{\,t+3}}.\frac{1}{1-\frac{1}{\mathrm{p}_s}} = \frac{1}{\mathrm{p}_s^{\,t+2}(\mathrm{p}_s-1)} \le \frac{1}{\mathrm{p}_s^{\,t+2}} \le$$

$$\frac{1}{\mathrm{p}_s^{\,t+1}} \le \frac{1}{t+1} = \frac{1}{n+1}$$ for $t = n$. We used inequality 2.6.4 $\mathrm{p}_s^{t+1} \ge t+1$ ($\mathrm{p}_s \ge 2$).

Thus the general term of the outer series $\lambda s.\sum_{j=0}^{\infty}\frac{1}{(j+2).\mathrm{p}_s^{\,j+2}}$ is $\mathcal{L}^2$-computable.

Therefore, to obtain $\mathcal{L}^2$-computability of the outer series we must look into the matter of speed of convergence. We have

$$\sum_{s=t+1}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(j+2).\mathrm{p}_s^{\,j+2}} \le \sum_{s=t+1}^{\infty}\sum_{j=0}^{\infty}\frac{1}{\mathrm{p}_s^{\,j+2}} = \sum_{s=t+1}^{\infty}\frac{1}{\mathrm{p}_s^{\,2}}.\frac{1}{1-\frac{1}{\mathrm{p}_s}} = \sum_{s=t+1}^{\infty}\frac{1}{\mathrm{p}_s.(\mathrm{p}_s-1)} \le$$

$$\sum_{s=t+1}^{\infty}\frac{1}{(s+1).s} = \sum_{s=t+1}^{\infty}\left(\frac{1}{s}-\frac{1}{s+1}\right) = \frac{1}{t+1} = \frac{1}{n+1}$$ for $t = n$.

We used inequality 2.6.9 $\mathrm{p}_s \ge s+1$. Finally, B_1 is $\mathcal{L}^2$-computable.

4.5. Addition to Merten's constant.

Let $w(n)$ be the number of different prime divisors of the positive integer n.
In [3, p. 94] the arithmetic mean of $w(1), \ldots, w(n)$ is examined:

$E_n(w) = \frac{1}{n}\sum_{i=1}^{n} w(i)$ and the variance: $\mathrm{Var}_n(w) = E_n(w^2) - E_n(w)^2$.

It can be proven that: $\lim_{n\to\infty}\left(\mathrm{Var}_n(w) - \ln(\ln(n))\right) = B_1 - N - \frac{\pi^2}{6}$,

where B_1 is Merten's constant, $N = \sum_{s=0}^{\infty}\frac{1}{\mathrm{p}_s^{\,2}}$ (p_s is $(s+1)$-th prime number).

We will show that the last limit is $\mathcal{L}^2$-computable. In section 4.4 we saw that B_1 is $\mathcal{L}^2$-computable, π is also $\mathcal{L}^2$-computable [cf. 8, p. 867] and so it is sufficient to check $\mathcal{L}^2$-computability of the number N.

We already pointed out that $\lambda s.\mathrm{p}_s \in \mathcal{L}^2$, so also $\lambda s.\mathrm{p}_s^{\,2} \in \mathcal{L}^2$ and therefore $\lambda s.\frac{1}{\mathrm{p}_s^{\,2}}$ is $\mathcal{L}^2$-expressible, hence $\mathcal{L}^2$-computable function. For the matter of convergence:

$$\sum_{s=t+1}^{\infty}\frac{1}{\mathrm{p}_s^{\,2}} \le \sum_{s=t+1}^{\infty}\frac{1}{\mathrm{p}_s.(\mathrm{p}_s-1)} \le \sum_{s=t+1}^{\infty}\frac{1}{(s+1).s} = \sum_{s=t+1}^{\infty}\left(\frac{1}{s}-\frac{1}{s+1}\right) = \frac{1}{t+1} = \frac{1}{n+1}$$ for $t = n$.

We used inequality 2.6.9 $\mathrm{p}_s \ge s+1$. Finally, N is an $\mathcal{L}^2$-computable number, so the limit in question has this property as well.

4.6. Lebesgue constants.

Let us fix a natural number n.
The Lebesgue's constant L_n has the following representation [3, p. 251]:

$$L_n = \frac{16}{\pi^2}\sum_{s=1}^{\infty}\sum_{j=1}^{(2n+1)s}\frac{1}{4s^2-1}\frac{1}{2j-1} = \frac{16}{\pi^2}\sum_{s=0}^{\infty}\sum_{j=0}^{(2n+1)(s+1)-1}\frac{1}{4(s+1)^2-1}\frac{1}{2j+1}.$$

Due to the fact that π is $\mathcal{L}^2$-computable, it is sufficient to prove $\mathcal{L}^2$-computability of the sum of the series.

Let $\theta : \mathbb{N}^2 \to \mathbb{R}$ be the function, defined by $\theta(s, j) = \frac{1}{4(s+1)^2-1}\frac{1}{2j+1}$ for $s, j \in \mathbb{N}$.

It is clear that θ is an $\mathcal{L}^2$-expressible function, so θ is $\mathcal{L}^2$-computable. Since the general term $\sum_{j=0}^{(2n+1)(s+1)-1}\theta(s,j)$ of the series is obtained by applying bounded summation to θ (the upper bound is a function from $\mathcal{L}^2$), we conlude that this general term is an $\mathcal{L}^2$-computable function (cf. [8, p. 865]). Now we turn to the matter of speed of convergence.

We have $\sum_{s=t+1}^{\infty}\sum_{j=0}^{(2n+1)(s+1)-1}\frac{1}{4(s+1)^2-1}\frac{1}{2j+1} = \sum_{s=t+1}^{\infty}\frac{1}{(2s+1)(2s+3)}\sum_{j=0}^{(2n+1)(s+1)-1}\frac{1}{2j+1}$.

The sequence $\sum_{j=1}^{s}\frac{1}{j}-\ln(s)$ converges (for $s \to \infty$) to the number γ – Euler's constant.

Therefore, it is bounded and let $\sum_{j=1}^{s}\frac{1}{j}-\ln(s) \le C$ for every natural s, where

C is a suitable natural number. We have

$$\sum_{j=0}^{(2n+1)(s+1)-1}\frac{1}{2j+1} \le \sum_{j=1}^{2((2n+1)(s+1)-1)+1}\frac{1}{j} = \sum_{j=1}^{(2n+1)(2s+2)-1}\frac{1}{j} \le C + \ln((2n+1)(2s+2)-1).$$

Thus $\sum_{s=t+1}^{\infty}\frac{1}{(2s+1)(2s+3)}\sum_{j=0}^{(2n+1)(s+1)-1}\frac{1}{2j+1} \le \sum_{s=t+1}^{\infty}\frac{C+\ln((2n+1)(2s+2)-1)}{(2s+1)(2s+3)}$.

Now we use that the sequence $\frac{C+\ln((2n+1)(2s+2)-1)}{(2s+1)(2s+3)}.s^{\frac{3}{2}}$ is convergent (for $s \to \infty$).

This holds because of the fact that the logarithmic function grows slower than the power function. So, the sequence in question is bounded and let D be a natural non-zero number, such that $\frac{C+\ln((2n+1)(2s+2)-1)}{(2s+1)(2s+3)}.s^{\frac{3}{2}} \le D$ for $s \in \mathbb{N}$.

In this situation, $\sum_{s=t+1}^{\infty}\frac{C+\ln((2n+1)(2s+2)-1)}{(2s+1)(2s+3)} \le \sum_{s=t+1}^{\infty}\frac{D}{s^{\frac{3}{2}}} = D.\sum_{s=t+1}^{\infty}\frac{1}{s^{\frac{3}{2}}}$.

To assess the last series we use proposition 2.6.2 and we obtain:

$$\sum_{s=t+1}^{\infty}\frac{1}{s^{\frac{3}{2}}} \le \frac{1}{(t+1)^{\frac{3}{2}}} + \int_{t+1}^{\infty}\frac{ds}{s^{\frac{3}{2}}} = \frac{1}{(t+1)^{\frac{3}{2}}} + \left.\frac{s^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right|_{t+1}^{\infty} = \frac{1}{(t+1)^{\frac{3}{2}}} + \frac{2}{(t+1)^{\frac{1}{2}}} \le \frac{3}{\sqrt{t+1}}.$$

Finally, $\sum_{s=t+1}^{\infty}\sum_{j=0}^{(2n+1)(s+1)-1}\frac{1}{4(s+1)^2-1}\frac{1}{2j+1} \le \frac{3D}{\sqrt{t+1}} = \frac{1}{n+1}$ for $t = (3Dn + 3D)^2 \dot{-} 1$.

So, L_n is an $\mathcal{L}^2$-computable real number.

4.7. Logarithm of Khinchin's constant.

Let K be Khinchin's constant.
For its logarithm we have the following representation [3, p. 60]:

$\ln K. \ln 2 = -\sum_{s=2}^{\infty} \ln(1-\frac{1}{s})\ln(1+\frac{1}{s}) = \sum_{s=0}^{\infty} (-\ln(1-\frac{1}{s+2}))\ln(1+\frac{1}{s+2})$.

We already saw in section 4.1 that ln 2 is an $\mathcal{L}^2$-computable real number. Therefore, it is sufficient to examine the sum of the series. Following the basic method, the first thing to prove is $\mathcal{L}^2$-computability of the general term of the series.
We use the following expansion:

$$-\ln(1-\frac{1}{s+2}) = \frac{1}{s+2}+\frac{1}{2(s+2)^2}+\frac{1}{3(s+2)^3}+\ldots = \sum_{j=0}^{\infty}\frac{1}{(j+1)(s+2)^{j+1}}.$$

Clearly, the function $\lambda s, j. \frac{1}{j+1}$ is $\mathcal{L}^2$-computable (even $\mathcal{L}^2$-expressible). The function $\lambda s, j.(s+2)^{j+1}$ has a lower elementary graph (property 2.1.5), and this fact gives $\mathcal{L}^2$-computability of the function $\lambda s, j. \frac{1}{(s+2)^{j+1}}$ (property 2.2.5). From proposition 2.3.3 it follows that the general term is $\mathcal{L}^2$-computable. For the matter of convergence:

$$\sum_{j=t+1}^{\infty}\frac{1}{(j+1)(s+2)^{j+1}} \le \sum_{j=t+1}^{\infty}\frac{1}{(s+2)^{j+1}} = \frac{1}{(s+2)^{t+2}}.\frac{1}{1-\frac{1}{s+2}} = \frac{1}{(s+2)^{t+1}}.\frac{1}{s+1} \le$$

$\frac{1}{(s+2)^{t+1}} \le \frac{1}{t+1} = \frac{1}{n+1}$ for $t = n$. We used inequality 2.6.4 $a^k \ge k$ for $a \ge 2$.

So, $\lambda s. -\ln(1-\frac{1}{s+2})$ is an $\mathcal{L}^2$-computable function.
Further, we use the following expansion:

$$\ln(1+\frac{1}{s+2}) = \frac{1}{s+2}-\frac{1}{2(s+2)^2}+\frac{1}{3(s+2)^3}-\ldots = \sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)(s+2)^{j+1}}.$$

The general term is a product of three bounded $\mathcal{L}^2$-computable functions (we saw this above for two of them and the function $\lambda j.(-1)^j$ is $\mathcal{L}^2$-expressible) and using proposition 2.3.3 we obtain that it is an $\mathcal{L}^2$-computable function. For the matter of convergence we use inequality 2.6.1: $\left|\sum_{j=t+1}^{\infty}\frac{(-1)^j}{(j+1)(s+2)^{j+1}}\right| \le \frac{1}{(t+2)(s+2)^{t+2}} \le \frac{1}{t+1} = \frac{1}{n+1}$ for $t = n$.

Therefore, $\lambda s. \ln(1+\frac{1}{s+2})$ is an $\mathcal{L}^2$-computable function.
The following estimates are true:

$0 \le -\ln(1-\frac{1}{s+2}) \le \ln 2$, $0 \le \ln(1+\frac{1}{s+2}) \le \ln\frac{3}{2}$ for any natural number s.

Therefore, the general term of the original series for ln K is a product of two bounded $\mathcal{L}^2$-computable functions, so it is also $\mathcal{L}^2$-computable (proposition 2.3.3).
The matter of convergence remains.
For this purpose we use the inequalities from proposition 2.6.11:

$-\ln(1-x) \le x.\frac{1}{1-x}$, $\ln(1+x) \le x$ for $x \in [0, 1)$. Thus for $s \in \mathbb{N}$ we have:

$$(-\ln(1-\frac{1}{s+2}))\ln(1+\frac{1}{s+2}) \le \frac{1}{s+2}.\frac{1}{1-\frac{1}{s+2}}.\frac{1}{s+2} = \frac{1}{s+2}.\frac{1}{s+1}.$$

Therefore, $\sum_{s=t+1}^{\infty}(-\ln(1-\frac{1}{s+2}))\ln(1+\frac{1}{s+2}) \le \sum_{s=t+1}^{\infty}\frac{1}{s+2}.\frac{1}{s+1} =$

$\sum_{s=t+1}^{\infty}\left(\frac{1}{s+1}-\frac{1}{s+2}\right) = \frac{1}{t+2} \le \frac{1}{t+1} = \frac{1}{n+1}$ for $t = n$.

In the end, we obtain that the constant ln K is $\mathcal{L}^2$-computable.

4.8. A constant related to quadratic non-residues.

Following [3, p. 96], let $q\,(p)$ be the least quadratic non-residue modulus p, where p is an odd prime number. For the mean A of q the following holds:

$$A = \lim_{n\to\infty}\frac{\sum_{p\le n} q(p)}{\sum_{p\le n} 1} \text{(the sums are for prime } p) = \sum_{s=0}^{\infty}\frac{\mathrm{p}_s}{2^{s+1}}.$$

We will show that the constant A is $\mathcal{L}^2$-computable, using the second representation.

Let $f : \mathbb{N}^2 \to \mathbb{N}$ be defined by $f\,(s,\ t) = \min\,(\mathrm{p}_s.(t+1), 2^{s+1})$ for $s,\ t \in \mathbb{N}$. Due to the fact that $\lambda s.2^{s+1}$ has a lower elementary graph (property 2.1.5), the function $\lambda s,\ z.\min\,(z,\ 2^{s+1}) \in \mathcal{L}^2$ from property 2.1.4. Using that $\lambda s.\mathrm{p}_s \in \mathcal{L}^2$ we apply substitution to obtain that $f \in \mathcal{L}^2$.

We will prove the following inequality: $\left|\frac{\mathrm{p}_s}{f(s,t)} - \frac{\mathrm{p}_s}{2^{s+1}}\right| \le \frac{1}{t+1}$ for all natural $s,\ t$.

Let us fix $s,\ t$. If $f\,(s,\ t) = 2^{s+1}$, the inequality is obviously true. Otherwise, we have

$f\,(s,\ t) = \mathrm{p}_s.(t+1)$ and $\mathrm{p}_s.(t+1) \le 2^{s+1}$. So in this situtation, $\left|\frac{\mathrm{p}_s}{f(s,t)} - \frac{\mathrm{p}_s}{2^{s+1}}\right| =$

$\left|\frac{\mathrm{p}_s}{\mathrm{p}_s.(t+1)} - \frac{\mathrm{p}_s}{2^{s+1}}\right| = \frac{\mathrm{p}_s}{\mathrm{p}_s.(t+1)} - \frac{\mathrm{p}_s}{2^{s+1}} = \frac{1}{t+1} - \frac{\mathrm{p}_s}{2^{s+1}} \le \frac{1}{t+1}$ and the inequality again holds.

From the proven inequality and the fact that the function $\lambda s,\ t.\frac{\mathrm{p}_s}{f(t,s)}$ is $\mathcal{L}^2$-expressible we obtain that the general term of the series is an $\mathcal{L}^2$-computable function.

It remains to examine the matter of convergence. For this goal, we use the inequality 2.6.10: $\mathrm{p}_s < (s+1)^2$, which holds for $s > g$, where g is a fixed natural number.

Then for $t \ge g$, $\sum_{s=t+1}^{\infty}\frac{\mathrm{p}_s}{2^{s+1}} \le \sum_{s=t+1}^{\infty}\frac{(s+1)^2}{2^{s+1}}$. Now we use the fact that the sequence

$\lambda s.\frac{(s+1)^2.s.(s+1)}{2^{s+1}}$ is convergent (for $s \to \infty$). This follows from the fact that the power function grows slowlier than the exponential function. Therefore, the sequence in question is bounded, so for a suitable non-zero natural number C, we have

$\frac{(s+1)^2}{2^{s+1}} \le \frac{C}{s.(s+1)}$ for all natural $s \ge t+1$.

Hence, for $t \ge g$, $\sum_{s=t+1}^{\infty}\frac{\mathrm{p}_s}{2^{s+1}} \le \sum_{s=t+1}^{\infty}\frac{C}{s.(s+1)} = C.\sum_{s=t+1}^{\infty}\left(\frac{1}{s}-\frac{1}{s+1}\right) = \frac{C}{t+1}$.

Thus $\sum_{s=t+1}^{\infty}\frac{p_s}{2^{s+1}} \le \frac{1}{n+1}$ for $t = \max(g, (Cn + C) \dot{-} 1)$.

4.9. Riemann's zeta function and Apéry's constant.

The Riemann's zeta function is defined by: $\zeta(x) = \sum_{n=1}^{\infty}\frac{1}{n^x}$ for $x > 1$.

Our goal is to show that the function $f_R : \mathbb{N} \to \mathbb{R}$, defined by $f_R(k) = \zeta(k+2)$ is $\mathcal{L}^2$-computable and to deduce some of its properties, which will be useful later.

We have that $f_R(k) = \sum_{s=1}^{\infty}\frac{1}{s^{k+2}} = \sum_{s=0}^{\infty}\frac{1}{(s+1)^{k+2}}$. The function $\lambda s, k.(s+1)^{k+2}$ has a lower elementary graph (property 2.1.5). Hence, from property 2.2.5 we obtain $\mathcal{L}^2$-computability of the general term of the series for f_R (as a function of s and k).
For the matter of convergence:

$$\sum_{s=t+1}^{\infty}\frac{1}{(s+1)^{k+2}} \le \sum_{s=t+1}^{\infty}\frac{1}{(s+1)^2} \le \sum_{s=t+1}^{\infty}\frac{1}{s(s+1)} = \sum_{s=t+1}^{\infty}\left(\frac{1}{s}-\frac{1}{s+1}\right) = \frac{1}{t+1} = \frac{1}{n+1} \text{ for } t = n.$$

Thus f_R is an $\mathcal{L}^2$-computable function.

From this fact it follows that all values of f_R are $\mathcal{L}^2$-computable real numbers.

In particular, the Apéry's constant $\zeta(3) = f_R(1)$ is $\mathcal{L}^2$-computable.

Property 4.9.1. $f_R(k+1) < f_R(k)$ for all natural k.

Proof. We have $f_R(k) - f_R(k+1) = \sum_{s=0}^{\infty}\frac{1}{(s+1)^{k+2}} - \sum_{s=0}^{\infty}\frac{1}{(s+1)^{k+3}} =$

$$\sum_{s=0}^{\infty}\left(\frac{1}{(s+1)^{k+2}}-\frac{1}{(s+1)^{k+3}}\right) = \sum_{s=0}^{\infty}\frac{1}{(s+1)^{k+2}}\left(1-\frac{1}{s+1}\right) = \sum_{s=0}^{\infty}\frac{s}{(s+1)^{k+3}} > 0.$$

Property 4.9.2. $1 < f_R(k) < 2$ for all natural k.

Proof. We have $f_R(k) = \sum_{s=1}^{\infty}\frac{1}{s^{k+2}} = 1 + \sum_{s=2}^{\infty}\frac{1}{s^{k+2}} > 1$. It is well-known that

$f_R(0) = \sum_{s=1}^{\infty}\frac{1}{s^2} = \frac{\pi^2}{6} < 2$. From property 4.9.1 f_R is monotonically decreasing, so $f_R(k) < 2$ for all natural k.

Property 4.9.3. $f_R(k) \le 1+\frac{1}{2^{k+2}}\frac{k+3}{k+1}$ for all natural k.

Proof. We will use proposition 2.6.2. We have

$$f_R(k) = \sum_{s=1}^{\infty}\frac{1}{s^{k+2}} = 1+\sum_{s=2}^{\infty}\frac{1}{s^{k+2}} \le 1+\frac{1}{2^{k+2}}+\int_2^{\infty}\frac{ds}{s^{k+2}} = 1+\frac{1}{2^{k+2}}+\left.\frac{s^{-k-2+1}}{-k-2+1}\right|_2^{\infty} =$$

$$1+\frac{1}{2^{k+2}}+\frac{2^{-k-1}}{k+1} = 1+\frac{1}{2^{k+2}}(1+\frac{2}{k+1}) = 1+\frac{1}{2^{k+2}}\frac{k+3}{k+1}.$$

4.10. Logarithm of π.

The following representation is true [3, p. 44]:

$\ln\pi - \ln 2 = \sum_{s=1}^{\infty}\frac{\zeta(2s)}{2s2^{2s-1}} = \sum_{s=0}^{\infty}\frac{\zeta(2(s+1))}{2(s+1)2^{2(s+1)-1}} = \sum_{s=0}^{\infty}\frac{f_R(2s)}{2(s+1)2^{2s+1}}$, where ζ and f_R are defined in section 4.9. We will show that $\ln\pi$ is an $\mathcal{L}^2$-computable real number.

We have that ln2 is $\mathcal{L}^2$-computable (section 4.1) and sum of $\mathcal{L}^2$-computable numbers is again an $\mathcal{L}^2$-computable number, so it is sufficient to the examine the sum of the series.

In section 4.9 we showed that f_R is an $\mathcal{L}^2$-computable function. It follows that $\lambda s.f_R(2s)$ is also $\mathcal{L}^2$-computable, since it is obtained by substitution with $\lambda s.2s \in \mathcal{L}^2$.

Moreover, from property 4.9.2 we derive that $\lambda s.f_R(2s)$ is bounded.

The function $\lambda s.\frac{1}{2(s+1)}$ is $\mathcal{L}^2$-expressible, hence $\mathcal{L}^2$-computable.

The function $\lambda s.2^{2s+1}$ has a lower elementary graph (property 2.1.5). From property 2.2.5 we conclude that the function $\lambda s.\frac{1}{2^{2s+1}}$ is $\mathcal{L}^2$-computable.

Now we use proposition 2.3.3 and thus the general term of the series is $\mathcal{L}^2$-computable.

For the matter of convergence:

$$\sum_{s=t+1}^{\infty}\frac{f_R(2s)}{2(s+1)2^{2s+1}} \le \sum_{s=t+1}^{\infty}\frac{2}{2(s+1)2^{2s+1}} \le \sum_{s=t+1}^{\infty}\frac{1}{2^{2s+1}} = \frac{1}{2^{2t+3}}.\frac{1}{1-\frac{1}{4}} = \frac{4}{3.2^{2t+3}} = \frac{1}{3.2^{2t+1}} \le$$

$\frac{1}{3.(2t+1)} = \frac{1}{6t+3} \le \frac{1}{t+1} = \frac{1}{n+1}$ for $t = n$. We used property 4.9.2 and then inequality 2.6.4 $2^x \ge x$ for any natural number x.

Finally, $\ln\pi$ is an $\mathcal{L}^2$-computable real number.

4.11. Niven's constant.

The Niven's constant C has the following representation [3, p. 112]:

$C = 1+\sum_{s=2}^{\infty}\left(1-\frac{1}{\zeta(s)}\right) = 1+\sum_{s=0}^{\infty}\left(1-\frac{1}{f_R(s)}\right)$, where ζ and f_R are defined in section 4.9.

We will show that C is $\mathcal{L}^2$-computable. It is sufficient to examine the sum of the series.

Using property 4.9.2 we have that $1 \le f_R(s) \le 2$ for all natural s. Therefore, $\frac{1}{2} \le \frac{1}{f_R(s)} \le 1$ for all natural s. We saw in section 4.9 that f_R is $\mathcal{L}^2$-computable, so using proposition 2.3.4 we obtain that $\lambda s.\frac{1}{f_R(s)}$ is an $\mathcal{L}^2$-computable function. Now from propositions 2.3.1 and 2.3.2 it follows that the general term of the series is $\mathcal{L}^2$-computable.

Further, let us examine the matter of convergence: $\sum_{s=t+1}^{\infty}\left(1-\frac{1}{f_R(s)}\right) = \sum_{s=t+1}^{\infty}\frac{f_R(s)-1}{f_R(s)} \le$

$$\sum_{s=t+1}^{\infty}\left(f_R(s)-1\right) \le \sum_{s=t+1}^{\infty}\frac{1}{2^{s+2}}\frac{s+3}{s+1} \le \sum_{s=t+1}^{\infty}\frac{1}{2^{s+1}} = \frac{1}{2^{t+2}}.\frac{1}{1-\frac{1}{2}} = \frac{1}{2^{t+1}} \le \frac{1}{t+1} = \frac{1}{n+1}$$ for $t = n$.

We used properties 4.9.2 and 4.9.3, the inequality $\frac{s+3}{s+1} \leq 2$ for $s \geq 1$ and the inequality 2.6.4 $2^x \geq x$ for all $x \in \mathbb{N}$.

In the end, the Niven's constant C is $\mathcal{L}^2$-computable.

4.12. A constant from number theory.

Following [3, p. 112-113], let m be a positive integer with factorization $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ of prime numbers. We suppose that the prime numbers $p_1, \ldots, p_k$ are pairwise different and that the exponents $a_1, \ldots, a_k$ are non-zero.

Let us define $H(m) = \begin{cases} 1 & \text{if } m = 1 \\ \max(a_1, \ldots, a_k) & \text{if } m > 1 \end{cases}$. The following is true:

$\lim\limits_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \frac{1}{H(m)} = \sum_{s=2}^{\infty} \frac{1}{s(s-1)\zeta(s)} = \sum_{s=0}^{\infty} \frac{1}{(s+2)(s+1)f_R(s)}$, where ζ and f_R are defined in section 4.9. We will show $\mathcal{L}^2$-computability of the sum of the series. In section 4.11 we saw that the function $\lambda s.\frac{1}{f_R(s)}$ is $\mathcal{L}^2$-computable. The function $\lambda s.\frac{1}{(s+2)(s+1)}$ is $\mathcal{L}^2$-expressible, hence $\mathcal{L}^2$-computable. From proposition 2.3.3, the general term of the series is $\mathcal{L}^2$-computable. For the matter of convergence: $\sum_{s=t+1}^{\infty} \frac{1}{(s+2)(s+1)f_R(s)} \leq$

$\sum_{s=t+1}^{\infty} \frac{1}{(s+2)(s+1)} = \sum_{s=t+1}^{\infty} \left(\frac{1}{s+1} - \frac{1}{s+2} \right) = \frac{1}{t+2} \leq \frac{1}{t+1} = \frac{1}{n+1}$ for $t = n$.

We used property 4.9.2. So the limit in question is an $\mathcal{L}^2$-computable real number.

4.13. Another constant from number theory.

The constant, which we will examine has the following representation [3, p. 113]:

$\sum_{s=0}^{\infty} (\zeta(\mathrm{p}_s) - 1) = \sum_{s=0}^{\infty} (f_R(\mathrm{p}_s - 2) - 1) = 0.8928945714\ldots$. Here ζ and f_R are defined in section 4.9 and p_s is the (s+1)-th prime number. We will show that this constant is $\mathcal{L}^2$-computable. From section 4.9 we know that f_R is $\mathcal{L}^2$-computable, we also have $\lambda s.\mathrm{p}_s - 2 = \lambda s.(\mathrm{p}_s \dot{-} 2) \in \mathcal{L}^2$. Therefore, $\lambda s.f_R(\mathrm{p}_s - 2)$ is $\mathcal{L}^2$-computable, because it is obtained by substitution. Moreover, the constant 1 is $\mathcal{L}^2$-computable and difference of $\mathcal{L}^2$-computable functions is again $\mathcal{L}^2$-computable. Hence, the general term of the series is an $\mathcal{L}^2$-computable function.

For the matter of convergence:

$\sum_{s=t+1}^{\infty} (f_R(\mathrm{p}_s - 2) - 1) \leq \sum_{s=t+1}^{\infty} \frac{1}{2^{\mathrm{p}_s - 2 + 2}} \frac{\mathrm{p}_s - 2 + 3}{\mathrm{p}_s - 2 + 1} = \sum_{s=t+1}^{\infty} \frac{1}{2^{\mathrm{p}_s}} \frac{\mathrm{p}_s + 1}{\mathrm{p}_s - 1} \leq \sum_{s=t+1}^{\infty} \frac{1}{2^{\mathrm{p}_s - 1}} \leq \sum_{s=t+1}^{\infty} \frac{1}{2^s} =$

$\frac{1}{2^{t+1}} . \frac{1}{1 - \frac{1}{2}} = \frac{1}{2^t} \leq \frac{1}{n+1}$ for $t = n+1$. We used property 4.9.3, the inequality $\frac{\mathrm{p}_s + 1}{\mathrm{p}_s - 1} \leq 2$ for natural $s \geq 1$ (for $s \geq 1$ we have $\mathrm{p}_s \geq 3$), the inequality 2.6.9 $\mathrm{p}_k \geq k+1$ for $k \in \mathbb{N}$ and the inequality 2.6.4 $2^x \geq x$ for $x \in \mathbb{N}$.

Finally, the constant in question is $\mathcal{L}^2$-computable.

Chapter 5

$\mathcal{M}^2$-computability of famous constants.

In this chapter we will apply the method from section 2.5 to prove $\mathcal{M}^2$-computability of different constants.

5.1. The number *e*.

In fact, the $\mathcal{M}^2$-computability of e is proven in [9], but the method used there is essentially different from the basic method from section 2.5.

The following representation is well-known: $e = \sum_{s=0}^{\infty} \frac{1}{s!}$. Using property 2.1.7 it is easily seen that the graph of the function $\lambda s.s!$ is Δ_0-definable. From this fact and from property 2.2.5 we obtain that the general term of the series $\lambda s.\frac{1}{s!}$ is an $\mathcal{M}^2$-computable function.

For the matter of convergence: $\sum_{s=n+1}^{\infty} \frac{1}{s!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots =$

$\frac{1}{(n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+2).(n+3)} + \ldots) \le \frac{1}{(n+1)!}(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots) = \frac{2}{(n+1)!} \le \frac{2}{2^n}$ for any natural n. Here we used inequality 2.6.7 $2^n \le (n+1)!$ for $n \in \mathbb{N}$.

Therefore, $\sum_{s=[\log_2(t+1)]+1}^{\infty} \frac{1}{s!} \le \frac{2}{2^{[\log_2(t+1)]}} = \frac{4}{2^{[\log_2(t+1)]+1}} \le \frac{4}{2^{\log_2(t+1)}} = \frac{4}{t+1} = \frac{1}{n+1}$

for $t = 4n+3$, $n \in \mathbb{N}$. Thus we obtain $\mathcal{M}^2$-computability of e.

5.2. Liouville's number *L*.

We will prove $\mathcal{M}^2$-computability of L, despite the fact that this is proven in [9] by another method. We use the representation $L = \sum_{s=1}^{\infty} \frac{1}{10^{s!}} = \sum_{s=0}^{\infty} \frac{1}{10^{(s+1)!}}$. The graph of the function $\lambda s.10^{(s+1)!}$ is Δ_0-definable: $y = 10^{(s+1)!} \leftrightarrow \exists k \le y\ (y = 10^k \wedge k = (s+1)!)$. Here we used property 2.1.7, which yields that the relations $y = 10^k$ and $k = (s+1)!$ are Δ_0-definable. Moreover, we bounded the quantifier with the help of inequality 2.6.4 for $a = 10$.

Now, from property 2.2.5 we obtain that the general term of the series $\lambda s.\frac{1}{10^{(s+1)!}}$ is an $\mathcal{M}^2$-computable function. For the matter of convergence:

$\sum_{s=n+1}^{\infty} \frac{1}{10^{(s+1)!}} \le \sum_{s=n+1}^{\infty} \frac{1}{(s+1)!} = \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \frac{1}{(n+4)!} + \ldots =$

$\frac{1}{(n+2)!}(1 + \frac{1}{n+3} + \frac{1}{(n+3).(n+4)} + \ldots) \le \frac{1}{(n+2)!}(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots) = \frac{2}{(n+2)!} \le \frac{2}{2^{n+1}}$ for all natural n. Here we used inequality 2.6.4 (for $a = 10$) and inequality 2.6.7.

Therefore, $\sum_{s=[\log_2(t+1)]+1}^{\infty} \frac{1}{10^{(s+1)!}} \le \frac{2}{2^{[\log_2(t+1)]+1}} \le \frac{2}{2^{\log_2(t+1)}} = \frac{2}{t+1} = \frac{1}{n+1}$ for $t = 2n+1$ and $n \in \mathbb{N}$.

Finally, L is $\mathcal{M}^2$-computable.

5.3. Constant of Erdös-Borwein.

The constant of Erdös-Borwein has the following representation: $E = \sum_{s=0}^{\infty} \frac{1}{2^{s+1}-1}$.

The graph of $\lambda s.2^{s+1} - 1$ is Δ_0-definable: $y = 2^{s+1} - 1$ if and only if $y + 1 = 2^{s+1}$.
Here we use the fact that the relation $z = 2^t$ is Δ_0-definable, which is easily seen with property 2.1.7. Therefore from property 2.2.5 we derive that the general term of the series $\lambda s.\frac{1}{2^{s+1}-1}$ is an $\mathcal{M}^2$-computable function. Now let us examine the matter of convergence.

Let us define $S(n) = \sum_{s=n+1}^{\infty} \frac{1}{2^{s+1}-1}$ for natural n.

Then, $S(n) \le \sum_{s=n+1}^{\infty} \frac{1}{2^{s+1}-2} = \frac{1}{2}.\sum_{s=n+1}^{\infty} \frac{1}{2^{s}-1} = \frac{1}{2}.\left(\frac{1}{2^{n+1}-1} + S(n)\right)$.

Thus $2.S(n) \le \frac{1}{2^{n+1}-1} + S(n)$ and hence, $S(n) \le \frac{1}{2^{n+1}-1}$ for all natural n.

We obtained $\sum_{s=n+1}^{\infty} \frac{1}{2^{s+1}-1} \le \frac{1}{2^{n+1}-1}$. Therefore, $\sum_{s=[\log_2(t+1)]+1}^{\infty} \frac{1}{2^{s+1}-1} \le \frac{1}{2^{[\log_2(t+1)]+1}-1} \le$ $\frac{1}{2^{\log_2(t+1)}-1} = \frac{1}{t}$ for $t > 0$. In the end, $\sum_{s=[\log_2(t+1)]+1}^{\infty} \frac{1}{2^{s+1}-1} \le \frac{1}{n+1}$ for $t = n+1$, $n \in \mathbb{N}$.

So, the constant E has the property $\mathcal{M}^2$-computability.

5.4. The number π.

For the number π we use the following representation [3, p. 20]:

$$\frac{\pi^2}{18} = \sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} = \sum_{s=0}^{\infty} \frac{1}{(s+1)^2\binom{2s+2}{s+1}}.$$

If we prove $\mathcal{M}^2$-computability of the sum of the series, then it will follow $\mathcal{M}^2$-computability of the number π, since the $\mathcal{M}^2$-computable numbers constitute a filed, which contains the real roots of the polynomials with $\mathcal{M}^2$-computable coefficients (cf. [6]). To prove $\mathcal{M}^2$-computability of the general term of the series we use [2, p. 22], where it is shown that the relation $y = \binom{n}{k}$ is Δ_0-definable.

Using this fact and substitution we obtain that the graph of the function $\lambda s.\binom{2s+2}{s+1}$ is Δ_0-definable. Therefore, using property 2.2.5, the function $\lambda s.\frac{1}{\binom{2s+2}{s+1}}$ is $\mathcal{M}^2$-computable.

Moreover, the function $\lambda s.\frac{1}{(s+1)^2}$ is obviously $\mathcal{M}^2$-expressible.

Thus from proposition 2.3.3 if follows that the general term is $\mathcal{M}^2$-computable.

For the matter of convergence we use inequality 2.6.8 $2^s \le \binom{2s}{s}$ for $s \in \mathbb{N}$.

We have $\sum_{s=n+1}^{\infty} \frac{1}{(s+1)^2 \binom{2s+2}{s+1}} \le \sum_{s=n+1}^{\infty} \frac{1}{\binom{2s+2}{s+1}} \le \sum_{s=n+1}^{\infty} \frac{1}{2^{s+1}} = \frac{1}{2^{n+2}} \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{n+1}}$.

Therefore, $\sum_{s=[\log_2(t+1)]+1}^{\infty} \frac{1}{(s+1)^2 \binom{2s+2}{s+1}} \le \frac{1}{2^{[\log_2(t+1)]+1}} \le \frac{1}{2^{\log_2(t+1)}} = \frac{1}{t+1} = \frac{1}{n+1}$ for $t = n$.

5.5. Logarithm of the golden section.

In this section we examine the constant $\ln\varphi$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden section.
For this purpose we use the following representation [3, p. 20]:
$2(\ln\varphi)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2.\binom{2n}{n}} = \sum_{s=0}^{\infty} \frac{(-1)^s}{(s+1)^2.\binom{2s+2}{s+1}}$. As in section 5.4, it is sufficient to prove $\mathcal{M}^2$-computability of the sum of the series. The general term of the series is a product of three $\mathcal{M}^2$-computable functions (we saw this for $\lambda s.\frac{1}{\binom{2s+2}{s+1}}$ in section 5.4, the other two functions $\lambda s.(-1)^s$ и $\lambda s.\frac{1}{(s+1)^2}$ are even $\mathcal{M}^2$-expressible), which are bounded, so using proposition 2.3.3 we obtain that the general term is an $\mathcal{M}^2$-computable function of s.
For the matter of convergence we use inequalities 2.6.1 and 2.6.8:
$\left| \sum_{s=n+1}^{\infty} \frac{(-1)^s}{(s+1)^2.\binom{2s+2}{s+1}} \right| \le \frac{1}{(n+2)^2.\binom{2n+4}{n+2}} \le \frac{1}{\binom{2n+4}{n+2}} \le \frac{1}{2^{n+2}} \le \frac{1}{2^{n+1}}$.

Therefore, $\left| \sum_{s=[\log_2(t+1)]+1}^{\infty} \frac{(-1)^s}{(s+1)^2.\binom{2s+2}{s+1}} \right| \le \frac{1}{2^{[\log_2(t+1)]+1}} \le \frac{1}{2^{\log_2(t+1)}} = \frac{1}{t+1} = \frac{1}{n+1}$ for $t = n$.

5.6. Paper folding constant.

Following [3, p. 439-440], let us imagine that we fold a sheet of paper through its middle in two equal parts, putting the right side over the left side. When we complete this process several times, we obtain a sequence of folds on the paper. When we unfold the paper, this folds look as lows (1) or peaks (0).
The paper folding sequence $\{s_n\}$, $n \ge 1$ is defined by the sequence of 0 and 1, corresponding to the sequence of folds. It is seen that $\{s_n\}$ = 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, ... and $s_{4n-3} = 1$, $s_{4n-1} = 0$, $s_{2n} = s_n$ for natural $n \ge 1$. It also can be proven that $\sigma = \sum_{n=1}^{\infty} \frac{s_n}{2^n} = \sum_{s=0}^{\infty} \frac{1}{2^{2^s}} (1 - \frac{1}{2^{2^{s+2}}})^{-1}$. We will show that the constant σ is $\mathcal{M}^2$-computable by using the second representation. Since $\lambda s.2^s$ has a Δ_0-definable graph (property 2.1.7), $\lambda s.2^{2^s}$ also has a Δ_0-definable graph: $y = 2^{2^s}$ if and only if $\exists z \le y\ (z = 2^s \wedge y = 2^z)$.
Here we bounded the quantifier with the help of inequality 2.6.4 for $a = 2$.

From property 2.2.5 it follows that $\lambda s.\frac{1}{2^{2^s}}$ is an $\mathcal{M}^2$-computable function. Further, using substitution with $\lambda s.s+2 \in \mathcal{M}^2$ we obtain that $\lambda s.\frac{1}{2^{2^{s+2}}}$ is also an $\mathcal{M}^2$-computable function. Therefore, $\lambda s.(1 - \frac{1}{2^{2^{s+2}}})$ is $\mathcal{M}^2$-computable. Moreover, $1 - \frac{1}{2^{2^{0+2}}} \leq 1 - \frac{1}{2^{2^{s+2}}} \leq 1$, so $\frac{15}{16} \leq 1 - \frac{1}{2^{2^{s+2}}} \leq 1$ for all natural s. Using this fact we obtain that $1 \leq (1-\frac{1}{2^{2^{s+2}}})^{-1} \leq \frac{16}{15}$ for all natural s. Now from proposition 2.3.4 we conclude that $\lambda s.(1-\frac{1}{2^{2^{s+2}}})^{-1}$ is an $\mathcal{M}^2$-computable function. Thus using proposition 2.3.3 we conclude that the general term of the series is an $\mathcal{M}^2$-computable function.

For the matter of convergence:

$$\sum_{s=n+1}^{\infty} \frac{1}{2^{2^s}}(1-\frac{1}{2^{2^{s+2}}})^{-1} = \sum_{s=n+1}^{\infty} \frac{1}{2^{2^s}}.\frac{2^{2^{s+2}}}{2^{2^{s+2}}-1} = \sum_{s=n+1}^{\infty} \frac{1}{2^{2^s}}.\frac{2^{4.2^s}}{2^{2^{s+2}}-1} = \sum_{s=n+1}^{\infty} \frac{2^{3.2^s}}{2^{4.2^s}-1} = \sum_{s=n+1}^{\infty} \frac{8^{2^s}}{16^{2^s}-1}.$$

Now we use inequality 2.6.6 and thus obtain:

$$\sum_{s=n+1}^{\infty} \frac{1}{2^{2^s}}(1-\frac{1}{2^{2^{s+2}}})^{-1} = \sum_{s=n+1}^{\infty} \frac{8^{2^s}}{16^{2^s}-1} \leq \sum_{s=n+1}^{\infty} \frac{8^{2^s}}{8^{2^s+s-1}} =$$

$$\sum_{s=n+1}^{\infty} \frac{1}{8^{s-1}} = \frac{1}{8^n}.\frac{1}{1-\frac{1}{8}} = \frac{8}{7.8^n} = \frac{64}{7.8^{n+1}} \leq \frac{10}{8^{n+1}}.$$ Therefore,

$$\sum_{s=[\log_2(t+1)]+1}^{\infty} \frac{1}{2^{2^s}}(1-\frac{1}{2^{2^{s+2}}})^{-1} \leq \frac{10}{8^{[\log_2(t+1)]+1}} \leq \frac{10}{8^{\log_2(t+1)}} = \frac{10}{(t+1)^3} \leq \frac{10}{t+1} = \frac{1}{n+1}$$ for $t = 10n+9$.

Finally, σ is $\mathcal{M}^2$-computable.

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