SOFIA UNIVERSITY ST. KLIMENT OHRIDSKI

FACULTY OF MATHEMATICS AND INFORMATICS
DEPARTMENT OF MATHEMATICAL LOGIC AND APPLICATIONS

M.Sc. Logic and Algorithms, Mathematics

MASTER THESIS

On some elementary theories for rotation in the line-based Euclidean plane

Author:

Lyuba Konova

Faculty number: 0MI3100010

Supervisor:

Prof. Tinko Tinchev

Contents

1 Introduction

The theory of the Euclidean plane has long been a topic of interest. A well-known result states that the Euclidean plane cannot be fully axiomatized in a first-order language using only binary predicates. Consequently, any forthcoming theories will necessarily be weaker than the full theory of the Euclidean plane. To avoid relying on complex theorems, we demonstrate instead that the primary object, the "point", cannot be explicitly defined. If it could, we would also be able to define the predicate of co-punctuality, which is not possible.

By incorporating a binary relation to define the angle between two lines, the line-based fragment of the Euclidean plane naturally extends the frameworks established in [1] and [2].

Throughout the different chapters, we explore the expressive power of various formal languages:

- Chapter 2 introduces preliminary knowledge and establishes fundamental notations that will be used throughout this work.
- Chapter 3 establishes that formal equality is essential.
- Chapter 4 focuses on axiomatizing the plane when angles are directed and may or may not be co-measurable with π.
- Chapter 5 examines the case where angles are undirected and co-measurable with π.

Additionally, some propositions are proved multiple times using different techniques. This approach not only showcases various problem-solving strategies but also, in some cases, leads to stronger corollaries.

2 Preliminaries and notations

2.1 The main structure

The main object of investigation in this work will be the Euclidean plane, consisting of all lines with the binary property "the angle between two lines equals ϕ", where ϕ is a given angle in $(0, \pi)$. We will use $\mathcal{F}^2_{\mathbb{R}}(\phi)$ to denote this structure in case the angle is directed (that is in the plane is fixed one orientation). Otherwise, the notation will be $\boldsymbol{\mathcal{F}}^2_{\mathbb{R}}(\phi)$.

To start with, let us define all relations included in the following sections:

1) We say that two lines a and b are parallel, and we write $a \parallel b$, whenever they do not intersect each other at only one point. We will associate a binary predicate P to denote it as follows:

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models P(a, b) \iff a \parallel b$$

2) When we want to say that two lines are parallel but different, we will use the letter P', so

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models P'(a,b) \iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models P(a,b) \wedge a \neq b$$

3) We fix the direction for measuring angles to be anti-clockwise. When we want to say that the angle between two lines a and b has measure θ, we will write: $\measuredangle(a,b) = \theta$. Keep in mind, that fixing a direction means that $\measuredangle(b,a) = -\theta$. The relation "the angle between two lines has measure ϕ" will be denoted by the letter R_ϕ and:

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models R_\phi(a,b) \iff \measuredangle(a,b) = \phi$$

4) At some point we include the relation $\mathbf{R}_\phi$, which is very similar to R_ϕ, but here the angle between two lines is not directed.

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models \mathbf{R}_\phi(a,b) \iff \measuredangle(a,b) = \phi \text{ or } \measuredangle(b,a) = \phi$$

5) With Co we will denote the 3-ary predicate for co-punctuality of three lines, namely:

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models Co(a,b,c) \iff a,b,c \text{ are different and have a common point.}$$

2.2 Languages and formulas

It is now time to meet the first-order languages we will be working with. We will investigate the expressivity of different languages, but we will stick to the following notations:
1) With $x, y, z, \ldots x_1, x_2, \ldots$ we will denote variables.
2) With $\mathcal{M}, \mathcal{N}, \mathcal{F}, \ldots$ we will denote models.
3) With $M, N, F, \ldots$ we will denote the universes of the models.
4) With $a, b, c \ldots, a_1, a_2 \ldots$ we will denote the elements of the universes of the models.
5) When some object is called a line, then it is element of $\mathcal{F}^2_{\mathbb{R}}(\phi)$.

Our line-based first-order theory is based on the idea of associating with parallelism and rotation the binary predicates P and R_ϕ, with the formulas $P(x,y)$ and $R_\phi(x,y)$ being read "x is parallel to y" and "the angle between x and y is ϕ". The formulas are given by the rule:

$$\varphi ::= (x = y) \mid R_\phi(x,y) \mid P(x,y) \mid \neg\varphi \mid (\varphi \vee \psi) \mid \forall x \varphi$$

Whenever it is clear that the angle ϕ is fixed, then for the sake of brevity we use the letter R.

2.3 Games:

Let $\mathcal{L}$ be a language without function symbols and with finite number of relation symbols and individual constants. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures and let k be a natural number. The Ehrenfeucht-Fraïssé game with length k on the structures $\mathcal{A}$ and $\mathcal{B}$ is played by two players, who will be called Abelard (duplicator) and Eloise (spoiler) and who know everything about the game. On each round Eloise chooses a structure and then an element from this structure. Abelard, then, picks an element from the structure that has not been chosen by Eloise. Let $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ be the elements until the end of the game which are chosen from $\mathcal{A}$ and from $\mathcal{B}$ respectively. Abelard wins the game if for every $1 \leq i \leq k$ the substructure of $\mathcal{A}$ generated by $a_1, \ldots, a_i$ is isomorphic to the substructure of $\mathcal{B}$ generated by $b_1, b_2, \ldots, b_k$.

Let us remind that:

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two models. We say that $\mathcal{A} \equiv_k \mathcal{B}$ if and only if for each sentence φ with $rk(\varphi) \leq k$ it holds that $\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi$.

Theorem 2.2. ***(Ehrenfeucht-Fraïssé:)*** *Let $\mathcal{L}$ be a language without function symbols and with finite number of relation symbols and constants. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{L}$-structures and let k be a natural number. The following are equivalent:*

1. *$\mathcal{A} \equiv_k \mathcal{B}$*
2. *Abelard has a winning strategy for the game with length k played on $\mathcal{A}$ and $\mathcal{B}$.*

The last theorem is Th 3.9 in [3].

2.4 Model theory

Let us remind the popular criteria for completeness:

Vaugth's test: Let $\mathcal{L}$ be at most countable language. If a $\mathcal{L}$-theory has only infinite models and is α-categorical for some infinite cardinal α, then it is complete.

3 Expressivity of $\mathcal{L}(R_\phi)$

In this chapter our main aim is to show that the inclusion of a predicate for equality is a necessity.

Let ϕ be an angle in $(0;\pi)$. Let us consider the language $\mathcal{L} = \{R_\phi\}$, where R_ϕ is a binary predicate, interpreted in the Euclidean plane $\mathcal{F}^2_{\mathbb{R}}(\phi)$ with the binary relation R_ϕ, defined in 2.1.

Proposition 3.1. *The two predicates R_{ϕ_1} and R_{ϕ_2}, where $\phi_1 = \frac{k}{m}\pi$ and $\phi_2 = \frac{1}{m}\pi$ are equally expressive, where k and m are co-prime integers.*

Proof. Let $\mathcal{L}$ be any language. Denote $\mathcal{L}_1 = \mathcal{L} \cup \{R_{\phi_1}\}$ and $\mathcal{L}_2 = \mathcal{L} \cup \{R_{\phi_2}\}$. Then the predicate R_{ϕ_2} is definable in $\mathcal{L}_1$:

$$R_{\phi_2}(x, y) \iff \exists z_1 \exists z_2 \ldots \exists z_{k_0-1}\big(R_{\phi_1}(x, z_1) \wedge R_{\phi_1}(z_1, z_2) \wedge \cdots \wedge R_{\phi_1}(z_{k_0-1}, y)\big),$$

where k_0 is the unique solution of $kx \equiv 1 \pmod{m}$ in $\{1, 2 \ldots, m\}$.
Similarly, R_{ϕ_1} is definable in $\mathcal{L}_2$ as follows:

$$R_{\phi_1}(x, y) \iff \exists z_1 \exists z_2 \ldots \exists z_{k-1}\big(R_{\phi_2}(x, z_1) \wedge R_{\phi_2}(z_1, z_2) \wedge \cdots \wedge R_{\phi_2}(z_{k-1}, y)\big).$$

□

Remark 3.2. We need to keep in mind that ϕ is not necessarily a rational multiple of π, or also co-measurable with π. This will significantly influence the nature of the theories.

Remark 3.3. From now on, whenever we work with a co-measurable with π angle, we can assume that $\phi = \frac{1}{m}\pi$ for some $m \geq 2$.

Before we continue, we remind the following definitions:

Definition 3.4. Let $\mathcal{L}$ be a language and $\mathcal{F}$ be a structure. We say that a n-ary predicate $S \subseteq F^n$ is *definable* in $\mathcal{F}$ if there is a $\mathcal{L}$-formula $\theta[x_1, x_2, \dots, x_n]$ such that for any $a_1, a_2, \dots, a_n \in F$:

$$(a_1, a_2, \dots, a_n) \in S \text{ if and only if } \mathcal{F} \models \theta[\![a_1, a_2, \dots, a_n]\!]$$

Definition 3.5. Let $\mathcal{L}$ be a language and $\mathcal{F}$ be a structure. We say that a n-ary predicate S is *parametrically definable* in $\mathcal{F}$ if there are a $\mathcal{L}$-formula $\theta[x_1, x_2, \dots, x_n, y_1, y_2, \dots y_s]$ and elements $b_1, b_2, \dots, b_s \in F$, such that for any $a_1, a_2, \dots, a_n \in F$:

$$(a_1, a_2, \dots, a_n) \in S \;\text{ if and only if } \mathcal{F} \models \theta[\![a_1, a_2, \dots, a_n, b_1, \dots, b_s]\!]$$

Proposition 3.6. *The following hold:*
(i) The binary predicate = is not parametrically definable in $\mathcal{F}^2_{\mathbb{R}}(\phi)$.
(ii) For any angle ϕ the predicate $P(a, b)$ is definable in $\mathcal{F}^2_{\mathbb{R}}(\phi)$ by the formula $\forall z(R(z, x) \iff R(z, y))$, i.e

$$P(a, b) \textit{ if and only if } \mathcal{F}^2_{\mathbb{R}}(\phi) \models \forall z(R(z, x) \iff R(z, y))[\![a, b]\!]$$

Proof. (i) Let $a \neq b$ be two parallel lines. For the sake of contradiction suppose that there is a formula $\theta[x_1, \dots, x_n, x, y]$ and lines $a_1, a_2, \dots, a_n$, such that for any a, b:

$$a = b \;\text{ if and only if } \mathcal{F}^2_{\mathbb{R}}(\phi) \models \theta[\![a_1, a_2 \dots, a_n, a, b]\!]$$

We shall prove the following claim:

For any $\mathcal{L}$-formula $\varphi[x_1, x_2, \dots, x_n, y]$, any $n \in \mathbb{N}$ and any lines $c_1, c_2 \dots, c_n$ one can verify that:

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models \varphi[\![c_1, \dots, c_n, a]\!] \iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models \varphi[\![c_1, \dots, c_n, b]\!],$$

whenever a and b are parallel. In words, every two parallel lines have the same type.

To prove this statement we will use induction on the complexity of the formula φ.

- If $\varphi \overset{\circ}{=} R(x, y)$, then we would like to prove the following equivalence: $\mathcal{F}^2_{\mathbb{R}}(\phi) \models R(c_1, a) \iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models R(c_1, b)$. However, this is a direct application of the axiom for corresponding angles and holds for the Euclidean plane. Furthermore, since $\mathcal{F}^2_{\mathbb{R}}(\phi) \models R(a, c_1) \iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models R(b, c_1)$, the position of the parameter c_1 does not matter.

- If $\varphi \overset{\circ}{=} \neg\psi$, then:

$$\begin{aligned}\mathcal{F}^2_{\mathbb{R}}(\phi) \models (\neg\psi)[\![c_1, \dots, c_n, a]\!] &\iff \mathcal{F}^2_{\mathbb{R}}(\phi) \not\models \psi[\![c_1, \dots, c_n, a]\!] \\ &\iff \mathcal{F}^2_{\mathbb{R}}(\phi) \not\models \psi[\![c_1, \dots, c_n, b]\!] \\ &\iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models (\neg\psi)[\![c_1, \dots, c_n, b]\!]\end{aligned}$$

- If $\varphi \overset{\circ}{=} (\varphi_1 \wedge \varphi_2)$ or $\varphi \overset{\circ}{=} (\varphi_1 \vee \varphi_2)$, the steps are similar.

- If $\varphi \overset{\circ}{=} \exists x\, \psi$, where the induction hypothesis holds for $\psi[x, x_1, \dots, x_n, y]$ and $n \geq 1$, then:

$$\begin{aligned}\mathcal{F}^2_{\mathbb{R}}(\phi) \models (\exists x\psi)[\![b_1, b_2, \dots, b_{n-1}, a]\!] &\iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models \psi[\![b_0, b_1, b_2, \dots, b_{n-1}, a]\!] \text{ for some line } b_0 \\ &\iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models \psi[\![b_0, b_1, b_2, \dots, b_{n-1}, b]\!] \text{ for some line } b_0 \\ &\iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models (\exists x\psi)[\![b_1, b_2, \dots, b_{n-1}, b]\!]\end{aligned}$$

As a consequence of the claim, we get that:

$$\begin{aligned}a = a &\iff \theta[\![a_1, a_2, \dots, a_n, a, a]\!] \\ &\iff \theta[\![a_1, a_2, \dots, a_n, a, b]\!] \\ &\iff a = b\end{aligned}$$

Clearly, this is a contradiction with our assumption and the binary predicate = is not parametrically definable.

(*ii*) The direction from left to right follows directly from the axiom for corresponding angles. Now, assume a and b are lines such that $\forall z(R_\phi(z, a) \iff R_\phi(z, b))$ and $\neg P(a, b)$. If $R_\phi(a, b)$, then take z to be the line a. Then, from $R_\phi(a, b)$ it must follow that $R_\phi(a, a)$ which contradicts with the irreflexivity of R_ϕ. Similarly, when $R_\phi(b, a)$, take z to be b. Now, if $\measuredangle(a, b) = \psi \neq \phi$, take z be the line c for which $R_\phi(c, a)$. Then $\measuredangle(c, b) = \phi + \psi \in (\phi, \phi + \pi)$. Thus, $\neg R_\phi(c, b)$ - a contradiction. Thus, the predicate P is always definable.

□

Remark 3.7. From now on in section 4 we shall consider the language $\mathcal{L}(P, R, =)$ as the extension of $\mathcal{L}(R, =)$ by definitions, namely the binary predicate symbol P is associated with the axiom:

$$\forall x \forall y (P(x, y) \iff \forall z (R(z, x) \iff R(z, y)))$$

4 Expressivity of $L(=, R)$

4.1 Definable predicates in the language in $\mathcal{F}^2_{\mathbb{R}}(\phi)$:

1. When ϕ is co-measurable with π, then

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models \forall x \forall y (P(x,y) \iff \exists x_1 \exists x_2 \dots \exists x_{m-1}(R(x,x_1) \wedge R(x_1,x_2) \wedge \dots \wedge R(x_{m-1},y)))$$

2. If $k \in \mathbb{N} \setminus \{0\}$, then:

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models \forall x \forall y (R_{k\phi}(x,y) \iff \exists x_1 \exists x_2 \dots \exists x_{k-1}(R(x,x_1) \wedge R(x_1,x_2) \wedge \dots \wedge R(x_{k-1},y)))$$

3. If $k \in \mathbb{N} \setminus \{0\}$, then: $\mathcal{F}^2_{\mathbb{R}}(\phi) \models \forall x \forall y (R_{-k\phi}(x,y) \iff R_{k\phi}(y,x))$

In order to see the expressive power of this language, one should verify whether one can define points. However, if that was possible, then the following predicate would have been definable in $\mathcal{F}^2_{\mathbb{R}}(\phi)$. On the contrary, we get that:

4.2 Non-definable predicates in the language in $\mathcal{F}^2_{\mathbb{R}}(\phi)$:

Proposition 4.1. *The predicate* Co *is not definable in* $\mathcal{F}^2_{\mathbb{R}}(\phi)$.

Proof. Suppose that there exists a formula φ such that:

$$\mathrm{Co}(a,b,c) \text{ if and only if } \mathcal{F}^2_{\mathbb{R}}(\phi) \models \varphi(x,y,z)[\![a,b,c]\!]$$

Fix two different parallel lines - a_0 and a_1. Define the map $h : \mathcal{F}^2_{\mathbb{R}}(\phi) \to \mathcal{F}^2_{\mathbb{R}}(\phi)$, as follows:

$$h(a) = \begin{cases} a_0, & \text{if } a = a_1 \\ a_1, & \text{if } a = a_0 \\ a, & \text{otherwise} \end{cases}$$

In order to prove that h is an automorphism, we need to check whether it preserves the validity of R, namely:

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models R(b,c) \iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models R(h(b),h(c))$$

Let $R_\phi(b,c)$. If b and c are different from a_0 and a_1, then it holds that $R_\phi(h(b),h(c))$. Obviously, it is impossible for both b and c to be in $\{a_0,a_1\}$, so assume that $b = a_1$. By construction $h(b) = h(a_1) = a_0$ and by the axiom for the corresponding angles we know $R_\phi(a_1,c) \iff R_\phi(a_0,c)$, so $R_\phi(h(b),h(c))$ holds. Any symmetrical situation can the tackled similarly.
Now, let $R_\phi(h(b),h(c))$. As before, if both b and c are not in $\{a_0,a_1\}$, then $R_\phi(b,c)$ holds. The other case is the same as before.

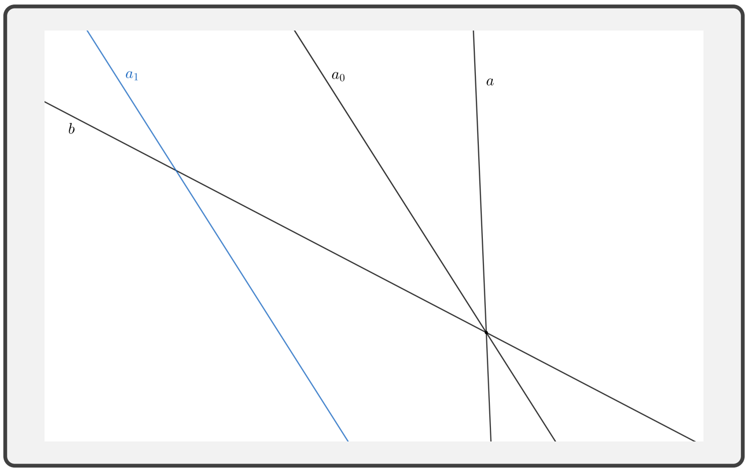

Figure 1

Let a, b be lines such that $R_\phi(a, a_0)$, $R_\phi(a_0, b)$ and $\mathrm{Co}(a_0, a, b)$ hold as shown in the picture: On the other hand:

$$\begin{aligned} \mathrm{Co}(a_0, a, b) &\iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models \phi[\![a_0, a, b]\!] \\ &\iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models \phi[\![h(a_0), h(a), h(b)]\!] \\ &\iff \mathrm{Co}(a_1, a, b) \end{aligned}$$

However, this is a clear contradiction, thereby we get that Co is not expressible. □

We could have also used the statement from Prop. 3.6, namely that any two parallel lines have the same type. However, if a and b are parallel, it does not hold that a and b have the same type in the language with the predicate Co.

4.3 Axiomatization:

Our next goal is to axiomatize the theory of the Euclidean plane in this language. In order to do this, we denote by Σ the following set of axioms . Note that P is used as definable predicate symbol as λ_7 shows:

$\lambda_{1,n}:\ \forall x_1\forall x_2\dots\forall x_n\exists y(\neg P(x_1,y)\wedge\neg P(x_2,y)\cdots\wedge\neg P(x_n,y))$

$\lambda_{2,n}:\ \forall x_1\dots\forall x_n\exists y(P(x_1,x_2)\wedge\cdots\wedge P(x_{n-1},x_n)\implies P'(x_1,y)\wedge\cdots\wedge P'(x_n,y))$

$\lambda_3:\quad \forall x\forall y(P(x,y)\implies\neg R(x,y))$

$\lambda_4:\quad \forall x\forall y(R(x,y)\implies\neg R(y,x))$

$\lambda_5:\quad \forall x\exists y R(x,y)$

$\lambda_6:\quad \forall x\exists y R(y,x)$

$\lambda_7:\quad \forall x\forall y(P(x,y)\iff\forall z(R(z,x)\implies R(z,y)))$

$\lambda_8:\quad \forall x\forall y(P(x,y)\implies\forall z(R(x,z)\implies R(y,z)))$

$\lambda_9:\quad \forall x\forall y\forall z(R(x,y)\wedge R(x,z)\implies P(y,z))$

$\lambda_{10}:\quad \forall x\forall y\forall z(R(y,x)\wedge R(z,x)\implies P(y,z))$

Remark 4.2. It should be clear that the following three sentences are derivable:

$\lambda_{11}:\quad \forall x(P(x,x))$

$\lambda_{12}:\quad \forall x\forall y(P(x,y)\implies P(y,x))$

$\lambda_{13}:\quad \forall x\forall y\forall z(P(x,y)\wedge P(y,z)\implies P(x,z))$

The theory containing Σ and the m axioms $\lambda_{rat,1},\dots,\lambda_{rat,m}$ will be called $LRot^m$, where:

$\lambda_{rat,k}:\quad \forall x\forall x_1\forall x_2\dots\forall x_k(R(x,x_1)\wedge R(x_1,x_2)\cdots\wedge R(x_{k-1},x_k)\implies\neg P(x,x_k))\qquad 1\leq k\leq m-1$

$$\lambda_{rat,m}:\quad \forall x\forall x_1\forall x_2\dots\forall x_m(R(x,x_1)\wedge R(x_1,x_2)\cdots\wedge R(x_{m-1},x_m)\implies P(x,x_m))$$

The theory containing Σ and the axiom scheme $\lambda_{irrat,k}$ will be called $LRot^\infty$, where:

$\lambda_{irrat,k}:\ \forall x\forall x_1\forall x_2\dots\forall x_k(R(x,x_1)\wedge R(x_1,x_2)\cdots\wedge R(x_{k-1},x_k)\implies\neg P(x,x_k))\qquad ,k\geq 1$

Remark 4.3. For every model $\mathcal{M}$ of Σ the last three axioms $\lambda_{11},\lambda_{12},\lambda_{13}$ imply that $P^{\mathcal{M}}$ is an equivalence relation on M. Moreover, $P^{\mathcal{M}}$ is a congruence as follows from λ_7 and λ_8.

Definition 4.4. Let $\mathcal{M}$ be a model of Σ. For every $a\in M$ we will use the following notation:

$$[a]_P=\{b\in M\mid P^{\mathcal{M}}(a,b)\}$$

Proposition 4.5. *Every structure $\mathcal{M}$ such that $\mathcal{M}\models\Sigma$ is divided by the relation $P^{\mathcal{M}}$ into an infinite number of equivalence classes. Furthermore, each P-equivalence class is infinite.*

Proof. Assume that there is a finite number of P-equivalence classes, namely $[a_1]_P, [a_2]_P, \dots [a_n]_P$. Then since $\mathcal{M} \models \lambda_{1,n}$, there is an element b such that $b \notin [a_i]_P$ for any $i \in \{1, \dots, n\}$. Thus, $[b]_P \neq [a_i]_P$, which is a contradiction.

For the second claim assume that there is a class $[a_1]_P$ with finite number of elements $a_1, a_2, \dots, a_k$. Since $\mathcal{M} \models \lambda_{2,k}$, and $a_1, a_2, \dots a_k$ are mutually parallel, then there is an element b such that $P'(a_i, b)$ for all $i \in \{1, \dots, k\}$. This implies that $b \neq a_i$ for any $i \in \{1, \dots, k\}$, which is a contradiction.

□

In order to prove that this is a proper candidate for an axiomatization, we start with the following claim:

Proposition 4.6. *The following are true:*
(i) If ϕ is co-measurable with π, then $\mathcal{F}^2_{\mathbb{R}}(\phi) \models LRot^m$
(ii) If ϕ is not co-measurable with π, then $\mathcal{F}^2_{\mathbb{R}}(\phi) \models LRot^\infty$

Proof. We will briefly comment on the axioms, since anyone familiar with the Euclidean plane can verify them quite easily. It is clear that one can find an infinite set with lines non-parallel to each other. In the same way one can construct infinitely many parallel lines. In the case with rational angle after rotating a line m times one will get a parallel line, and in the case of irrational angle one will never obtain a parallel line no matter the number of rotations. λ_3 eliminates lines that are both P and R related. λ_4 guarantees that R is asymmetric. λ_5 and λ_6 provide seriality. From λ_7 to λ_{10} are the well-known axioms for corresponding angles and the last three state that P is an equivalence relation. □

Definition 4.7. Let $\mathcal{M}$ be a model for Σ and $a \in M$. For any $i \in \mathbb{Z}$ denote:
1) $[a]^0_{\mathcal{M}} := [a]_P$
2) $[a]^i_{\mathcal{M}} := [b]_P$, where $R_{i\phi}(a, b)$ and $i \neq 0$;

Remark 4.8. Definition 4.7 is correct.

Proof. We need to verify that $[a_1]_P = [a_2]_P$ for any two elements $a_1, a_2 \in M$, such that $R_{i\phi}(a, a_1)$ and $R_{i\phi}(a, a_2)$. Clearly $[a]^0_{\mathcal{M}}$ is correctly defined. First, let $i > 0$. Assume, that $[a]^{i-1}_{\Updownarrow} = [c]_P$, where $R_{(i-1)\phi}(a, c)$. By the properties of $R_{i\phi}$ there are elements c_1 and c_2, such that $R_{(i-1)\phi}(a, c_1) \wedge R_\phi(c_1, a_1)$ and $R_{(i-1)\phi}(a, c_2) \wedge R_\phi(c_2, a_2)$. From the assumption we know that $P(c_1, c_2)$. Applying axiom λ_8 we get $R_\phi(c_2, a_1)$. Then, applying axiom λ_9 we get that $P(a_1, a_2)$. Thus, by induction on i, we prove that $[a_1]_P = [a_2]_P$. For $i < 0$ we follow the same idea but we use axioms λ_7 and λ_{10}. □

Definition 4.9. Let $\mathcal{M}$ be a model of Σ. For any element $a \in M$ the set $O_{\mathcal{M}}(a) = \{[a]^i_{\mathcal{M}} \mid i \in \mathbb{Z}\}$ will be called *the orbit* of a.

Definition 4.10. Let $\mathcal{M}$ be a model of Σ. Define the binary relation $\sim_{\mathcal{M}}$ as follows:

$$a \sim_{\mathcal{M}} b \iff O_{\mathcal{M}}(a) = O_{\mathcal{M}}(b)$$

Remark 4.11. For the sake of brevity when the model is unique we will omit the index $\mathcal{M}$ in $[a]^i_{\mathcal{M}}$, $O_{\mathcal{M}}(a)$ and $\sim_{\mathcal{M}}$.

4.4 Completeness of $LRot^m$

Let us investigate some basic properties of the orbits necessary for the proving the completeness of $LRot^m$.

Proposition 4.12. *Let $\mathcal{M}$ be a model for $LRot^m$ and let $a \in M$. Then:*

1. *for every integer i the set $[a]^i$ is infinite.*
2. *for all integers i and j, such that $0 \leq i < j \leq m$ it holds that $[a]^i \cap [a]^j = \emptyset$.*
3. $[a]^0 = [a]^m$
4. *for any $b \in M$ it holds that:*

$$b \notin \bigcup O(a)$$

$$\iff$$

$$\mathcal{M} \models \neg P(x, y) \wedge \neg R(x, y) \wedge \bigwedge_{k=2}^{m-1} (\neg(\exists x_1 \dots \exists x_{k-1}(R(x, x_1) \wedge R(x_1, x_2) \wedge R(x_{k-1}, y)))[\![a, b]\!]$$

5. *for all $b \in M$ it holds that:*

$$O(a) = O(b) \iff O(a) \cap O(b) \neq \emptyset \iff \bigcup O(a) \cap \bigcup O(b) \neq \emptyset$$

Remark 4.13. For any model $\mathcal{M}$ and any $a \in M$ the set $O(a)$ with operation $[a]^i \mapsto [a]^{i+1}$ can be considered as a cyclic group of order m.

Lemma 4.14. *Let $\mathcal{M}$ be a countable model of $LRot^m$. Then for any i:*

1. *$[a]^i$ is countable;*
2. *$M/\sim$ is countable;*

Proof. Both follow from Prop. 4.5, i.e from axioms $\lambda_{1,n}$ and $\lambda_{2,n}$. □

Theorem 4.15. *The theory $LRot^m$ is ω-categorical.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be two countable models of $LRot^m$. Let $O_{\mathcal{M}}(a_0), O_{\mathcal{M}}(a_1), \dots, O_{\mathcal{M}}(a_n), \dots$ and $O_{\mathcal{N}}(b_0), O_{\mathcal{N}}(b_1), \dots, O_{\mathcal{N}}(b_n), \dots$ be enumerations respectively of $M/\sim_{\mathcal{M}}$ and $N/\sim_{\mathcal{N}}$.

Let $n < \omega$. Then:

$$O(a_n) = \{[a_n]^i \mid 0 \leq i < m\}$$

$$O(b_n) = \{[b_n]^i \mid 0 \leq i < m\}$$

Let $0 \leq i < m$ and let $h_{n,i}$ be a bijection between $[a_n]^i$ and $[b_n]^i$. Define

$$h := \bigcup_{n<\omega} \bigcup_{0 \leq i < m} h_{n,i}$$

Clearly, h is a bijective map. We need to verify that this is indeed isomorphism, or that:

$$\langle s, t \rangle \in R^{\mathcal{M}} \iff \langle h(s), h(t) \rangle \in R^{\mathcal{N}}$$

" $\implies$ " Let $s \in \bigcup O_{\mathcal{M}}(a_i)$. Clearly, $s \in \bigcup O_{\mathcal{M}}(s)$, so $\bigcup O_{\mathcal{M}}(s) \cup \bigcup O_{\mathcal{M}}(a_i) \neq \emptyset$, so by 4.11.5: $O_{\mathcal{M}}(s) = O_{\mathcal{M}}(a_i)$.

Since $\mathcal{M} \models R_\phi(s, t)$, then 4.11.4 implies that $t \in \bigcup O_{\mathcal{M}}(s)$, so $t \in \bigcup O_{\mathcal{M}}(a_i)$. Furthermore, if $s \in [a_i]^j_{\mathcal{M}}$, then $t \in [a_i]^{j+1}_{\mathcal{M}}$. Thus, $h(s) \in [b_i]^j_{\mathcal{N}}$ and $h(t) \in [b_i]^{j+1}_{\mathcal{N}}$. Therefore, $\langle h(s), h(t) \rangle \in R^{\mathcal{N}}$.

The reverse direction is analogous. □

As a consequence of the Vaught test and Prop. 4.6 , we get:

Theorem 4.16. *The theory $LRot^m$ is complete. Moreover, $LRot^m = Th(\mathcal{F}^2_{\mathbb{R}}(\phi))$ where $\phi = \frac{1}{m}\pi$.*

Now, it is time to inspect the other theory.

4.5 Completeness of $LRot^\infty$

We denote by $\mathcal{S}$ the following structure for $\mathcal{L}(=, R)$:

- the universe is $\mathbb{Z} \times \mathbb{Z}$
- for all $\langle s_1, t_1\rangle, \langle s_2, t_2\rangle \in \mathbb{Z} \times \mathbb{Z}$ it holds that:

$$\langle s_1, t_1\rangle, \langle s_2, t_2\rangle \in R^{\mathcal{S}} \iff s_2 = s_1 + 1$$

Proposition 4.17. *$\mathcal{S}$ has the following properties:*

1. $\mathcal{S} \models \forall z(R(x, z) \iff R(y, z))[\![\langle s_1, t_1\rangle, \langle s_2, t_2\rangle]\!]$ *if and only if* $s_1 = s_2$
2. $\mathcal{S} \models LRot^\infty$
3. $[\langle 0, 0\rangle]^0_{\mathcal{S}} = \{\langle 0, t\rangle \mid t \in \mathbb{Z}\}$ *and* $[\langle 0, 0\rangle]^i_{\mathcal{S}} = \{\langle i, t\rangle \mid t \in \mathbb{Z}\}$ *for* $i \neq 0$.
4. $O_{\mathcal{S}}(\langle 0, 0\rangle) = \mathbb{Z} \times \mathbb{Z}$.

Proof. By straightforward verification. □

Definition 4.18. A countable model of $LRot^\infty$ that consists of the orbit of only one line will be called *star model.*

It should be clear that the star model is unique up to isomorphism.

We will now prove that for each model $\mathcal{M}$ of $LRot^{irrat}$ it holds that $\mathcal{S} \equiv_n \mathcal{M}$ for each n, and thus it would follow, that $\mathcal{S} \equiv \mathcal{M}$.

Proposition 4.19. *Let $\mathcal{M}$ be a model for $LRot^\infty$. For all integers n Abelard has a winning strategy for the Ehrenfeucht-Fraïssé game with length n played on S and M.*

Proof. Consider a Ehrenfeucht-Fraïssé game with length n played on S and M. *Dangerous zone* for an element x when k rounds are remaining to the end of the game will be called the set of elements $[x]^i$, where $-2^{k-1} \leq i \leq 2^{k-1}$. *Vicinity* of a line x will be called the set of elements $[x]^i$, where $-2^k \leq i \leq 2^k$.

A move will be called *dangerous* if the newly chosen element is in the dangerous zone of a previously picked element. A sequence of moves will be called *dangerous* if each element has been picked from the dangerous zone of a previous one. Take into account, that the vicinity of a element contains all the possible dangerous sequences starting with this element, since $\sum_{i=0}^{k-1} 2^i < 2^k$.

Consider the i-th round of the game. Let $s_1, s_2, \ldots, s_{i-1}$ be the elements that have been chosen from the star model up to this moment of the game and $m_1, m_2, \ldots, m_{i-1}$- the corresponding elements from $\mathcal{M}$. Let Eloise choose an element x on the i-th round.

The strategy for Abelard should be executed consecutively. In other words, each step is applied only when the conditions for the previous one do not hold.

If x is from the star model, consider the following strategy for Abelard:

- If $x = s_j$ for some $1 \leq j \leq i-1$, then Abelard chooses the element m_j.
- If $P'(x, s_j)$ for some j, $1 \leq j \leq i-1$, and $x \notin \{s_1, s_2, \ldots, s_{i-1}\}$, then Abelard chooses a element y such that $P'(y, m_j)$ and $y \notin \{m_1, m_2, \ldots, m_{i-1}\}$.
- If x is in the dangerous zone of the element s_j, then Abelard must find the orbit $O(m_j)$ and find the element that relates to m_j the same way as x to s_j, i.e since there is a unique t such that $R_{t\phi}(s_j, x)$, then Abelard chooses y so that $R_{t\phi}(m_j, y)$.
- If x is not in the dangerous zone of any element s_j, $1 \leq j \leq i-1$, then Abelard can choose one arbitrary orbit, say $O(m_1)$ and take the first element that is not in the vicinity of any previously chosen line from $\bigcup O(m_1)$. Such element exists since the union of vicinities of finite number of elements is finite set.

If x is from $\mathcal{M}$, consider the following strategy for Abelard:

- If $x = m_j$ for some j, such that $1 \leq j \leq i-1$, then Abelard chooses the element s_j.
- If $P'(x, m_j)$ for some j, such that $1 \leq j \leq i-1$ and $x \notin \{m_1, m_2, \ldots, m_{i-1}\}$, then Abelard chooses a element y such that $P'(y, s_j)$ and $y \notin \{s_1, s_2, \ldots, s_{i-1}\}$.
- If x is in the dangerous zone of the element m_j, then Abelard must find the element from S that relates to s_j the same way as x to m_j, since there is a unique t such that $R_{t\phi}(m_j, x)$, then Abelard chooses y so that $R_{t\phi}(s_j, y)$
- If x is not in the dangerous zone of any element m_j, $1 \leq j \leq i-1$, then Abelard should do the following: since $s_j \in \bigcup O(s_1)$ for each j, then $s_j \in [s_1]^t$ for some t. Abelard finds the maximum t, say it belongs to s_{j_0}, and takes the first element that is out the **vicinity** of s_{j_0}.

Now, we will prove simultaneously that:
(i) Abelard could always execute his strategy;
(ii) The two generated models at any round of game are isomorphic, which indicates that his strategy is a winning strategy.

We will prove (i) and (ii) by induction. Assume i rounds of the game are remaining:

Proof of (i)*:* Assume that Eloise picks an element x and Abelard cannot follow his strategy. Then it is clear that x is a new element that is not parallel to the previous ones and it must be in the dangerous zone of some element. However, if m_i and m_j are from different orbits, then the dangerous zones of their corresponding s_i and s_j do not overlap. Thus, if Eloise chooses a element from the dangerous zone of l elements, then they all must be from one orbit and Abelard knows that there is only one orbit to choose from. Furthermore, that element cannot have already been chosen, since the dangerous zones of every moves grow smaller.

Proof of (ii): Let S' and M' be the substructures of S and M respectively chosen after all k rounds of the game. Then the isomorphism between them is the correspondence between the chosen elements on each move.

- It is clear that $s_i = s_j \iff m_i = m_j$.
- If $R(s_i, s_j)$ or $R(s_j, s_i)$ is true in S for some j less than i, then on the j-th move, s_j has been in the dangerous zone of s_i, therefore due to the strategy of Abelard m_i and m_j would be in the same orbit and $R(m_i, m_j)$. The reverse holds as well.

□

This theorem implies that any two models of $LRot^\infty$ are elementary equivalent, so we can conclude the following:

Theorem 4.20. *The theory $LRot^\infty$ is complete. Moreover, $LRot^\infty = Th(\mathcal{F}^2_{\mathbb{R}}(\phi))$ when ϕ is not co-measurable with π.*

4.6 Categoricity

Proposition 4.21. *Let α, β be infinite cardinals such that $\alpha > \omega$ and $\beta \geq \omega$. Then:*
(i) $LRot^m$ is not α-categorical.
(ii) $LRot^\infty$ is not β-categorical.

Proof. (i) Consider two models of $LRot^m$ $\mathcal{M}$ and $\mathcal{N}$, both with universe $\alpha \times \mathbb{Z} \times \mathbb{Z}_m$ and the following interpretations of the predicate R:

$$((a_1, b_1, c_1), (a_2, b_2, c_2)) \in R^{\mathcal{M}} \iff a_1 = a_2 \text{ and } c_1 + \overline{1} = c_2$$

$$((a_1, b_1, c_1), (a_2, b_2, c_2)) \in R^{\mathcal{N}} \iff b_1 = b_2 \text{ and } c_1 + \overline{1} = c_2$$

Both models have cardinality α. Consider the equivalence relation $P^{\mathcal{M}}$ and $P^{\mathcal{N}}$. In M there are α equivalence classes, each with cardinality ω, and in N there are ω equivalence classes, each with cardinality α. Thus, $\mathcal{M} \ncong \mathcal{N}$.

(ii) Similarly, for $LRot^\infty$ consider $\mathcal{M}$ and $\mathcal{N}$ with universes respectively $\beta \times \mathbb{Z} \times \mathbb{Z}$ and $\{1\} \times \beta \times \mathbb{Z}$.

$$((a_1, b_1, c_1), (a_2, b_2, c_2)) \in R^{\mathcal{M}} \iff a_1 = a_2 \text{ and } c_1 + 1 = c_2$$

$$((1, b_1, c_1), (1, b_2, c_2)) \in R^{\mathcal{N}} \iff c_1 + 1 = c_2$$

The first coordinate shows the number of different orbits, the second the cardinality of each P- equivalence class and the third one- the cardinality of the orbit. Thus, we can conclude that

$\mathcal{M}$ consists of β orbits each with cardinality ω and $\mathcal{N}$ consists of one orbit with cardinality β. □

4.7 Finite axiomatization

We shall present two proofs that the theories are not finitely axiomatizable. The first one is an application of a classic technique relying on the compactness theorem, while the second one has more interesting corollaries.

Proposition 4.22. *Let Γ be an infinite set of sentences with the following property: For each finite subset Γ_0 there is a structure $\mathcal{A}$ such that $\mathcal{A} \models \Gamma_0$ but $\mathcal{A} \not\models \Gamma$. Then, Γ is not finitely axiomatizable.*

Proof. Assume that ψ is a sentence axiomatizing Γ. Then :

$$\forall \mathcal{M}(\mathcal{M} \models \psi \iff \mathcal{M} \models \Gamma)$$

Therefore, $\Gamma \models \psi$ and from the compactness theorem there is a finite $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \models \psi$. However, there is a model $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \Gamma_0$ and $\mathcal{M}_0 \not\models \Gamma$. The first one implies that $\mathcal{M}_0 \models \psi$ and the second - $\mathcal{M}_0 \not\models \psi$. Contradiction. □

We can apply the previous proposition for the set Σ. For each finite subset of Σ we need to construct a model $\mathcal{M}$ such that $\mathcal{M} \models \Sigma_0$ but $\mathcal{M} \not\models \Sigma$.

Proposition 4.23. *The set of axioms Σ has the property that for each finite subset of Σ we need to construct a model $\mathcal{M}$ such that $\mathcal{M} \models \Sigma_0$ but $\mathcal{M} \not\models \Sigma$.*

Proof. Let Σ_0 be a finite subset of Σ. Then, there is a $n \in \mathbb{N}$ such that for each number $m > n$: $\lambda_{2,m} \notin \Sigma_0$. In other words, we can construct a model $\mathcal{M}$ such that each line has exactly n parallel lines. We preserve all other properties of $\mathcal{F}^2_{\mathbb{R}}(\phi)$. Clearly, $\mathcal{M} \not\models \Sigma$. □

Theorem 4.24. *Both $LRot^m$ and $LRot^\infty$ are not finitely axiomatizable.*

Now, we restate the previous theorem and present a second proof:

Theorem 4.25. *$LRot^m$ is not finitely axiomatizable.*

Proof. For the sake of contradiction assume that there is finite set of axioms Λ such that for every model $\mathcal{M}$:

$$\mathcal{M} \models \Lambda \iff \mathcal{M} \models LRot^m$$

Let n be the maximal rank of a formula in Λ and let $k = n + 1$. We are going to construct a finite model $\mathcal{M}$, such that $\mathcal{M} \equiv_k \mathcal{F}^2_{\mathbb{R}}(\phi)$. Then it would follow that $\mathcal{M} \models \Lambda$, which from the assumption would imply that $\mathcal{M} \models LRot^m$. However, no finite structure could model $LRot^m$.

Let $I = \{1, \dots k-1, k\}$ and $A = \{0, 1, \dots, m-1\}$. Define the structure $\mathcal{M}$ with universe $M := I \times A \times I$, with the following interpretation of R and P:

$$\mathcal{M} \models (a_1, a_2, a_3) = (b_1, b_2, b_3) \iff a_1 = b_1 \wedge a_2 = b_2 \wedge a_3 = b_3$$

$$\mathcal{M} \models R_{l\phi}((a_1, a_2, a_3), (b_1, b_2, b_3)) \iff b_2 = a_2 + l \qquad ,1 \leq l \leq m$$

$$\mathcal{M} \models P((a_1, a_2, a_3), (b_1, b_2, b_3)) \iff a_1 = b_1 \wedge b_2 = a_2$$

Let us consider the i-th move in the Ehrenfeucht-Fraïssé game with length k, played by Eloise and Abelard on $\mathcal{M}$ and $\mathcal{F}^2_{\mathbb{R}}(\phi)$. Let $s_1, s_2, \dots, s_{i-1}$ be the lines that have been chosen from $\mathcal{F}^2_{\mathbb{R}}(\phi)$ up to this moment of the game and $m_1, m_2, \dots, m_{i-1}$- the tuples from $\mathcal{M}$. Let Eloise choose a element x on the i-th move.

If x is from $\mathcal{M}$, consider the following strategy for Abelard:

- If $x = m_i$, then Abelard chooses s_i.
- If x is parallel to m_i, then Abelard chooses a parallel line to s_i.
- If x is in the same orbit as m_i and it holds $R_{l\phi}(m_i, x)$, then Abelard chooses one line y, such that $\angle(s_i, y) = l\phi$
- In all other cases, Abelard chooses a random line different from the orbits of the already chosen ones.

If x is from $\mathcal{F}^2_{\mathbb{Q}}$, consider the following strategy for Abelard:

- If $x = s_i$, then Abelard chooses m_i.
- If x is parallel to s_i, then Abelard chooses a parallel to m_i line.
- If x is in the same orbit as s_i and it holds $R_{l\phi}(s_i, x)$, then Abelard chooses one line y, such that $R_{l\phi}(m_i, y)$.
- In all other cases, Abelard chooses a tuple with different first coordinate from these of the already chosen tuples.

□

Now, we must prove that:
(i) Abelard could always execute his strategy;
(ii) The two generated models at the end of the game are isomorphic;

Remark 4.26. In (*) all coefficients are reduced modulo m.

Proof of *(i)*: The only case when Abelard would not be able to make a move would be if there are not enough lines in $\mathcal{M}$. However, the length of the game is k and $|M| = k.m.k$.

Proof of *(ii)*: Let F be the generated submodel of $\mathcal{F}^2_{\mathbb{R}}(\phi)$ by $s_1, \dots, s_k$. Now, if $R(m_i, m_j)$, where $i < j$, then s_j must have been from the same orbit as s_i and $R(s_i, s_j)$.

4.8 Complexity of the membership problem for $LRot^m$

From the proof of the previous theorem we can infer that the problem "does a sentence φ belong to $LRot^m$" is in $PSPACE$. This is true because for each sentence φ with rank k one can find a finite model with size which is $O(k^2)$.

However, we present a second proof that makes a correspondence between the formulas in $\mathcal{L}(=,R)$ and the formulas in $\mathcal{L}(=)$

Let us denote by $\mathcal{F}$ the set of lines from $\mathcal{F}^2_{\mathbb{R}}(\phi)$ that are not in the orbits in the two axis lines and do not pass through the center of the plane. Denote $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$

Proposition 4.27. *$\mathcal{F} \models LRot^m$.*

Proof. $\mathcal{F} \models \lambda_{1,n} \wedge \lambda_{2,n}$ since we have removed only two whole orbits and from all the other orbits we have removed finite number of lines.

Since $\mathcal{F}$ is a subset of $\mathcal{F}^2_{\mathbb{R}}(\phi)$, it models all $\forall$-axioms.

Lastly, $\mathcal{F} \models \lambda_5 \wedge \lambda_6$ because we have removed either whole orbits or finitely many lines.

□

For every $\mathcal{L}(=,R)$-formula φ our goal to find a formula $\hat{\varphi}$ from $\mathcal{L}(=)$ such that:

$$\mathcal{F} \models \varphi \iff \mathbb{R}_0 \models \hat{\varphi}$$

In $\mathcal{F}$ every line g has equation of the form $y = ax + c$ with $a, c \in \mathbb{R}_0$ and we can match it with a sequence $(a, a', \dots, a^{(m-1)}, c)$ with length $m+1$, where $a^{(i)}$ is the coefficient in front of x of the line obtained by rotating i times the line g.

In order to construct $\hat{\varphi}$, we need to define the following predicates in $\mathcal{L}(=)$:

1. $\text{line}(a_1, a_2, \dots, a_m, c) \iff \neg \bigvee_{1 \le i < j \le m} (a_i = a_j)$
   In order for one $(m+1)$-tuple to be a line, the first m coefficients must be different.
2. $\text{common}_{i,j}(a_1, a_2, \dots, a_m, c, b_1, b_2 \dots, b_m, d) \iff a_i = b_j$
   Here we define m^2 predicates of this type, $1 \le i, j \le m$.
3. $\hat{R}(a_1, a_2, \dots, a_m, c, b_1, b_2 \dots, b_m, d) \iff a_2 = b_1 \wedge a_3 = b_2 \wedge \dots \wedge a_1 = b_m$
   We shift the first m coefficients one to the left in order to preserve the structure of the orbit.
4. $\hat{R}_{k\phi}(a_1, a_2, \dots, a_m, c, b_1, b_2 \dots, b_m, d) \iff a_{1+k} = b_1 \wedge a_{2+k} = b_2 \wedge \dots \wedge a_k = b_m,$
   Here all indices here are reduced modulo m and $-m \le k \le m$;
5. $(a_1 \dots, a_m, c) = (b_1, \dots, b_m, d) \iff \bigwedge_{1 \le i \le m} a_i = b_i \wedge c = d$
6. $P(a_1 \dots, a_m, c, b_1, \dots, b_m, d) \iff \bigwedge_{1 \le i \le m} a_i = b_i$

Let $x_1, x_2, \dots$ be an enumeration of the variables. For better visibility with $\overline{x_i}$ we will abbreviate the sequence $x_{m.i}, x_{m.i+1} \dots, x_{m.i+i-1}$. The construction of the formula $\hat{\varphi}$ is given below:

If $\varphi \overset{\circ}{=} (x_i = x_j)$, then $\hat{\varphi} \overset{\circ}{=} \overline{x_i} = \overline{x_j} \wedge \text{line}(\overline{x_i})$.

If $\varphi \overset{\circ}{=} R(x_i, x_j)$, then $\hat{\varphi} \overset{\circ}{=} \hat{R}(\overline{x_i}, \overline{x_j}) \wedge \text{line}(\overline{x_i}) \wedge \text{line}(\overline{x_j})$.

If $\varphi \overset{\circ}{=} \varphi_1 \wedge (\vee)\varphi_2$, then $\hat{\varphi} \overset{\circ}{=} \hat{\varphi}_1 \wedge (\vee)\hat{\varphi}_2$.

If $\varphi \overset{\circ}{=} \neg\psi[x_1, \dots, x_n]$, then

$$\hat{\varphi} \overset{\circ}{=} \neg\hat{\psi} \wedge \text{line}(\overline{x_1}) \wedge \cdots \wedge \text{line}(\overline{x_n}) \wedge \bigwedge_{1 \le s,t}^{n} \bigwedge_{1 \le i,j}^{m} (\text{common}_{i,j}(\overline{x_t}, \overline{x_s}) \implies \hat{R}_{j-i}(\overline{x_t}, \overline{x_s}))$$

If $\varphi \overset{\circ}{=} \exists x_l \psi$, where $\psi[x_l, x_1, \dots, x_n]$ and $l > n$, then

$$\hat{\varphi} \overset{\circ}{=} \exists x_{ml} \exists x_{ml+1} \dots \exists x_{ml+m-1} \big(\text{line}(\overline{x_l}) \wedge \bigwedge_{s=1}^{n} \bigwedge_{1 \le i,j}^{m} (\text{common}_{i,j}(\overline{x_l}, \overline{x_s}) \implies \hat{R}_{j-i}(\overline{x_l}, \overline{x_s})) \wedge \hat{\psi} \big)$$

Remark 4.28. The conjunction $\bigwedge_{s=1}^{n} \bigwedge_{1 \le i,j}^{m} (\text{common}_{i,j}(\overline{x_l}, \overline{x_s}) \implies \hat{R}_{j-i}(\overline{x_l}, \overline{x_s}))$ is necessary in order to preserve the equivalence classes of the relation O.

Proposition 4.29. *Any set of lines $g_1, g_2, \dots, g_n$ satisfy the formula:*

$$\bigwedge_{1 \le s,t}^{n} \bigwedge_{1 \le i,j}^{m} (\text{common}_{i,j}(\overline{g_t}, \overline{g_s}) \implies \hat{R}_{j-i}(\overline{g_t}, \overline{g_s}))$$

Proposition 4.30. *For any $\mathcal{L}(=, R)$-formula $\varphi[x_1, \dots, x_n]$ it holds that:*

$$\mathcal{F} \models \varphi \iff \mathbb{R}_0 \models \hat{\varphi}$$

Proof. " $\implies$ " We are going to use induction on the formula φ proving a stronger claim, namely that if $\mathcal{F} \models \varphi[\![g_1, g_2 \dots, g_n]\!]$ for some lines $g_1, g_2, \dots, g_n$, then $\mathbb{R}_0 \models \hat{\varphi}[\![\overline{g_1}, \dots, \overline{g_n}]\!]$, where $\overline{g_i}$ is the corresponding sequence of g_i.

- Let $\varphi \overset{\circ}{=} R(x_i, x_j)$ and assume that $\mathcal{F} \models R(x_i, x_j)[\![g, h]\!]$ for some lines g and h with equations: $y = g_1 x + c_1$ and $y = h_1 x + c_2$. Then, if the corresponding sequence to g is $(g_1, g_2, \dots, g_m, c_1)$, the corresponding sequence to h will be $(g_2, g_3 \dots, g_1, c_2)$, where $g_1, g_2, \dots, g_m$ are different non-zero real numbers. Thus, $\mathbb{R}_0 \models \hat{R}(g_1, \dots, g_m, c_1, g_2, \dots, g_1, c_2)$ and $\mathbb{R}_0 \models \text{line}(g_1, g_2, \dots, c_1)$.
- Let $\varphi \overset{\circ}{=} \varphi_1 \wedge \varphi_2$ and assume that $\mathcal{F} \models \varphi[\![g_1, g_2, \dots, g_n]\!]$. Then, $\mathcal{F} \models \varphi_1[\![g_1, g_2, \dots, g_n]\!]$ and $\mathcal{F} \models \varphi_1[\![g_1, g_2, \dots, g_n]\!]$. By the induction hypotheses, $\mathbb{R}_0 \models \varphi_1[\![\overline{g_1}, \overline{g_2}, \dots, \overline{g_n}]\!]$ and $\mathbb{R}_0 \models \varphi_2[\![\overline{g_1}, \overline{g_2}, \dots, \overline{g_n}]\!]$, so $\mathbb{R}_0 \models \varphi[\![\overline{g_1}, \overline{g_2}, \dots, \overline{g_n}]\!]$. Similar proof can be executed for $\varphi \overset{\circ}{=} \varphi_1 \wedge \varphi_2$.

- Let $\varphi \stackrel{\circ}{=} \neg\psi$ and assume $\mathcal{F} \models \neg\psi[\![g_1, g_2, \ldots, g_n]\!]$ for some lines $g_1, g_2, \ldots, g_n$. Clearly, all their corresponding sequences are lines and the formula

$$\bigwedge_{1\leq s,t}^{n} \bigwedge_{1\leq i,j}^{m} (\text{common}_{i,j}(\overline{g_t}, \overline{g_s}) \implies \hat{R}_{j-i}(\overline{g_t}, \overline{g_s}))$$

holds as well. By the induction hypotheses $\mathbb{R}_0 \models \neg\psi[\![\overline{g_1}, \overline{g_2}, \ldots, \overline{g_n}]\!]$. Thus, $\mathbb{R}_0 \models \varphi[\![\overline{g_1}, \overline{g_2}, \ldots, \overline{g_n}]\!]$

- Let $\varphi \stackrel{\circ}{=} \exists x\psi[\![x, x_1, x_2, \ldots, x_m]\!]$. Assume that $\mathcal{F} \models \psi[\![g_0, g_1, g_2, \ldots, g_n]\!]$ for some lines $g_0, g_1, g_2, \ldots, g_n$ and we can take their corresponding sequences. Then the predicate line$(\overline{g_i})$ would be true for all $i \in \{0, 1, \ldots, n\}$.

  Since those are real lines $\mathbb{R}_0 \models \bigwedge_{s=1}^{n} \bigwedge_{1\leq i,j}^{m} (\text{common}_{i,j}(\overline{g_0}, \overline{g_s}) \implies \hat{R}_{j-i}(\overline{g_0}, \overline{g_s}))$ and by IH $\mathbb{R}_0 \models \hat{\psi}[\![\overline{g_0}, \overline{g_1}, \ldots, \overline{g_n}]\!]$, so $\mathbb{R}_0 \models \hat{\varphi}[\![\overline{g_1}, \overline{g_2}, \ldots, \overline{g_n}]\!]$

" $\Longleftarrow$ " Now, let $\mathbb{R}_0 \models \hat{\varphi}[\![g_{11}, g_{12}, \ldots, c_1, \ldots, g_{n1}, g_{n2}, \ldots, c_n]\!]$.
For all $(g_{i1}, \ldots, g_{im})$ and $(g_{j1}, \ldots, g_{jm})$ holds exactly one of the following:
1) $(g_{i1}, \ldots, g_{im})$ is a cyclic permutation of $(g_{j1}, \ldots, g_{jm})$
2) $(g_{i1}, \ldots, g_{im})$ and $(g_{j1}, \ldots, g_{jm})$ have no common elements.

Thus, we can construct a set C with corresponding $m-$ tuples of gradients such that:
1) If $(g_{i1}, \ldots, g_{im})$ is a cyclic permutation of $(g_{j1}, \ldots, g_{jm})$, then $(a_{i1}, \ldots, a_{im})$ is the same cyclic permutation of $(a_{j1}, \ldots, a_{jm})$.
2) If $(g_{i1}, \ldots, g_{im})$ and $(g_{i1}, \ldots, g_{im})$ have no common elements, then $(a_{i1}, \ldots, a_{im})$ and $(a_{j1}, \ldots, a_{jm})$ have no common elements.
3) $a_{11}, a_{12}, \ldots, a_{nm}$ are all completely new and different from $g_{11}, g_{12}, \ldots, g_{nm}$.

An $m-$tuple of gradients will be an $m-$tuple containing all gradients of the lines obtained by consecutive rotation by angle ϕ.

Our claim is that $\mathcal{F} \models \varphi[\![h_1, h_2, \ldots, h_n]\!]$, where $h_i : y = a_{i1}x + c_i$.

- Let $\varphi \stackrel{\circ}{=} R(x, y)$ and $\mathbb{R}_0 \models \hat{\varphi}[\![g_{11}, g_{12}, \ldots, g_{1m}, c_1, g_{21}, \ldots, g_{2m}, c_2]\!]$. Then, by $(g_{11}, g_{12}, \ldots, g_{1m})$ is a shifted one to to left permutation of $(g_{21}, \ldots, g_{2m})$, and so is $(a_{11}, a_{12}, \ldots, a_{1m})$ of $(a_{11}, a_{12}, \ldots, a_{1m})$. Thus, clearly $\mathcal{F} \models R(h_1, h_2)$.

- For $\varphi \stackrel{\circ}{=} \varphi_1 \wedge (\vee)\varphi_2$ or $\varphi \stackrel{\circ}{=} \neg\psi$ the proof is straightforward.

- Let $\varphi \stackrel{\circ}{=} \exists x\psi$ and $\psi[x, x_1, x_2, \ldots, x_n]$.

  Assume $\mathbb{R}_0 \models \psi[\![t_1, t_2, \ldots, t_m, c_0, g_{11}, g_{12}, \ldots, g_{1m}, c_1, g_{n1}, \ldots, g_{nm}, c_2]\!]$. Then we know the corresponding tuples $\overline{a_1}, \ldots, \overline{a_n}$ from C and need to show that we can construct a new one for $\bar{t}$.

  If $(t_1, t_2, \ldots, t_n)$ has something in common with $\overline{g}_i$, then it is a cyclic permutation of it (this is what the long formula with the predicate common forces). Then we build the same permutation of $\overline{a_i}$ and put it in C as the corresponding tuple of $(t_1, \dots, t_m)$. Otherwise, if $(t_1, \dots, t_n)$ has nothing in common with the other $g-$tuples, we choose one rondom line $h_0 : y = a_0 x + c_0$ and put $(a_0, a_0', \dots, a_0^{(m-1)}$ in C. A quick comment why there cannot b a collision. If $\bar{t}$ has something in common with two lines, say $\overline{g_1}$ and $\overline{g_2}$, then again by the long formula $\bar{t}$ is $\hat{R}-$connected with both of them, so they it turns out that $\overline{g_1}$ is a cyclic permutation of $\overline{g_2}$. In both cases $\mathcal{F} \models \varphi[\![h_1, h_2, \dots, h_n]\!]$.

□

Definition 4.31. $EQ^\infty := \{\varphi \mid \varphi \text{is a } \mathcal{L}(=) \text{ sentence true in all infinite structures}\}$.

Theorem 4.32. *The membership problem $\varphi \in EQ^\infty$ is $PSPACE$-complete [2]*

Proposition 4.33. *For any sentence φ in $\mathcal{L}(=)$ it holds that:*

$$\mathbb{R}_0 \models \varphi \iff \varphi \in EQ^\infty$$

Proof. Detailed proof can be found in [1]. □

Theorem 4.34. *The theory $LRot^m$ is $PSPACE$-complete.*

Proof. The problem $\varphi \in LRot^m$ is in $PSPACE$:
Let φ be a $\mathcal{L}(=, R)$- sentence. Since $LRot^m$ is complete, then $\varphi \in LRot^m \iff \mathcal{F} \models \phi$. By 4.30 the last one is equivalent to $\mathbb{R}_0 \models \hat{\varphi}$, which by 4.33 happens if and only if $\hat{\varphi} \in EQ^\infty$. Remark that $\hat{\varphi}$ is obtained algorithmically for polynomial space from φ, and the problem $\hat{\varphi} \in EQ^\infty$ is in $PSPACE$ [2].

The problem $\varphi \in LRot^m$ is $PSPACE$-hard:
Let φ be a $\mathcal{L}(=)$-sentence. Then $\varphi \in EQ^\infty \iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models \varphi \iff \varphi \in LRot^m$.
To verify the the first equivalence, assume that $\varphi \in EQ^\infty$, so φ is true in all infinite models including $\mathcal{F}^2_{\mathbb{R}}(\phi)$. Now, let $\mathcal{F}^2_{\mathbb{R}}(\phi) \models \varphi$ and $\mathcal{A}$ be an infinite model. Since $LRot^m$ is ω-categorical, then $\mathcal{A} \models \varphi$. □

4.9 Complexity of the membership problem for $LRot^\infty$

Let $\mathcal{S}$ be the star model and let us fix an element $a \in S$. Then, as mentioned in remark 4.17, each element can be represented by a pair $\langle s, t\rangle \in \mathbb{Z} \times \mathbb{Z}$. Our goal is for every formula φ to construct a formula $\hat{\varphi}$ such that:

$$\varphi \in LRot^\infty \iff \hat{\varphi} \in Th((\mathbb{Z}, \leq)),$$

where $(\mathbb{Z}, \leq)$ is the structure with universe containing the integers and binary predicate $\leq$, defined as usual. It will be denoted by $\mathcal{Z}$.

In $\mathcal{Z}$ we can define the binary predicate s in the following way:

$$s(x, y) \iff x \leq y \wedge x \neq y \wedge \forall z(x \leq z \wedge z \leq y \implies y = z \vee x = z)$$

So, $s(x, y)$ says that y is the immediate successor of x.

Let $x_1, x_2, \ldots$ is an enumeration of all variables. Now, we can translate the formulas in $\mathcal{L}(R, =)$ in the language $\mathcal{L}(\leq, s)$ in in the following way:

- If $\varphi[x_i, x_j] \overset{\circ}{=} (x_i = x_j)$, then $\hat{\varphi}[x_{2i}, x_{2i+1}, x_{2j}, x_{2j+1}] \overset{\circ}{=} x_{2i} = x_{2j} \wedge x_{2i+1} = x_{2j+1}$;
- If $\varphi[x_i, x_j] \overset{\circ}{=} R(x_i, x_j)$, then $\hat{\varphi}[x_{2i}, x_{2i+1}, x_{2j}, x_{2j+1}] \overset{\circ}{=} s(x_{2i}, x_{2j})$;
- If $\varphi \overset{\circ}{=} \varphi_1 \wedge (\vee)\varphi_2$, then $\hat{\varphi} \overset{\circ}{=} \hat{\varphi}_1 \wedge (\vee)\hat{\varphi}_2$;
- If $\varphi \overset{\circ}{=} \neg\psi$, then $\hat{\varphi} \overset{\circ}{=} \neg\hat{\psi}$;
- If $\varphi \overset{\circ}{=} \exists x_i \psi$, then $\hat{\varphi} \overset{\circ}{=} \exists x_{2i} \exists x_{2i+1} \hat{\psi}$

Proposition 4.35. *For each formula $\varphi[x_1, x_2, \ldots, x_n]$ it holds that:*

$$\mathcal{S} \models \varphi[\![\langle s_1, t_1\rangle, \ldots, \langle s_n, t_n\rangle]\!] \iff \mathcal{Z} \models \hat{\varphi}[\![s_1, t_1, \ldots, s_n, t_n]\!]$$

Proof. We are going to prove the claim by induction on the complexity of φ. If:

- $\varphi[x_i, x_j] \overset{\circ}{=} (x_i = x_j)$, $1 \leq i, j \leq n$;

$$\begin{aligned} \mathcal{S} \models \varphi[\![\langle s_1, t_1\rangle, \ldots, \langle s_n, t_n\rangle]\!] &\iff \langle t_i, s_i\rangle = \langle t_j, s_j\rangle \\ &\iff t_i = t_j \text{ and } s_i = s_j \iff \mathcal{Z} \models \hat{\varphi}[\![s_1, t_1, \ldots, s_n, t_n]\!] \end{aligned}$$

- $\varphi[x_i, x_j] \overset{\circ}{=} R(x_i, x_j)$, $1 \leq i, j \leq n$;

$$\begin{aligned} \mathcal{S} \models \varphi[\![\langle s_1, t_1\rangle, \ldots, \langle s_n, t_n\rangle]\!] &\iff R_\phi(\langle s_i, t_i\rangle, \langle s_j, t_j\rangle) \\ &\iff s(s_{2i}, s_{2j}) \iff \mathcal{Z} \models \hat{\varphi}[\![s_1, t_1, \ldots, s_n, t_n]\!] \end{aligned}$$

- The binary cases are clear;
- $\varphi \overset{\circ}{=} \exists x_i \varphi$, $\varphi[x_1, \ldots, x_n]$ and $i > n$. Then $\psi[x_i, x_1, \ldots, x_n]$;

$$\begin{aligned}
\mathcal{S} \models \varphi[\![\langle s_1, t_1\rangle, \ldots, \langle s_n, t_n\rangle]\!] &\iff \text{for some } \langle s, t\rangle \in \mathbb{Z} \times \mathbb{Z} :\ \mathcal{S} \models \psi[\![\langle s, t\rangle, \langle s_1, t_1\rangle, \ldots, \langle s_n, t_n\rangle]\!] \\
&\iff \text{for some } s \text{ and some } t : \mathcal{Z} \models \hat{\psi}[\![s, t, s_1, t_1, \ldots s_n, t_n]\!] \\
&\iff \mathcal{Z} \models \exists x_{2i} \exists x_{2i+1} \hat{\psi}[\![s_1, t_1, \ldots s_n, t_n]\!] \\
&\iff \mathcal{Z} \models \hat{\psi}[\![s_1, t_1, \ldots s_n, t_n]\!]
\end{aligned}$$

□

4.10 Complexity of $(\mathbb{Z}, \leq)$

Let $\mathcal{N}$ be a structure with universe $\mathbb{N}$ and binary relation $\leq$. Then clearly, the elements 0 and 1 are definable. We define the binary predicate *Int* the following way:

$$Int(x, y) \iff y = 0 \vee y = 1$$

Now, to each integer number i we correlate a pair of its natural part ($|i|$ and its sign written as 0 or 1. For example: $4 \to (4, 1)$, $0 \to (0, 1)$ and $-5 \to (5, 0)$. We translate as following:

If $\varphi[x_i, x_j] \overset{\circ}{=} x_i \leq x_j$, then:

$$\begin{aligned}
\hat{\varphi} \overset{\circ}{=} Int(x_{2i}, x_{2i+1}) \wedge Int(x_{2j}, x_{2j+1}) \wedge \big((x_{2i} \leq x_{2j} \wedge x_{2i+1} = 1 \wedge x_{2j+1} = 1) \\
\vee (x_{2i} \geq x_{2j} \wedge x_{2i+1} = 0 \wedge x_{2j+1} = 0) \\
\vee (x_{2i+1} = 0 \wedge x_{2j+1} = 1)\big)
\end{aligned}$$

If $\varphi \overset{\circ}{=} \psi_1 \vee (\wedge) \psi_2$, then: $\hat{\varphi} \overset{\circ}{=} \hat{\psi}_1 \vee (\wedge) \hat{\psi}_2$.

If $\varphi \overset{\circ}{=} \neg\psi$ and $\varphi[x_1, x_2, \ldots, x_n]$, then $\hat{\varphi} \overset{\circ}{=} \neg\hat{\psi} \wedge Int(x_2, x_3) \wedge \cdots \wedge Int(x_{2n}, x_{2n+1})$.

If $\varphi \overset{\circ}{=} \exists x_i \psi$, then:

$$\hat{\varphi} \overset{\circ}{=} \exists x_{2i} \exists x_{2i+1} (Int(x_{2i}, x_{2i+1}) \wedge \hat{\psi})$$

We will prove the following claim: Let $a_1, a_2, \ldots, a_n$ be some integers and let $i_1, s_1, i_2, \ldots, i_n, s_n$ be their natural parts and 0-1 signs. Then for any formula φ:

$$\mathbb{Z} \models \varphi[\![a_1, a_2, \ldots, a_n]\!] \iff \mathcal{N} \models \hat{\varphi}[\![i_1, s_1, i_2 \ldots, i_n, s_n]\!]$$

Proof: For the first case, when $\varphi \overset{\circ}{=} x_i \leq x_j$:

$$\begin{aligned}\mathbb{Z} \models (x_i \leq x_j)[\![a_1, a_2]\!] &\iff a_1 \leq a_2 \\ &\iff \text{either } a_2 \text{ is non-negative and } a_1 \text{ is negative} \\ &\text{or both } a_1, a_2 \text{ are non-negative and } a_1 \leq a_2 \\ &\text{or both } a_1, a_2 \text{ are negative and } a_1 \geq a_2 \\ &\iff \mathcal{N} \models \hat{\varphi}[\![i_1, s_1, i_2, s_2]\!]\end{aligned}$$

The other binary cases are clear. Let us prove $\varphi \overset{\circ}{=} \exists x_t \psi$, where $\psi[x_t, x_1, x_2, \ldots, x_n]$ and $t > n$. Then:

$$\begin{aligned}\mathbb{Z} \models \exists x_t \psi[\![a_1, a_2, \ldots, a_n]\!] &\iff \mathbb{Z} \models \psi[\![a_t, a_1, a_2, \ldots, a_n]\!] \text{ for some integer } a_t \\ &\iff \mathcal{N} \models \hat{\psi}[\![i_t, s_t, i_1, s_1, \ldots, i_n, s_n]\!] \text{ for the nat. part and the 0-1 sign of some } a_t \\ &\iff \mathcal{N} \models \exists x_{2t} \exists x_{2t+1} (\hat{\psi} \wedge Int(x_{2t}, x_{2t+1}))[\![i_1, s_1, \ldots, i_n, s_n]\!]\end{aligned}$$

Thus, we get that for any sentence φ: $\mathbb{Z} \models \varphi \iff \mathcal{N} \models \hat{\varphi}$. However $\hat{\varphi}$ is obtained by polynomial "expansion" from φ and the question "Is a sentence in $\mathrm{Th}(\mathbb{N}, \leq)$" is PSPACE-complete (Theorem 5.32 Fer), then the question "Is a sentence in $\mathrm{Th}(\mathbb{Z}, \leq)$" is in PSPACE as well. The problem is $PSPACE$ hard because the predicate $=$ is definable.

Corollary 4.35.1. *Let φ be $\mathcal{L}(R, =)$-sentence. Then:*

$$\mathcal{S} \models \varphi \iff \mathbb{Z} \models \hat{\varphi}$$

Theorem 4.36. *The theory $LRot^{\infty}$ is $PSPACE$-complete.*

Proof. The problem "is a sentence $\varphi \in LRot^{\infty}$" is in $PSPACE$:
We have the following equivalences:

$$\varphi \in LRot^{\infty} \iff \mathcal{S} \models \varphi \iff \mathbb{Z} \models \hat{\varphi} \iff \hat{\varphi} \in Th((Z, \leq)),$$

where the last problem is $PSPACE$-complete [**?**], Theorem 1.7.

The problem is $PSPACE$-hard: Let φ be a sentence in $\mathcal{L}(=)$. Then φ is also in $\mathcal{L}(R, =)$, so:

$$\varphi \in EQ^{\infty} \iff \mathcal{F}^2_{\mathbb{Q}} \models \varphi \iff \mathcal{S} \models \varphi \iff \varphi \in LRot^{\infty}$$

□

In the next section we explore another language, which at first hand is quite similar to the previous one. However, it turns out that there are some minor differences and interesting cases.

5 Expressivity of $\mathcal{L}(\mathbf{R}_\phi, =)$

In this section we will investigate the line-based Euclidean plane without fixed orientation. We will denote it with $\mathcal{F}^2_{\mathbb{R}}(\phi)$. Let ϕ be an angle between 0 and $\frac{\pi}{2}$. We define the binary relation $\mathbf{R}_\phi$ as follows:

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models \mathbf{R}_\phi(x, y) \text{ if and only if the angle between } x \text{ and } y \text{ is } \phi$$

Note that it holds that:

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models \mathbf{R}_\phi(x, y)[\![a, b]\!] \text{ if and only if } \mathcal{F}^2_{\mathbb{R}}(\phi) \models R_\phi(x, y) \vee R_\phi(y, x)[\![a, b]\!]$$

Again, we associate the binary predicate P with parallelism. It turns out that depending on the angle ϕ it is now always the case that P is definable in $\mathcal{F}^2_{\mathbb{R}}(\phi)$:

Proposition 5.1. *It holds that:*
(i) If $\phi = \frac{1}{4}\pi$, then P is not definable in $\mathcal{F}^2_{\mathbb{R}}(\phi)$;
(ii) In all other cases:

$$\mathcal{F}^2_{\mathbb{R}}(\phi) \models P(a, b) \iff \mathcal{F}^2_{\mathbb{R}}(\phi) \models \forall c\, (\boldsymbol{R}(a, c) \implies \boldsymbol{R}(b, c))$$

Proof. For the sake of contradiction assume that P is definable via $\psi \in \mathcal{L}$. Then $\mathbf{R}_{\frac{\pi}{2}}$ is definable since:

$$\mathbf{R}_{\frac{\pi}{2}}(a, b) \iff \exists c \Big(\mathbf{R}_{\frac{\pi}{4}}(a, c) \wedge \mathbf{R}_{\frac{\pi}{4}}(c, b) \wedge \neg P(a, b)\Big)$$

Consider the lines a and b as in the graphic.

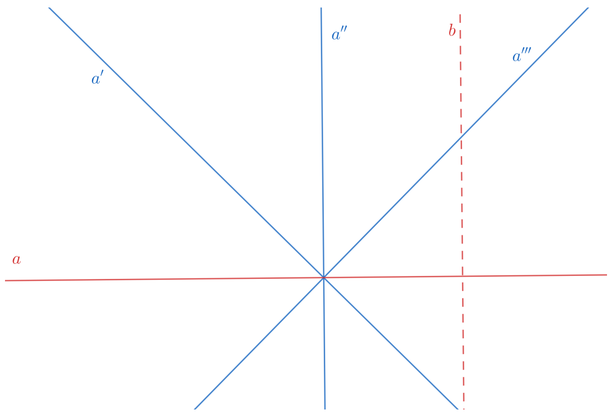

Figure 2

Then the following map is an automorphism:

$$h(x) = \begin{cases} a, & \text{if } x = b \\ b, & \text{if } x = a \\ x, & \text{otherwise} \end{cases}$$

Let us first check if the validity of the predicate $\mathbf{R}$ is preserved. Let $\mathbf{R}(x, y)$ for some lines x and y.

- if $x \notin \{a, b\}$ and $y \notin \{a, b\}$, then $\mathbf{R}(h(x), h(y)) \iff \mathbf{R}(x, y)$
- If $x = a$ and $\mathbf{R}(x, y)$, then it holds that $P(y, a') \vee P(y, a')$. Then $\mathbf{R}(h(a), h(y)) \iff \mathbf{R}(b, y)$ which is visible by 2
- If $x = b$ the situation is similar.

However, we know that $\mathbf{R}_{\frac{\pi}{2}}(a, a'')$, but $\neg\mathbf{R}_{\frac{\pi}{2}}(h(a), h(a''))$. We obtain the contradiction. □

5.1 Theory for $\mathbf{R}_{\frac{\pi}{4}}$ or L4Rot

Consider the following set of axioms:
$\mu_{1,n} : \forall x_1 \ldots \forall x_n \exists y(\bigwedge_{1 \leq i < j \leq n}(\neg\mathbf{R}(x_i, x_j) \wedge (x_j \neq x_j)) \implies \bigwedge_{1 \leq i \leq n}(\neg\mathbf{R}(x_i, y)) \wedge (x_i \neq y))$
$\mu_{2,n} : \forall x \exists y_1 \exists y_2 \ldots \exists y_n(\mathbf{R}(x, y_1) \wedge \mathbf{R}(x, y_2) \wedge \cdots \wedge \mathbf{R}(x, y_k) \wedge \bigwedge_{1 \leq i < j \leq n} y_i \neq y_j)$
$\mu_3 : \forall x \forall y(\mathbf{R}(x, y) \implies \mathbf{R}(y, x))$
$\mu_4 : \forall x(\neg\mathbf{R}(x, x))$
$\mu_5 : \forall x \exists y\, \mathbf{R}(x, y)$
$\mu_6 : \forall x \forall y \forall z(\mathbf{R}(x, y) \wedge \mathbf{R}(x, z) \implies \neg\mathbf{R}(y, z))$
$\mu_7 : \forall x \forall y \forall z \forall u(\mathbf{R}(x, y) \wedge \mathbf{R}(y, z) \wedge \mathbf{R}(z, u) \implies \mathbf{R}(x, u))$
$\mu_{8,n} : \forall x_1 \ldots \forall x_n \exists y(\bigwedge_{1 \leq i < j \leq n} \neg(\mathbf{R}(x_i, y) \vee \exists z(\mathbf{R}(x_i, z) \wedge \mathbf{R}(z, y)))$

Let **L4Rot** be the theory containing those axioms.

We begin with a proposition that shows that **L4Rot** is a proper candidate for an axiomatization.

Proposition 5.2. $\mathcal{F}^2_{\mathbb{R}}(\frac{\pi}{4}) \models$ ***L4Rot***

Proof. Let us briefly comment on the axioms: The three schemes $\mu_{1,n}, \mu_{2,n}$ and $\mu_{8,n}$ follow from the following two facts: first, there is an infinite set of lines, such that no two lines in it are $\mathbf{R}$-related and second, for each line a there is an infinite set of lines, each of which is $\mathbf{R}$- related with a. The relation $\mathbf{R}_\phi$ is anti-reflexive, symmetric and serial. The most interesting axiom to check is μ_7.

Let a, b, c, d be four lines such that $\mathbf{R}_\phi(a, b), \mathbf{R}_\phi(b, c), \mathbf{R}_\phi(c, d)$. We need to check the whether $\mathbf{R}_\phi(a, d)$ is true. We have two cases: a and c are parallel or perpendicular. In the first case, it is clear that $\mathbf{R}_\phi(a, d)$. In the second the line d must be the angle bisector of one of the 90-degree angles formed by a and c. However, again $\mathbf{R}_\phi(a, d)$ follows. □

Definition 5.3. Let $\mathcal{M}$ be a model of **L4Rot** and $a \in M$. Then:

$$[a]_R = \{b \in M | \mathbf{R}(a, b) \text{ or there is an element } c : \mathbf{R}(a, c) \text{ and } \mathbf{R}(c, b)\}$$

Proposition 5.4. *Let $\mathcal{M} \models$ **L4Rot** and $a \in M$. The following hold:*

1. *$[a]_R$ is non-empty;*
2. *$a \in [a]_R$;*
3. *if $b \in [a]_R$, then $a \in [b]_R$;*
4. *for any $b \in [a]_R$ we have that $[b]_R = [a]_R$;*
5. *for any two elements a and b we have that either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$*

Proof. 1) From μ_5 there is an element b such that $\mathbf{R}(a, b)$, so $b \in [a]_R$.

2) By μ_5 there is an element b there is an element b, such that $\mathbf{R}(a, b)$. Using μ_3 we have that $\mathbf{R}(b, a)$ as well. Now, $\mathbf{R}(a, b)$ and $\mathbf{R}(b, a)$, so $a \in [a]_R$.

3) It follows directly from the symmetricity of $\mathbf{R}$.

4) Assume that $b \in [a]_R$. Let $c \in [b]_R$.

4.1) $\mathbf{R}(b, a)$ and $\mathbf{R}(c, b)$ hold. Then, $c \in [a]_R$.

4.2) $\mathbf{R}(b, d)$, $\mathbf{R}(d, a)$ hold for some d and $\mathbf{R}(c, b)$. Then, by μ_7 $\mathbf{R}(c, a)$ holds as well and $c \in [a]_R$.

4.3) $\mathbf{R}(b, d)$, $\mathbf{R}(d, a)$ hold for some d and $\mathbf{R}(c, e)$, $\mathbf{R}(e, b)$ hold for some e. Then, combining $\mathbf{R}(e, b)$, $\mathbf{R}(b, d)$, $\mathbf{R}(d, a)$, by μ_7 we get that $\mathbf{R}(e, a)$. Thus, $\mathbf{R}(c, e)$ and $\mathbf{R}(e, a)$ imply that $c \in [a]^R$.

4.4) $\mathbf{R}(b, a)$ and $\mathbf{R}(c, e)$, $\mathbf{R}(e, b)$ hold for some e. Then, $\mathbf{R}(c, a)$ and $c \in [a]_R$.

All these cases prove that $[b]_R \subseteq [a]_R$.

By 3) we have that $a \in [b]_R$, so similarly, we prove that $[a]_R \subseteq [b]_R$.

5) If $[a]_R \cap [b]_R = c$, then by 4) $[a]_R = [c]_R = [b]_R$. □

In other words, this proposition says that for each element $a \in M$, the set $[a]_R$ is exactly the equivalence class of a when the following equivalence relation is defined:

$$a \sim_R b \iff \mathbf{R}(a, b) \text{ or there is an element } c \in M : \mathbf{R}(a, c) \text{ and } \mathbf{R}(c, b)$$

Lemma 5.5. *Let $\mathcal{M} \models$ **L4Rot**. Then $M/\sim_R$ is infinite.*

Proof. Assume the contrary and let $[a_1]_R, \ldots, [a_n]_R$ be all the equivalence classes in $M/\sim_R$. Then, by axiom $\mu_{8,n}$ applied for elements $a_1, a_2, \ldots, a_n$, there is an element b such that $\neg\mathbf{R}(a_i, b)$ and there is no element c such that $\mathbf{R}(a_i, c)$ and $\mathbf{R}(c, b)$. Then, $a_i \not\sim_R b$ for any $1 \leq i \leq n$, so $b \notin [a_i]_R$ for any $1 \leq i \leq n$. Clearly, we get a contradiction. □

Definition 5.6. Let $\mathcal{M}$ be a model of **L4Rot** and $a \in M$. In $[a]_R$ we fix a red-blue colouring as follows:

- the colour of a is blue.
- if $\mathbf{R}(a, b)$, then the colour of b is red, otherwise the colour of b is blue.

Lemma 5.7. *1. The colouring in is correct.*

2. When coloured, each $\sim_R$- equivalence class contains infinite number of blue lines and infinite number of red lines.

Proof. 1. Assume that there is an element b that can be colored both in red and blue. Then, $\mathbf{R}(b, a)$, $\mathbf{R}(b, c)$ and $\mathbf{R}(c, a)$. By μ_7 we get $\mathbf{R}(a, a)$ - contradicition.

2. Take one equivalence class. Since it is non-empty we can find one element b, which by the colouring is blue. By axiom μ_5 there is also one element r, whose colour should be red since $\mathbf{R}(b, r)$. Applying the scheme $\mu_{2,n}$ for the blue element b we generate an infinite number of red elements. Vice versa, appling it for the red element r we generate an infinite number of blue elements.

□

Corollary 5.7.1. *Let $\mathcal{M}$ be a countable model of **L4Rot**. Then:*
1) the number of different classes in $M/\sim_R$ is countable;
2) in each class there are countable number of blue and countable number of red elements.

Theorem 5.8. *The following claims hold:*
*(i) **L4Rot** is ω-categorical.*
*(ii) For all any infinite cardinal number $\alpha > \omega$ **L4Rot** is not α-categorical.*

Proof. (i) Let $\mathcal{M}$ and $\mathcal{N}$ be two countable models of **L4Rot**. Then, by the previous corollary we know that $M/\sim_R$ and $N/\sim_R$ are countable. Let $[m_1]_R^{\mathcal{M}}, [m_2]_R^{\mathcal{M}}, \ldots,$ and $[n_1]_R^{\mathcal{N}}, [n_2]_R^{\mathcal{N}}, \ldots,$ be the different equivalence classes in M and N respectively. Furthermore, we know that the red and blue elements in $[m_i]_R^{\mathcal{M}}$ and $[n_i]_R^{\mathcal{N}}$ are countable and can be enumerated: say $a_i^1 \ldots, a_i^n, \ldots,$ are the red elements and $b_i^1 \ldots, b_i^n, \ldots,$ are the blue ones in $[m_i]_R^{\mathcal{M}}$, while $c_i^1 \ldots, c_i^n, \ldots,$ are the red elements and $d_i^1 \ldots, d_i^n, \ldots,$ are the blue ones in $[n_i]_R^{\mathcal{N}}$.

For each integer i define $h_i : [m_i]_R^{\mathcal{M}} \rightarrow [n_i]_R^{\mathcal{N}}$ as follows : $h_i(a_i^n) = c_i^n$ and $h_i(b_i^n) = d_i^n$ for each $1 \leq n < \omega$.

Now, $h = \bigcup_{i \geq 1} h(i)$. Clearly, h is bijection between M and N. It is left to prove that for any two elements a and b from M : $\mathbf{R}(a, b) \iff \mathbf{R}(h(a), h(b))$.

Let $\mathbf{R}(a, b)$ for some elements a and b from M. Then $b \in [a]_R^{\mathcal{M}}$ and moreover their colours are different. Since h preserves the colouring, $h(a)$ and $h(b)$ will also have different colours, implying $\mathbf{R}(h(a), h(b))$. The reverse direction is the same.

(ii) We can construct a model with α R- classes each with cardinality ω and another with ω $R-$ classes each with cardinality α. □

As a corollary we obtain the following theorem:

Theorem 5.9. ***L4Rot*** *is complete. Furthermore,* ***L4Rot*** $= Th(\boldsymbol{\mathcal{F}}^2_{\mathbb{R}}(\frac{\pi}{4}))$.

The membership problem in this section is in $PSPACE$, because we can use the following equivalence:

$$\boldsymbol{\mathcal{F}}^2_{\mathbb{R}}(\phi) \models \mathbf{R}_\phi(x,y)[\![a,b]\!] \text{ if and only if } \mathcal{F}^2_{\mathbb{R}}(\phi) \models R_\phi(x,y) \vee R_\phi(y,x)[\![a,b]\!]$$

and translate a formula $\varphi \in \mathcal{L}_{\{=\mathbf{R}\}}$ naturally in a formula $\hat{\varphi} \in \mathcal{L}_{\{=,R,P\}}$.

However, since our language contains the binary predicate $=$, then the membership problem is $PSPACE-$hard.

So, we can formulate the following theorem:

Theorem 5.10. *The problem if a sentence $\varphi \in Th($**L4Rot**$)$ is PSPACE- complete.*

Last, but not least we present a proper axiomatization for the $Th(\boldsymbol{\mathcal{F}}^2_{\mathbb{R}}(\phi))$ when $\phi \neq \frac{\pi}{2}$ and $\phi \neq \frac{\pi}{4}$.

5.2 Theory of $\mathbf{R}_\phi$

Let $\phi = \frac{1}{m}\pi$ where $m \geq 2$ and $m \neq 4$. Consider the following set of axioms Λ:

$\mu_1 : \forall x \forall y (P(x, y) \iff \forall z(\mathbf{R}(x, z) \implies \mathbf{R}(y, z)))$
$\mu_{2,n} : \forall x_1 \forall x_2 \ldots \forall x_n \exists y (\neg P(x_1, y) \wedge \neg P(x_2, y) \cdots \wedge \neg P(x_n, y))$
$\mu_{3,n} : \forall x_1 \forall x_2 \ldots \forall x_n \exists y (P(x_1, x_2) \wedge \cdots \wedge P(x_{n-1}, x_n) \implies P'(x_1, y) \wedge P'(x_2, y) \cdots \wedge P'(x_n, y))$
$\mu_4 : \forall x \exists y \exists z (\mathbf{R}(x, y) \wedge \mathbf{R}(x, z) \wedge \neg P(y, z))$
$\mu_5 : \forall x \forall y (P(x, y) \implies \neg \mathbf{R}(x, y))$
$\mu_6 : \forall x \forall y (\mathbf{R}(x, y) \implies \mathbf{R}(y, x))$
μ_7 : P is an equivalence relation
$\mu_8 : \forall y_0 \forall y_1 \forall y_2 \ldots \forall y_m (\mathbf{R}(x, y_1) \wedge \ldots \mathbf{R}(y_{m-1}, y_m) \wedge \neg (\bigvee_{0 \leq i < j \leq m-1} P(y_i, y_j)) \implies P(y_0, y_m))$
$\mu_{9,k}$ $\forall y_0 \exists y_1 \ldots \exists y_k (\mathbf{R}(x, y_1) \wedge \mathbf{R}(y_1, y_2) \wedge \cdots \wedge \mathbf{R}(y_{k-1}, y_k) \wedge \bigwedge_{0 \leq i < j \leq n} \neg P(y_i, y_j))$,
where $1 \leq k \leq m - 1$
$\mu_{10} : \forall x \forall y \forall z (\mathbf{R}(x, y) \wedge \mathbf{R}(x, z) \wedge \neg P(y, z) \implies \forall u (\mathbf{R}(x, u) \implies P(y, u) \vee P(z, u)))$

Remark 5.11. Note that the axiom μ_7 follows from μ_1 and μ_6.

Let $\mathbf{LRot^m}$ be the theory containing Λ. In order to present a better visualization of its models, we will consider the following definition and some of its properties:

Definition 5.12. Let $\mathcal{M}$ be a model of $\mathbf{LRot^m}$ and $a \in \mathcal{M}$. A path with start a will be called any sequence of elements $a, a_1, , \ldots, a_{m-1}$ from M, such that $\mathbf{R}_\phi(a, a_1)$, $\mathbf{R}_\phi(a, a_1)$, $\mathbf{R}_\phi(a_1, a_2), \ldots$ and $\mathbf{R}_\phi(a_{m-2}, a_{m-1})$ and no two elements are parallel.

Proposition 5.13. *Let $\mathcal{M} \models$ **LRot^{m}**. Then it holds that:*

1. *For each $a \in M$ there is an R-path with start a;*
2. *If $(a_0, a_1, \ldots, a_{m-1})$ is an R-path, then for each $1 \leq i \leq m-1$, $(a_i, a_{i+1}, \ldots, a_{m-1}, a_0, a_1, \ldots, a_{i-1})$ is R-path as well.*
3. *If $(a_0, a_1, \ldots, a_{m-1})$ is an R-path and for each $0 \leq i \leq m - 1$ holds that $P(a_i, b_i)$, then $(b_0, b_1 \ldots, b_{m-1})$ is also an R−path with start b_0.*
4. *If $(a_0, a_1, \ldots, a_{m-1})$ and $(b_0, b_1, \ldots, b_{m-1})$ are R-paths and $P(a_0, b_0)$ and $P(a_1, b_1)$, then for each $2 \leq k \leq m - 1$ it holds that $P(a_k, b_k)$.*
5. *If $(a_0, a_1 \ldots, a_{m-1})$ is an R-path, then so is $(a_0, a_{m-1}, \ldots, a_1)$.*
6. *If $(a_0, a_1, \ldots, a_{m-1})$ and $(b_0, b_1, \ldots, b_{m-1})$ are R-paths and $P(a_0, b_0)$, then either for all $1 \leq k \leq m - 1$ $P(a_k, b_k)$ or for all $1 \leq k \leq m - 1$ $P(a_k, b_{m-k})$.*
7. *If $(a_0, a_1, \ldots, a_{m-1})$ and $(b_0, b_1, \ldots, b_{m-1})$ are R-paths, then $\{[a_i] \mid 0 \leq i \leq m - 1\} = \{[b_i] \mid 0 \leq i \leq m - 1\}$.*

Definition 5.14. Let $\mathcal{M}$ be a model of $\mathbf{LRot^m}$ and $a \in \mathcal{M}$. Let $(a, a_1, a_2, \ldots a_{m-2})$ be a R-path with start a. Then, the set $O(a) := \{[a], [a_1], [a_2], \ldots, [a_{m-1}]\}$ will be called the orbit of a.

Remark 5.15. The correctness of this definition follows from 5.13.

Remark 5.16. We can consider that when constructing a path there are two directions to follow. However, since we construct the orbit as a set, then, both directions generate the same P-equivalence classes.

Similarly to the previous theories each model of $\mathbf{LRot^m}$ can be divided into non-intersecting orbits.

Definition 5.17. Let $\mathcal{M}$ be a model of $\mathbf{LRot^m}$. Define the binary relation $\sim_{\mathcal{M}}$ as follows:

$$a \sim_{\mathcal{M}} b \iff O^{\mathcal{M}}(a) = O^{\mathcal{M}}(b)$$

Lemma 5.18. *Let $\mathcal{M}$ be a model of $\boldsymbol{LRot^m}$. The following hold:*

1. *$M/\sim$ is infinite.*
2. *Each $\sim -$ equivalence class is infinite.*

Proposition 5.19. *(i) $\boldsymbol{LRot^m}$ is ω-categorical*
(ii) $\boldsymbol{LRot^m}$ is not α-categorical for any cardinal number alpha $> \omega$

Proof. (i) Let $\mathcal{M}$ and $\mathcal{N}$ be two countable models of $\mathbf{LRot^m}$. Then, by the previous lemma $M/\sim_{\mathcal{M}}$ and $N/\sim_{\mathcal{N}}$ are countable. Let $O(a_1), O(a_2), \dots,$ and $O(b_1), O(b_2), \dots,$ be enumerations of the different $\sim$- equivalence classes in M and N respectively. We assume that we have fixed the elements $a_1, a_2, \dots, b_1, b_2, \dots$ and in each orbit we have fixed a path with start a_i and a path with start b_i. Now, define a map $h_i : O(a_i) \to O(b_i)$, as following:

1. Take the two fixed paths, one starting from a_i: $(a_i, c_1, c_2, \dots c_{m-1})$, and one, starting from b_i: $(b_i, d_1, d_2, \dots d_{m-1})$.
2. Let $h_{i,0}$ be a bijection between $[a_i]$ and $[b_i]$;
3. For any number $1 \leq k \leq m-1$ take one bijection $h_{i,k} : [c_k] \to [d_k]$.
4. Let $h_i = \bigcup_{0 \leq k \leq m-1} h_{i,k}$.

Finally, the map $h := \bigcup_{i \geq 1} h_i$ is the desired isomorphism. Let us verify that it preserves the validity of $\mathbf{R}$.

Let c, d be two elements from M such that $\mathbf{R}(c, d)$. Then c and d are in the same orbit, say $O(a_1)$. Furthermore, for any path, starting from a_1, c and d are in two consecutive P-classes. In other words, $a_1, \dots, c, d, \dots$ is one possible path. Thus, $b_1, \dots, h(c), h(d), \dots,$will also be a path starting from b_1, so $\mathbf{R}(h(c), h(d))$ will hold. The reverse direction is the same.

(ii) Again, consider two models $\mathcal{M}$ and $\mathcal{N}$ where the first one has α orbits each countable, and the second one consists of ω orbits, each with cardinality equal to α. $\square$

As a consequence of the Vaught test we get the following theorem:

Corollary 5.19.1. *$\boldsymbol{LRot^m}$ is complete. Moreover, $\boldsymbol{LRot^m} = \boldsymbol{\mathcal{F}}^2_{\mathbb{R}}(\phi))$ for $\phi = \frac{1}{m}\pi$.*

Similarly to $LRot^m$ we can prove that:

Theorem 5.20. *$\boldsymbol{LRot^m}$ is not finitely axiomatizable.*

Theorem 5.21. *The membership problem "Is a sentence $\varphi \in \boldsymbol{LRot^m}$ " is PSPACE-complete.*

Proof. The same argument as for **L4Rot** holds for the complexity of the theory $\mathbf{LRot^m}$.

$$\boldsymbol{\mathcal{F}}^2_{\mathbb{R}}(\phi) \models \mathbf{R}_\phi(x,y)[\![a,b]\!] \text{ if and only if } \mathcal{F}^2_{\mathbb{R}}(\phi) \models R_\phi(x,y) \vee R_\phi(y,x)[\![a,b]\!]$$

and translate a formula $\varphi \in \mathcal{L}(= \mathbf{R})$ naturally in a formula $\hat{\varphi} \in \mathcal{L}_(=, R, P)$.

However, since our language contains the binary predicate $=$, then the membership problem is $PSPACE-$hard.

□

References

[1] Tatyana Ivanova and Tinko Tinchev. The fragment of elementary plane euclidean geometry based on perpendicularity alone with complexity pspace-complete. *Transactions of the American Mathematical Society*, November 2024. ISSN 1088-6850. doi: 10.1090/tran/9302. URL `http://dx.doi.org/10.1090/tran/9302`.

[2] Philippe Balbiani and Tinko Tinchev. Line-based affine reasoning in euclidean plane. *Journal of Applied Logic*, 5(3):421–434, 2007.

[3] Leonid Libkin. *Elements of finite model theory*. Springer, 2004.

[4] Jeanne Ferrante and Charles Rackoff. *The computational complexity of logical theories*, volume 718 of *Lecture Notes in Mathematics*. Springer-Verlag, 1979.