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Category-Theoretic Methods in Recursion Theory and Inductive Definitions

Finite Models by Resolution with Termoids

by Anton Kirilov Zinoviev

THESIS

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IMPORTANT NOTATION

Notation	Meaning	Defined in
$\mathcal{P}X$	power set, indexed power set	(2C),(10C)
$\mathcal{P}\mathbf{M}$	power structure	(10L)
$h^{\mathcal{P}}$	power homomorphism	(10M)
$\lvert\mathbf{M}\rvert$	universe of structure	(10D)
$\mathbf{M}_\kappa$	carrier of sort κ	(10D)
$\mathtt{f}^{\mathbf{M}}$	interpretation of operation symbol in structure	(10D)
$\ulcorner\xi\urcorner$	the name for ξ	(11C)
$[X], [f]$	termal structure and renaming morphism	(11E)
$[\![X]\!], [\![f]\!]$	termoidal algebra and renaming morphism	(14I)
$\tau^{\mathbf{M}}$	value of termal expression in structure	(11J)
$\tau[v]^{\mathbf{M}}$	value of termal expression with assignment	(11M)
$\tau^{\mathcal{P}\mathbf{M}}$	values of termoidal expression in structure	(14P)
$\tau[\![v]\!]^{\mathcal{P}\mathbf{M}}$	values of termoidal expression with assignment	(14P)
$\tau^{[\![X]\!]}$		(14P)
$\tau[\![s]\!]^{[\![X]\!]}$	application of termoidal substitution	(14P)
$\partial\mathbf{M}$	algebraic fragment of structure	(12C3)
∂h	algebraic fragment of homomorphism	(12C4)
$\int_{\mathbf{M}}$	catamorphism $\partial\mathbf{M}\to\mathbf{M}$	(12L)
X°, f°		(12R)
$[\![\mathrm{nam}_X]\!]^{[X]}$	homomorphism converting termoids to terms	(16A)
$[\mathrm{Nam}_X]^{[\![X]\!]}$	homomorphism converting terms to termoids	(16A)
$\overline{\lambda}$	literal or literaloid contrary to λ	(20A2)

$$\mathrm{val}_{\mathbf{M}} : [\lvert\mathbf{M}\rvert] \to \mathbf{M} \qquad \mathrm{nam}_X : X \to \lvert[X]\rvert \qquad \mathrm{Nam}_X : X^\circ \to \lvert[\![X]\!]\rvert$$

$$\mathrm{val}_{\mathbf{M}}\,\tau = \tau^{\mathbf{M}} \qquad \mathrm{nam}_X\,\xi = \ulcorner\xi\urcorner \qquad \mathrm{Nam}_X\,\xi = \ulcorner\xi\urcorner$$

$$\mathrm{Val}_{\mathbf{M}} : [\![\lvert\mathbf{M}\rvert]\!] \to \mathcal{P}\mathbf{M} \qquad \mathrm{Valt}_{\mathbf{M}} : [\![\lvert\mathbf{M}\rvert]\!] \to \mathbf{M} \qquad \mathrm{Vals}_X : [\![\lvert[\![X]\!]\rvert]\!] \to [\![X]\!]$$

$$\mathrm{Val}_{\mathbf{M}}\,\tau = \tau^{\mathcal{P}\mathbf{M}} \qquad \mathrm{Valt}_{\mathbf{M}}\,\tau = \tau^{\mathbf{M}} \qquad \mathrm{Vals}_X\,\tau = \tau^{[\![X]\!]}$$

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I would like to thank my colleagues from the Department of mathematical logic for their never ending support.

Contents

§1. INTRODUCTION

The resolution refutation is both sound and complete. If a set of clauses is satisfiable, no contradiction is derivable. And, if a set of clauses is not satisfiable, it is possible to derive a contradiction by resolution.

In some cases the resolution algorithm saturates after generating only finitely many clauses. In other words, we reach a set of clauses such that there is no contradiction and no new clauses can be generated by resolution. In such situations we may wish to use somehow the information contained in the generated finite set of clauses in order to build a finite model of the initial set of clauses.

It turns out, this is indeed possible if we modify the resolution to use the so called *termoids* instead of terms. If the resolution with termoids saturates after generating finitely many clauses, then a finite model exists. Moreover, the information contained in the generated set of clauses can be used in order to build such finite model algorithmically.

No formal algorithms will be specified in this work. Nevertheless, all proofs about the existence of finite structures will be constructive and it should not be difficult to extract practical algorithms from the proofs.

We can use the resolution with termoids in order to obtain some purely theoretical results. For example, we can prove that the class VED [11, p. 47] has the finite model property, that is, any satisfiable finite set of clauses belonging to VED has a finite model. A clause belongs to the class VED if it is a Horn clause and for any variable `x` all occurrences of `x` in the clause are at equal depth.

Another interesting result is that the deductive machinery of Prolog also has the finite model property. Let Γ be a finite set of Horn clauses. Suppose we ask Prolog if φ follows from Γ and after some finite computation Prolog answers with “no”. In this case there exists a finite model of Γ in which φ is not valid.

§2. CONVENTIONS

A) For the *null tuple*, for the *singletons*, *pairs*, *triplets* and in general for any *n-tuple* we are going to use the notation $\langle\rangle$, $\langle\alpha\rangle$, $\langle\alpha', \alpha''\rangle$, $\langle\alpha', \alpha'', \alpha'''\rangle$, $\langle\alpha_1, \dots, \alpha_n\rangle$, etc. For a set X and a nonnegative integer n we define X^n to be the set of all n-tuples of elements of X.

B) An n-ary function f on A is a function whose domain contains only n-tuples; we also say that n is the arity of f. A function is *nullary*, *unary*, *binary* or *ternary* if its arity is 0, 1, 2, 3, respectively. The image of α under function f will be denoted by $f\alpha$.

Given a function $f : X \to Y$, X is called *domain* of f, Y is called *codomain* of f and the set $\{fx : x \in X\}$ is called *image* of f. We write $\operatorname{Dom} f$ for the domain of f and $\operatorname{Cod} f$ for the codomain.

The identity function (or homomorphism) of a set (or a structure) A will be denoted by id_A. Sometimes the lower index $_A$ will be omitted.

There is unique function from $\varnothing$ to any set. There is no function from X to $\varnothing$ unless $X = \varnothing$.

For the composition of functions the following notation will be used: $(f \circ g)\xi = f(g\xi)$.

C) The power set of X will be denoted by $\mathcal{P}X$. For arbitrary function $f : X \to Y$, a function $f^{\mathcal{P}} : \mathcal{P}X \to \mathcal{P}Y$ can be defined as follows:

$$f^{\mathcal{P}}(A) = \{f(\alpha) : \alpha \in A\}$$

Notice that $\mathcal{P}$ is an endofunctor in the category $\mathfrak{Set}$ of all sets; particularly, $(f \circ g)^{\mathcal{P}} = f^{\mathcal{P}} \circ g^{\mathcal{P}}$.

D) Given a function $f : X \to Y$, if $X' \subseteq X$, then the restriction of f to X' will be denoted by $f \restriction X'$.

E) 0 is a natural number.

F) For any finite set of numbers A, $\max A$ is its greatest element.

G) The following system of references is used: Proposition (L) of §20 is referenced as "Proposition (20L)" or simply as "(20L)". Item (3) of the same proposition is referenced as "(20L3)". Proposition (L) from the current section is referenced as "Proposition (L)" or simply as "(L)". Item (3) of this proposition is referenced as "(L3)".

Theorems, lemmas, definitions, etc. are referenced analogously.

H) No knowledge of category theory is assumed in this work. Occasionally I am going to state in remarks that something is a category, or a functor, or a natural transformation, etc. but the reader is free to ignore such remarks. In fact, often I am going to leave such remarks without a formal proof.

Capital latin letters are used for objects of categories. Regular letters (i.e. A, B, X) for sets and letters in boldface (i.e. $\mathbf{A}, \mathbf{M}, \mathbf{K}$) for structures.

Small greek letters are used for elements of objects of categories. Usually $\alpha, \alpha_1, \alpha'$ are elements of A, μ, μ_1, μ' are elements of M, etc.

Small latin letters (i.e. f, g, h) are used for arrows of categories (usually functions and homomorphisms).

Non-letter symbols are used for functors — $[X], [f], \mathcal{P}A, f^{\mathcal{P}}, \partial\mathbf{M}, \partial h$.

Words of straight latin letters are used for natural transformations — $\mathrm{Nam}_X, \mathrm{val}_{\mathbf{M}}, \mathrm{Vals}_X$.

Capitalised words of gothic letters are used for categories — $\mathfrak{Set}, \mathfrak{Str}$.

Typewriter letters are used for formal symbols — $\mathtt{f}, \mathtt{c}, \mathtt{p}, \mathtt{x}$.

The letters i, j, k, l, m, n are reserved for natural numbers.

When it is not convenient to consider a set an object of category, capital greek letters (i.e. Γ, Δ, Θ) are used.

Small gothic letters (i.e. $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{e}, \mathfrak{f}, \mathfrak{g}$) are used if none of the above rules prescribes what has to be used.

Informal Sections

§3. HISTORICAL REMARKS

The Entscheidungsproblem

The *Entscheidungsproblem*, or the classical decision problem of David Hilbert was one of the founding problems of Mathematical logic. One way to state this problem is:

Given a first-order formula φ, decide if it is satisfiable.
Or, equivalently, decide if $\neg\varphi$ is valid/provable.

It is intriguing to read how highly this problem was esteemed by the logicians during the first half of the 20^{th} century. For example:[1]

- According to Hilbert and Ackermann: "The Entscheidungsproblem must be considered the main problem of mathematical logic." [16]
- According to Bernays and Schönfinkel: "The central problem of mathematical logic, which is also most closely related to the questions of axiomatics, is the Entscheidungsproblem." [3]
- According to Herbrand: "We could consider the fundamental problem of mathematics to be the following: Problem A: What is the necessary and sufficient condition for a theorem to be true in a given theory having only a finite number of hypotheses? [...] The solution of this problem would yield a general method in mathematics and would enable mathematical logic to play with respect to classical mathematics the role that analytic geometry plays with respect to ordinary geometry." [14]

[1] The citations are from [6].

At the time Gödel proved his famous incompleteness theorems, the area of Entscheidungsproblem had already acquired rich theory. There were several results showing that certain subclasses of the full predicate logic were decidable. For other classes it was possible to prove a negative result of the following form: if we were able to decide whether a formula belonging to a particular subclass is satisfiable, then we would be able to decide this for any formula.

For a long time one characteristic feature of this area was the great variety of methods. Usually, each new result about decidability or undecidability was obtained by entirely new and original method. This, however, was about to change in 1964 when Maslov used for first time a deductive system (the so called "inverse method"[2]) in order to obtain a new result about the decidability of the so called Maslov class[3] [22].

Suppose we are given some sound and complete deductive system $\mathfrak{d}$ about the predicate logic. Due to the undecidability of the whole predicate logic, $\mathfrak{d}$ has the following two properties:

- If φ is valid, then sooner or later $\mathfrak{d}$ is going to prove φ.
- There exists a formula φ which is not valid but $\mathfrak{d}$ is unable to tell this (i.e. $\mathfrak{d}$ searches for a proof of φ ad infinitum).

Suppose, however, that $\mathfrak{d}$ is crafted in such a way, that when φ belongs to a certain class Γ, then $\mathfrak{d}$ is unable to search for a proof infinitely. If this is so, then for any $\varphi \in \Gamma$ one of the following must happen:

- $\mathfrak{d}$ proves φ after finitely many steps.
- After "finitely many steps" $\mathfrak{d}$ is unable to continue the search for a proof of φ. Due to the completeness of $\mathfrak{d}$, we can conclude that φ is not valid/provable.

In 1976 it became apparent that this method is rather general. In a relatively short article Joyner [18] was able to define three resolution decision procedures which could decide several important classes of predicate formulae: the monadic class,[4] the class of Herbrand[5], of Bernays-Schönfinkel[6], of Ackermann,[7] of Gödel,[8] of Maslov and an extended version of the Skolem

[2] The method of resolution can be considered to be a method of implementation of a special case of the Maslov's inverse method. See [19].

[3] The Maslov class contains all prenex formulae of the form $\exists \mathtt{x}_1 \dots \exists \mathtt{x}_n \forall \mathtt{y}_1 \dots \forall \mathtt{y}_m \exists \mathtt{z}_1 \dots \exists \mathtt{z}_k \varphi$, where φ is a Krom formula.

[4] Formulae without functional symbols where all predicate symbols have arity one.

[5] Prenex formulae whose matrix is a conjunction of literals.

[6] Prenex formulae $\exists \mathtt{x}_1 \dots \exists \mathtt{x}_n \forall \mathtt{y}_1 \dots \forall \mathtt{y}_m \varphi$ without functional symbols.

[7] Prenex formulae $\exists \mathtt{x}_1 \dots \exists \mathtt{x}_n \forall \mathtt{y} \exists \mathtt{z}_1 \dots \exists \mathtt{z}_k \varphi$ without functional symbols.

[8] Prenex formulae $\exists \mathtt{x}_1 \dots \exists \mathtt{x}_n \forall \mathtt{y}_1 \forall \mathtt{y}_2 \exists \mathtt{z}_1 \dots \exists \mathtt{z}_k \varphi$ without functional symbols.

class. Although none of these results was new, before Joyner there existed no method that could be used to prove the decidability of so many different classes.

For any formula belonging to one of the mentioned classes, the resolutive decision procedures of Joyner will either produce the empty clause (in which case the formula is not satisfiable), or reach saturation when no new resolvents can be produced (in which case the formula is satisfiable).

Satisfiability in Finite Structures

It seems the first time the mathematicians became interested in satisfiability in finite structures was when they wanted to use this as a tool to prove the decidability of a class. In 1933 Gödel proved that the so called Gödel class (all prenex formulae $\exists \mathtt{x}_1 \dots \exists \mathtt{x}_n \forall \mathtt{y}_1 \forall \mathtt{y}_2 \exists \mathtt{z}_1 \dots \exists \mathtt{z}_k \varphi$ without functional symbols) had the finite model property. From this result Gödel concluded that this class was decidable.

Definition. A class of formulae has the *finite model property*, or, alternatively, is *finite controllable*, if any satisfiable formula of this class has a finite model.

All finite models in a finite language are enumerable (up to isomorphism). Besides that, the question whether a formula is true in a finite model is decidable. Therefore, the problem of the satisfiability of a formula in a finite structure is semidecidable.

Now, suppose that a class of closed predicate formulae has the finite model property. For any formula of this class we can run simultaneously the following two processes:

First, we try to find a finite structure where the formula is true.

Second, we try to prove the negation of the formula by means of some sound and complete deductive system.

If the formula is satisfiable, then it has a finite model, so the first process is going to stop. Otherwise, that is when the formula has no model, the negation of the formula will be valid/provable, so the second process is going to stop. In result, we obtain a decision procedure for the formulae of this class. Consequently, the following theorem is true:

Theorem. *If a class of predicate formulae has the finite model property, then it is a decidable class.*

Since the whole predicate logic is undecidable, the opposite is not true. As a matter of fact, it is not difficult to find predicate formulae which are satisfiable only in infinite models.

Example. Consider the following formula:[9]

$$\forall \mathtt{x} \exists \mathtt{y} \forall \mathtt{z} (\neg \mathtt{p}(\mathtt{x},\mathtt{x}) \wedge \mathtt{p}(\mathtt{x},\mathtt{y}) \wedge (\mathtt{p}(\mathtt{z},\mathtt{x}) \rightarrow \mathtt{p}(\mathtt{z},\mathtt{y}))) \tag{1}$$

After skolemisation we obtain the following three formulae:

$$\forall \mathtt{x} \neg \mathtt{p}(\mathtt{x},\mathtt{x}) \tag{2}$$
$$\forall \mathtt{x} \mathtt{p}(\mathtt{x},\mathtt{f}(\mathtt{x})) \tag{3}$$
$$\forall \mathtt{x} \forall \mathtt{z} (\mathtt{p}(\mathtt{z},\mathtt{x}) \rightarrow \mathtt{p}(\mathtt{z},\mathtt{f}(\mathtt{x}))) \tag{4}$$

Let the structure **M** be such that its universe is the set of the natural numbers and the symbol `f` is interpreted as the succession function and `p` is interpreted as "less-than". It is not difficult to see that **M** is a model of (1), (2), (3) and (4). On the other hand, from (3) and (4) it follows that in any model of these formulae the following formula also is true:

$$\forall \mathtt{x} \mathtt{p}(\mathtt{x},\mathtt{f}(\mathtt{f}(\mathtt{x}))) \tag{5}$$

From (5) and (4) it follows that

$$\forall \mathtt{x} \mathtt{p}(\mathtt{x},\mathtt{f}(\mathtt{f}(\mathtt{f}(\mathtt{x})))) \tag{6}$$

also is true and so on. All these formulae together with (2) show that (2), (3) and (4) have no finite model. Consequently, (1) also has no finite model.

Automated Finite Model Building

One way to prove that a class has the finite model property is to find an effective procedure which is able to compute a finite model of any satisfiable formula of the class. Since this is a major topic of this thesis, here I shall outline shortly the existing methods for automated finite model building. Generally speaking, all existing methods fall in one of the following three groups.

The first group of methods are conceptually the simplest. First we look for a model whose universe has only one element, then for a model with two elements, then with three and so on until a finite model is found. Due to the obvious combinatorial explosion these methods are unsuitable for finding even modestly large models. On other hand, when these methods find a model, it usually is the smallest.

[9] This example is from [6, p. 33]

Owing to some non-trivial techniques, the methods of this group are very fast when the goal is to find a small model. There are some algebraic problems about the existence of certain finite algebraic structures which were solved for first time by finite model builders of this type [23, 30]. The main examples of such systems are MACE by McCune [24] and Falcon by J. Zhang and H. Zhang [31].

The second group of methods use some variant of tableaux. Tableaux methods are close to the intuition. They can be used to prove the unsatisfiability of a formula but also to find models (sometimes partial models) when the formula is satisfiable. George Boolos [4] was the first one who in 1984 was able to find a tableaux method which is theoretically complete with respect to the finite satisfiability of the predicate formulae.

The usefulness of the tableaux methods is undeniable in the area of the non-classical logics. On the other hand, with respect to the first-order predicate logic, the usefulness of the conventional tableaux methods is much less apparent.[10]

Two tableaux based methods have been especially influential and served as a basis for several later improvements. These are SATCHMO by Manthey and Bry [21] and the hypertableaux of Baumgartner [2].

A new and interesting tableaux based method has been proposed by Brown and Smolka [5]. This method came as a result of a very prolonged struggle to create a complete tableaux method for the higher-order logic with sorts[11]. Despite that the tableaux of Brown and Smolka is designed for for an extension of the first-order logic, the authors were able to use it in order to prove the finite model property for some classes of first-order formulae, for example the class of Bernays-Schönfinkel with equality.

The third group of methods are the so called "transformation methods". The idea is to use some conventional method (for example resolution) in order to build a Herbrand model. Subsequently, Herbrand model is transformed into a finite model by means of some factorisations.

The first method of this group has been proposed by Tammet in 1991, see [27] or [28, ch. 6]. This method applies to a class of formulae which is an extension of both the monadic class and the class of Ackermann. Let Γ be a satisfiable set of formulae of this class. Then the algorithm of Tammet works in the following way:

1. $\Delta := \varnothing$.

[10] One explanation why this is so can be found in [1].

[11] Or, equivalently, of a complete cut-free sequent calculus for this logic.

2. Find a termal identity $\tau = \sigma$, such that $\tau = \sigma$ does not follow from Δ and $\Gamma \cup \Delta \cup \{\tau = \sigma\}$ is satisfiable.
3. In case this was impossible — return back with backtracking.
4. $\Delta := \Delta \cup \{\tau = \sigma\}$.
5. If Δ implies that all ground terms have finitely many different values, then stop. A factorisation of Herbrand universe based on Δ gives a finite model.
6. Otherwise go to 2.

Later Tammet found a way to remove the need for backtracking, see [11, ch. 7] or [28, ch. 5]. His new method is both more efficient and easier to implement.

Another method for finite model building is based on an idea by Fermüller and Leitsch (1996) [10], see also [8, ch. 3]. I will describe this method more thoroughly in §5.

§4. RESOLUTION

The Basic Method

There exist different definitions of what constitutes a clause. I am going to use the following one:

Definition. (1) *Literal* is an atomic formula or a negation of an atomic formula. *Positive literal* is an atomic formula and *negative literal* is a negation of an atomic formula.

(2) *Clause* is a formula which is either a literal, or a disjunction of two or more literals, or equal to $\bot$.

If δ is a clause and s is a substitution, I am going to denote the result of the application of s to δ by $\delta[s]^{[X]}$. The reasons behind this "strange" notation are going to become apparent later.

When the literal λ occurs in the clause δ, I am going to denote this by $\lambda \in \delta$.

For any set of literals Γ, the clause which is obtained from a clause δ by removing all literals belonging to Γ will be denoted by $\delta \setminus \Gamma$. If all literals of δ belong to Γ, then $\delta \setminus \Gamma = \bot$. When λ is a literal, I am going to write $\delta \setminus \lambda$ instead of $\delta \setminus \{\lambda\}$.

If φ is an atomic formula, then $\overline{\varphi} = \neg\varphi$ and $\overline{\neg\varphi} = \varphi$. For any literal λ, $\overline{\overline{\lambda}} = \lambda$.

Two clauses δ and ε are *variants*, if δ can be obtained from ε by means of bijective renaming of the variables in ε. Two clauses have *disjoint dependency* if no variable occurs in both of them. *Disjoint variants* of clauses δ and ε are variants δ' and ε' of δ and ε, such that δ' and ε' have disjoint dependency.

Definition. (1) A *factor* of a clause δ is a clause $\delta[s]^{[X]}$, such that s is most general unifier of a non-empty set of literals of δ.

(2) Given clauses δ and ε and literals $\lambda \in \delta$ and $\mu \in \varepsilon$, if s is the most general unifier of λ and $\overline{\mu}$, the clause $((\delta \setminus \lambda) \vee (\varepsilon \setminus \mu))[s]^{[X]}$ is called *binary resolvent* of δ and ε.

(3) The literal μ is called *resolved literal*.

(4) *Resolvent* of δ and ε is a binary resolvent of disjoint variants of factors of δ and ε.

The basic idea of the method of *resolution* is the following. We start with an arbitrary set Γ of clauses and then we add to Γ all resolvents of elements of Γ. We continue by adding to Γ all resolvents of the elements of the new, enlarged set. If we reach $\perp$ after finitely many steps, then the initial set Γ is not satisfiable (soundness of the resolution). Otherwise, that is if we never reach $\perp$, then the set Γ is satisfiable (completeness of the resolution).

This procedure is executable by a formal algorithm. Therefore, the resolution gives us an algorithm which semidecides the unsatisfiability of any finite set of clauses.[12] If the set is inconsistent, the algorithm is going to find this by producing $\perp$ after finitely many steps. But when the set is not inconsistent, i.e. it is satisfiable, the process of producing more and more resolvents usually continues ad infinitum.

Are there some theoretical estimations of how many steps are required of this method in order to prove the unsatisfiability of a set? Unfortunately, no. The number of the steps required by the resolution is not bounded by any polynomial, or exponent, or superexponent. In fact, it is not very difficult to see that the number of the steps required by an algorithm solving a semidecidable problem which is not decidable can not be bound by any total computable function. Therefore, the computational complexity of the resolution can not be measured by any of the numerical functions people normally work with.

The basic method of resolution is not suitable to decide the satisfiability of a set of clauses. Usually, when the initial set is satisfiable the method

[12] It is possible to adapt the method of resolution in order to semidecide the unsatisfiability of any recursively enumerable set of clauses.

continues ad infinitum generating more and more clauses. There are several resolution refinements aiming to restrict somehow the number of the produced resolvents in order to make the whole procedure more efficient. Surely, such refinements make the method of resolution significantly more efficient when used to prove unsatisfiability, but it turns out there is another benefit — some useful classes of formulae become decidable by such refinements.

Subsumption

The so called "subsumption" is useful resolution refinement. Suppose we have produced clauses $\mathtt{p} \vee \mathtt{q}$ and $\mathtt{p} \vee \mathtt{q} \vee \mathtt{r}$. The first clause is both shorter than the second clause and more informative than it (the second clause follows from it). So it makes sense to prefer the first clause to the second while producing resolvents. In fact we can remove the second clause from the set of the clauses without loss of the completeness.

Definition. A clause δ *subsumes* a clause ε if there exists a substitution s, such that all literals of $\delta[s]^{[X]}$ are literals of ε too.

Example. Assuming $\mathtt{x}$ is a variable and $\mathtt{c}$ is a constant symbol, the clause $\mathtt{p}(\mathtt{c}) \vee \mathtt{q}(\mathtt{x})$ is subsumed by the clause $\mathtt{p}(\mathtt{x})$.
The same literals occur in the clauses $\mathtt{p} \vee \mathtt{q}$ and $\mathtt{q} \vee \mathtt{q} \vee \mathtt{p}$. Therefore, each of them is subsumed by the other.

Suppose two clauses δ' and δ'' contain the same sets of literals. Then δ' subsumes δ'' and δ'' subsumes δ'. Therefore, if we use resolution with subsumption, we can remove one of these clauses without loss of completeness. Consequently, we can work with clauses as if they are sets of literals rather than formulae of special kind. For instance, usually there is no need to think that the clauses $\mathtt{p} \vee \mathtt{q}$ and $\mathtt{q} \vee \mathtt{q} \vee \mathtt{p}$ are two different clauses.

Condensation

In some cases, an instance of a clause can be stronger than the original clause. For example consider the clause $\mathtt{p}(\mathtt{x}) \vee \mathtt{p}(\mathtt{a})$. If we apply to it the substitution

$$s\xi = \begin{cases} \mathtt{a} & \text{if } \xi = \mathtt{x} \\ \xi & \text{if } \xi \neq \mathtt{x} \end{cases}$$

the result will be the clause $\mathtt{p}(\mathtt{a}) \vee \mathtt{p}(\mathtt{a})$ which can be simplified by subsumption to $\mathtt{p}(\mathtt{a})$. All literals of the new clause $\mathtt{p}(\mathtt{a})$ belong to the initial clause $\mathtt{p(x)} \vee \mathtt{p(a)}$ while the opposite is not true — the literal $\mathtt{p(x)}$ does not belong to the new clause. Therefore, it will be beneficial to replace the clause $\mathtt{p(x)} \vee \mathtt{p(a)}$ with $\mathtt{p(a)}$.

Definition. (1) A clause δ is *condensed* if no literal occurs more than once in δ and there exists no instance $\delta[s]^{[X]}$ of δ, such that all literals of $\delta[s]^{[X]}$ are literals of δ and not all literals of δ are literals of $\delta[s]^{[X]}$.

(2) A clause δ is called *condensation* of a clause ε if δ is condensed, all literals of δ are literals of ε and the set of the literals of δ is equal to the set of the literals of $\varepsilon[s]^{[X]}$ for some substitution s.

Each clause has unique condensation up to renaming of the variables and permutation of the literals. [18, p. 406]

Positive Resolution with Restricted Factoring

In order to produce a resolvent of two clauses, we first generate factors of these clauses and then we produce a binary resolvent. In order to restrict the number of the generated resolvents, it will be beneficial if we are able to restrict somehow the ways we generate factors. One method which preserves the completeness is the following: instead of arbitrary unifications when producing the factors, we permit only unifications of positive literals merged into the positive resolved literal.

Example. Consider the clauses $\mathtt{p(x)} \vee \mathtt{p(c)} \vee \mathtt{q(y)} \vee \mathtt{q(z)}$ and $\neg\mathtt{p(x)} \vee \neg\mathtt{p(c)}$. The restricted factoring means we are permitted to unify the literals $\mathtt{p(x)}$ and $\mathtt{p(c)}$ in order to produce a resolvent. We are not permitted to unify $\mathtt{q(y)}$ with $\mathtt{q(z)}$, since these literals are not resolved. It is possible to resolve the literals $\neg\mathtt{p(x)}$ and $\neg\mathtt{p(c)}$ if we unify them, but since they are not positive, we are not permitted to do so, either.

Another refinement of the basic resolution method is to require one of the clauses to be positive (i.e. without negative literals). The following definition formalises the simultaneous use of this refinement with the restricted factoring:

Definition. (1) A clause is *positive* if it contains no negative literals.

(2) Given a clause δ and a positive clause ε, a literal $\lambda \in \delta$ and a non-empty set Γ of literals of ε, if s is the most general unifier of $\{\overline{\lambda}\} \cup \Gamma$, then the clause $((\delta \setminus \lambda) \vee (\varepsilon \setminus \Gamma))[s]^{[X]}$ is called *positive resolvent (with restricted factoring)* of δ and ε.

(3) The elements of Γ are called *resolved literals*.

Positive Hyperresolution

Consider the following clauses:

$$\neg\mathtt{p} \vee \neg\mathtt{q} \vee \mathtt{r}_1 \tag{1}$$

$$\mathtt{p} \vee \mathtt{r}_2 \tag{2}$$

$$\mathtt{q} \vee \mathtt{r}_3 \tag{3}$$

There are two different ways to produce positive resolvents.

On one hand, we can start with (1) and (2) and produce a positive resolvent $\neg\mathtt{q} \vee \mathtt{r}_1 \vee \mathtt{r}_2$. Then from this clause and (3) we can produce the positive resolvent $\mathtt{r}_1 \vee \mathtt{r}_2 \vee \mathtt{r}_3$.

On the other hand, we can start with (1) and (3) and produce a positive resolvent $\neg\mathtt{p} \vee \mathtt{r}_1 \vee \mathtt{r}_3$. Then from this clause and (2) we can produce the positive resolvent $\mathtt{r}_1 \vee \mathtt{r}_3 \vee \mathtt{r}_2$.

Notice that the clauses $\mathtt{r}_1 \vee \mathtt{r}_2 \vee \mathtt{r}_3$ and $\mathtt{r}_1 \vee \mathtt{r}_3 \vee \mathtt{r}_2$ contain equal sets of literals. Such clauses can be considered essentially one clause. There is no point to produce such clauses in two different ways. In order to avoid this duplication, the notion "hyperresolution" is introduced. The idea is to produce the hyperresolvent $\mathtt{r}_1 \vee \mathtt{r}_2 \vee \mathtt{r}_3$ in just one step rather than by means two successive positive resolvents.

Definition. (1) A *clash sequence* is a sequence of clauses $\langle \varepsilon, \delta_1, \ldots, \delta_n \rangle$, such that $n \geq 1$, ε is not positive and $\delta_1, \ldots, \delta_n$ are positive clauses. The clause ε is called *nucleus* and $\delta_1, \ldots, \delta_n$ are *electrons*.

(2) Let $\langle \varepsilon, \delta_1, \ldots, \delta_n \rangle$ be a clash sequence. Let the clauses $\delta_0, \delta_1, \ldots, \delta_n$ be such that $\delta_0 = \delta$, and δ_{i+1} is condensation of a positive resolvent of variants of δ_i and ε_{i+1} having disjoint dependency. If δ_n is positive, then it is called *positive hyperresolvent* defined by the clash sequence $\langle \varepsilon, \delta_1, \ldots, \delta_n \rangle$.

It can be shown that if δ is a positive hyperresolvent defined by some clash sequence, then all clash sequences obtained by permutations of the electrons produce positive hyperresolvents having the same sets of literals as δ.

Returning to the previous example, notice that $\mathtt{r}_1 \vee \mathtt{r}_2 \vee \mathtt{r}_3$ is a positive hyperresolvent defined by the clash sequence $\langle \neg\mathtt{p} \vee \neg\mathtt{q} \vee \mathtt{r}_1, \mathtt{p} \vee \mathtt{r}_2, \mathtt{q} \vee \mathtt{r}_3 \rangle$ and $\mathtt{r}_1 \vee \mathtt{r}_3 \vee \mathtt{r}_2$ is a positive hyperresolvent defined by the clash sequence $\langle \neg\mathtt{p} \vee \neg\mathtt{q} \vee \mathtt{r}_1, \mathtt{q} \vee \mathtt{r}_3, \mathtt{p} \vee \mathtt{r}_2 \rangle$. Since the clash sequence $\langle \neg\mathtt{p} \vee \neg\mathtt{q} \vee \mathtt{r}_1, \mathtt{q} \vee \mathtt{r}_3, \mathtt{p} \vee \mathtt{r}_2 \rangle$ can be obtained from $\langle \neg\mathtt{p} \vee \neg\mathtt{q} \vee \mathtt{r}_1, \mathtt{p} \vee \mathtt{r}_2, \mathtt{q} \vee \mathtt{r}_3 \rangle$ by permutation of the electrons, we can chose arbitrarily one of these sequences and disregard the other one.[13]

[13]One alternative is to permit all permutations of the electrons but to require the

The positive hyperresolution can be combined with subsumption. We start with an arbitrary set Γ of clauses. Then we add to Γ all positive hyperresolvents defined by clash sequences whose elements are elements of Γ. If we reach $\perp$ after finitely many steps, then the initial set Γ is not satisfiable (soundness of the positive hyperresolution). Otherwise, that is if we never reach $\perp$, then the set Γ is satisfiable (completeness of the positive hyperresolution).

Definition. A *reducing function* $\mathfrak{g}$ is a function mapping each finite set Γ of clauses to a subset $\mathfrak{g}(\Gamma) \subseteq \Gamma$, such that any element of Γ is subsumed by some element of $\mathfrak{g}(\Gamma)$ and no element of $\mathfrak{g}(\Gamma)$ is subsumed by different element of $\mathfrak{g}(\Gamma)$.

The following definition formalises the simultaneous use of the positive hyperresolution with a reducing function:

Definition. (1) Let $\mathfrak{g}$ be some reducing function. For any set Γ of clauses, if Γ' is the set of all positive hyperresolvents defined by clash sequences whose elements are elements of Γ, then let $\mathfrak{res}(\mathfrak{g};\Gamma)$ be the set $\mathfrak{g}(\Gamma \cup \Gamma')$.

(2) For any natural number n, $\mathfrak{res}^n(\mathfrak{g};\Gamma)$ is the iterative application of the operator $\mathfrak{res}$. Namely, $\mathfrak{res}^0(\mathfrak{g};\Gamma) = \Gamma$ and $\mathfrak{res}^{n+1}(\mathfrak{g};\Gamma) = \mathfrak{res}(\mathfrak{g};\mathfrak{res}^n(\mathfrak{g};\Gamma))$.

(3) $\mathfrak{res}^*(\mathfrak{g};\Gamma)$ is the union of all $\mathfrak{res}^n(\mathfrak{g};\Gamma)$ for all n.

The following theorem states that the positive hyperresolution with subsumption is both sound and complete:

Theorem. *Given a reducing function* $\mathfrak{g}$*, a set* Γ *of clauses is satisfiable if and only if* $\perp \notin \mathfrak{res}^*(\mathfrak{g};\Gamma)$.

Notice that the reasoning with subsumption is non-monotonic. Usually, $\mathfrak{res}^n(\mathfrak{g};\Gamma)$ will not be a subset of $\mathfrak{res}^{n+1}(\mathfrak{g};\Gamma)$.

Orderings and Splitting

The *ordering refinements* are one especially effective additional restriction still preserving the completeness of the resolution.[14] The ordering is a partial ordering of the set of the literals satisfying some additional properties. The completeness of the positive hyperresolution will be pre-

negative literals of the nucleus to be resolved in an specific order.

[14] The orderings were already present in the first publication of Maslov [22] about his "inverse method" in 1964. Independently from Maslov, in 1969 Kowalski and Hayes [20] introduced the orderings for the method of resolution in somewhat more limited form.

served if we impose the following additional restriction: whenever λ is resolved literal from electron ε we require that no other literal of ε strongly precedes λ with respect to the ordering. An extensive treatment on orderings is given in [11, ch. 4] and [28, ch. 3].

The final resolution refinement I am going to mention is the so called *splitting*. [29] Let the clause $\delta \vee \varepsilon$ be such that no variable occurs simultaneously in δ and ε. Because of the disjoint variables of δ and ε, the universal quantification of the clause $\delta \vee \varepsilon$ is true in a structure $\mathbf{M}$ if and only if the disjunction of the universal quantifications of δ and of ε is true in $\mathbf{M}$. Therefore a set of clauses $\Gamma \cup \{\delta \vee \varepsilon\}$ is inconsistent if and only if both $\Gamma \cup \{\delta\}$ and $\Gamma \cup \{\varepsilon\}$ are inconsistent. Or, equivalently, a set of clauses $\Gamma \cup \{\delta \vee \varepsilon\}$ is satisfiable if and only if at least one of the sets $\Gamma \cup \{\delta\}$ and $\Gamma \cup \{\varepsilon\}$ is satisfiable.

Now, suppose that we have produced such a clause $\delta \vee \varepsilon$ by resolution. Then we are permitted to replace the clause $\delta \vee \varepsilon$ first by δ and then by ε. If the set with δ is satisfiable, the initial set is satisfiable too. If the set with ε is satisfiable, the initial set is satisfiable too. And, if we reach $\perp$ in both cases, then the initial set of clauses is inconsistent.

§5. MODEL BUILDING BY RESOLUTION

As it has been shown by Joyner [18], for several classes of predicate formulae the satisfiability problem can be decided by means of refinements of the method of resolution. The classes VED, PVD and OCC1N are interesting because they do not require specially crafted refinement but can be decided by a rather general method — the positive hyperresolution. In particular, no orderings are necessary in order to decide these classes.

The Class VED

Definition. (1) A clause is a *Horn clause*, if it contains at most one positive literal.

(2) A clause δ belongs to the class *VED* if it is a Horn clause and for any variable $\mathtt{x}$ in δ, all occurrences of $\mathtt{x}$ in δ are at equal depth.

The positive hyperresolution decides this class. Namely, the following is true:

Theorem. *Let $\mathfrak{g}$ be a reducing function. Let Γ be a finite set of clauses which is a subset of VED. Then:*

(1) *All elements of* $\mathfrak{res}^*(\mathfrak{g};\Gamma)$ *belong to VED.*

(2) *If* Γ *is inconsistent, then* $\bot \in \mathfrak{res}^*(\mathfrak{g};\Gamma)$.

(3) *If* Γ *is satisfiable, then there exists a natural number* n*, such that* $\mathfrak{res}^n(\mathfrak{g};\Gamma) = \mathfrak{res}^i(\mathfrak{g};\Gamma)$ *for any* $i \geq n$.[15]

Proof. See [8, ch. 1.2]. ■

Let Γ be a satisfiable finite set of VED clauses. Since all elements of VED are Horn clauses, all positive clauses belonging to the finite set $\mathfrak{res}^*(\mathfrak{g};\Gamma)$ are atomic formulae. It can be shown that these atomic formulae yield the following finite representation of a Herbrand model of Γ.[16]

Theorem. *Let* Γ *be a satisfiable finite set of VED clauses. Then the set* $\mathfrak{res}^*(\mathfrak{g};\Gamma)$ *is finite and all positive clauses belonging to it are atomic formulae. Let* $\mathbf{M}$ *be Herbrand structure with the following interpretation of the predicate symbols:* $\mathtt{p}(\tau_1,\dots,\tau_n)$ *is true if and only if* $\mathtt{p}(\tau_1,\dots,\tau_n)$ *is a ground instance of an element of* $\mathfrak{res}^*(\mathfrak{g};\Gamma)$*. Then* $\mathbf{M}$ *is a model of* Γ.[17]

Proof. See Theorem 1.1 in [8, ch. 1.1]. ■

The Classes PVD and OCC1N

Definition. (1) A clause δ belongs to the class *PVD* if for any occurrence of a varianbe in a positive literal of δ there exists an occurrence of the same variable in a negative literal of δ at greater or equal depth.

(2) A clause δ belongs to the class *OCC1N* if no variable occurs more than once in a positive literal and any occurrence of a varianbe in a positive literal of δ is at smaller or equal depth than any occurrence of the same variable in the negative literals of δ.

The positive hyperresolution decides both classes. Namely, the following is true:

Theorem. *Let* $\mathfrak{g}$ *be a reducing function. Let* Γ *be a finite set of clauses which is a subset of PVD or OCC1N. Then:*

(1) *All elements of* $\mathfrak{res}^*(\mathfrak{g};\Gamma)$ *belong to PVD or resp. to OCC1N.*

(2) *If* Γ *is inconsistent, then* $\bot \in \mathfrak{res}^*(\mathfrak{g};\Gamma)$.

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[15] In fact, with suitable reducing function, even when Γ is inconsistent, there exists a natural number n, such that for any $i \geq n$, $\mathfrak{res}^i(\mathfrak{g};\Gamma) = \{\bot\}$. This is so because $\bot$ subsumes any clause.

[16] I suppose the finite atomic representations have been introduced first in [7] but I don't have access to this work.

[17] In fact, $\mathbf{M}$ is the so called "minimal Herbrand model".

(3) *If* Γ *is satisfiable, then there exists a natural number* n*, such that* $\mathfrak{res}^n(\mathfrak{g};\Gamma) = \mathfrak{res}^i(\mathfrak{g};\Gamma)$ *for any* $i \geq n$*.*

Proof. See [11, ch. 3.2].[18] ■

Finite Satisfiability and Resolution

Since the positive hyperresolution is sound in any structure, if a set Γ of clauses is universally valid in a finite structure $\mathbf{M}$, all hyperresolvents obtained from Γ will be universally valid in $\mathbf{M}$, hence it will not be possible to derive $\perp$ from Γ. This means that the positive hyperresolution is sound with respect to the finite unsatisfiability.

Is the positive hyperresolution complete with respect to the finite unsatisfiability?

Suppose that the satisfiable finite set Γ is such that the positive hyperresolution stops after finitely many steps.[19] Can we conclude from this that Γ has a finite model?

Unfortunately, the answer is negative. The following example by Baaz (1996), cited in [8, proposition 1.1 in ch. 1.2], shows that the positive hyperresolution is not complete with respect to the finite unsatisfiability.

Example. Consider the set of the following three clauses:

$$\mathtt{p}(\mathtt{x},\mathtt{x}) \tag{1}$$

$$\neg\mathtt{p}(\mathtt{f}(\mathtt{x}),\mathtt{f}(\mathtt{y})) \vee \mathtt{p}(\mathtt{x},\mathtt{y}) \tag{2}$$

$$\neg\mathtt{p}(\mathtt{c},\mathtt{f}(\mathtt{x})) \tag{3}$$

The positive hyperresolution trivially terminates on Γ. There is only one possible positive hyperresolvent. From the clash sequence $\langle(2),(1)\rangle$ we can produce $\mathtt{p}(\mathtt{x},\mathtt{x})$ which is equal to (1).

Nevertheless, Γ has no finite models. Let us see why this is so. Let $\mathtt{p}^{\mathbf{M}}$, $\mathtt{f}^{\mathbf{M}}$ and $\mathtt{c}^{\mathbf{M}}$ be the respective interpretations in $\mathbf{M}$ of the symbols $\mathtt{p}$, $\mathtt{f}$ and $\mathtt{c}$. Let $|\mathbf{M}|$ be the universe of $\mathbf{M}$. Suppose $\mathbf{M}$ is a model of these clauses. Clause (3) implies that $\mathtt{p}^{\mathbf{M}}\langle\mathtt{c}^{\mathbf{M}},\mathtt{f}^{\mathbf{M}}(\mu)\rangle$ is false for any $\mu \in |\mathbf{M}|$. From this and (2) we obtain that $\mathtt{p}^{\mathbf{M}}\langle\mathtt{f}^{\mathbf{M}}(\mathtt{c}^{\mathbf{M}}),\mathtt{f}^{\mathbf{M}}(\mathtt{f}^{\mathbf{M}}(\mu))\rangle$ is false for any $\mu \in |\mathbf{M}|$. From this and (2) we obtain that $\mathtt{p}^{\mathbf{M}}\langle\mathtt{f}^{\mathbf{M}}(\mathtt{f}^{\mathbf{M}}(\mathtt{c}^{\mathbf{M}})),\mathtt{f}^{\mathbf{M}}(\mathtt{f}^{\mathbf{M}}(\mathtt{f}^{\mathbf{M}}(\mu)))\rangle$ is false for any $\mu \in |\mathbf{M}|$. And so on. If we compare these results with (1), we will be able to conclude that all elements of the sequence

$$\mathtt{c}^{\mathbf{M}},\mathtt{f}^{\mathbf{M}}(\mathtt{c}^{\mathbf{M}}),\mathtt{f}^{\mathbf{M}}(\mathtt{f}^{\mathbf{M}}(\mathtt{c}^{\mathbf{M}})),\mathtt{f}^{\mathbf{M}}(\mathtt{f}^{\mathbf{M}}(\mathtt{f}^{\mathbf{M}}(\mathtt{c}^{\mathbf{M}}))),\ldots$$

[18] The positive hyperresolution decides the class PVD even without condensation or restricted factoring. Both are required in order to decide OCC1N.

[19] This means $\perp \notin \mathfrak{res}^*(\mathfrak{g};\Gamma)$ and for some n, $\mathfrak{res}^n(\mathfrak{g};\Gamma) = \mathfrak{res}^i(\mathfrak{g};\Gamma)$ for any $i \geq n$.

are different elements of $|\mathbf{M}|$.

Finite Models by Linear Atomic Representations of Herbrand Models

Notice that the definition of PVD implies that all positive clauses belonging to PVD are ground. On the other hand, the definition of OCC1N implies that no variable may occur more than once in any positive clause belonging to OCC1N.

Definition. A clause is *linear* if no variable occurs more than once in it. In particular, all ground clauses are linear.

All positive clauses belonging to PVD or OCC1N are linear.

Recall that the rule of splitting allows us to decompose a clause if it contains two parts without common variables. No two literals in a linear clause may contain a common variable, so we can apply splitting to all such clauses, "decomposing" the positive hyperresolvents to atomic formulae. Since all positive hyperresolvents of clauses belonging to PVD or OCC1N are linear, we can decompose all of them to atomic formulae and, in result, from these atomic formulae we will obtain an atomic representation of a Herbrand model.

Theorem. *Let Γ be a satisfiable finite set of PVD or OCC1N clauses. Then, if we use the positive hyperresolution together with splitting, we will obtain a set Δ of clauses, such that $\bot \notin \Delta$, all clauses in Γ are subsumed by clauses in Δ, all positive clauses in Δ are atomic formulae and all positive hyperresolvents based on clash sequence whose elements belong to Δ also belong to Δ.*

Let $\mathbf{M}$ be Herbrand structure with the following interpretation of the predicate symbols: $\mathtt{p}^{\mathbf{M}}\langle\tau_1, \ldots, \tau_n\rangle$ is true if and only if $\mathtt{p}(\tau_1, \ldots, \tau_n)$ is a ground instance of an element of Δ. Then $\mathbf{M}$ will be a model of Γ.

In the atomic representations of Herbrand models of VED clauses any atomic formulae are possible. However, in the atomic representations of Herbrand models of clauses belonging to PVD or OCC1N, all atomic formulae defining the model are linear. Because of this it becomes possible to transform such Herbrand model to a finite model.

Proposition. *Let Γ be a finite set of linear atomic formulae. Then there exists a computable from Γ finite set Δ of atomic formulae, such that a ground atomic formula is an instance of a formula of Γ if and only if it* *is not an instance of a formula of* Δ.

Proof. See [8, ch. 1.4]. ■

Suppose that Γ is a linear finite atomic representation of a Herbrand structure. From this proposition we can obtain a finite set Δ representing all cases when the atomic formulae are not true in this Herbrand structure. Let $\mathbf{M}$ be a finite structure, such that all elements of its universe are values of ground terms and for no formulae $\mathtt{p}(\tau_1, \ldots, \tau_n) \in \Gamma$ and $\mathtt{p}(\sigma_1, \ldots, \sigma_n) \in \Delta$ there exist assignment functions v_1 and v_2, such that the n-tuple consisting of the values of the terms $\tau_1, \ldots, \tau_n$ in $\mathbf{M}$ with assignment v_1 is equal to the respective n-tuple for the terms $\sigma_1, \ldots, \sigma_n$ with assignment v_2. The existence of such structure follows from (29C).[20]

Now, interpret the predicate symbols in $\mathbf{M}$ in the following way:

For any μ belonging to the universe of $\mathbf{M}$, let $t(\mu)$ be a ground term, whose value in $\mathbf{M}$ is μ. Let $\mathtt{p}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle$ be true if and only if $\mathtt{p}(t(\mu_1), \ldots, t(\mu_n))$ is a ground instance of an element of Γ.

One can see that the function mapping each ground term to its value in $\mathbf{M}$ is a surjective homomorphism from Herbrand model represented by Γ to $\mathbf{M}$. Consequently, a clause is universally valid in the finite structure $\mathbf{M}$ if and only if it is universally valid in Herbrand structure represented by Γ.

§6. TOWARDS RESOLUTION WITH TERMOIDS

Resolution in Algebras

Definition. (1) The *universe* of a structure $\mathbf{M}$ will be denoted by $|\mathbf{M}|$.

(2) *Assignment function* in $\mathbf{M}$ is a function mapping each variable to an element of $|\mathbf{M}|$.

(3) We will need special constant symbols representing the elements of the universe of a structure. For any $\mu \in |\mathbf{M}|$ the symbol representing μ will be denoted by $\ulcorner\mu\urcorner$. Such symbols will be called *names*.

(4) If τ is a term or a clause and v is an assignment function in $\mathbf{M}$, the result of the replacement of each variable $\mathtt{x}$ in τ by $\ulcorner v\mathtt{x}\urcorner$ will be denoted by $\tau[v]$. By $\tau[v]^{\mathbf{M}}$ we will denote the *value* of τ in the structure $\mathbf{M}$ with assignment v.

(5) A clause δ is universally valid in a structure $\mathbf{M}$ if $\delta[v]^{\mathbf{M}}$ is true for any assignment v in $\mathbf{M}$. Obviously, this is so if and only if all ground clauses

[20] Since the elements of Γ are linear, we do not really have to use a strong proposition like (29C). In [8, ch. 1.4] the existence of a finite structure with this property is shown directly.

having the form $\delta[v]$ are true in $\mathbf{M}$.

It is not difficult to see that if two clauses δ and ε are universally valid in a structure $\mathbf{M}$, then all their resolvents will be universally valid in $\mathbf{M}$. Likewise, if the clauses $\delta, \varepsilon_1, \dots, \varepsilon_n$ are universally valid in $\mathbf{M}$, then all positive hyperresolvents defined by the clash sequence $\langle \delta, \varepsilon_1, \dots, \varepsilon_n \rangle$ are universally valid in $\mathbf{M}$. This means that the method of resolution and its modifications are sound not only with respect to the satisfiability, but also with respect to the universal validity in a structure.

Definition. (1) Informally, *algebra* is a structure without interpretation of the predicate symbols.

(2) The algebra corresponding to a structure $\mathbf{M}$ is called called *algebraic fragment* of $\mathbf{M}$. The algebraic fragment of $\mathbf{M}$ will be denoted by $\partial\mathbf{M}$.

(3) A clause δ is *universally satisfiable* in an algebra $\mathbf{A}$ if there exists a structure $\mathbf{M}$, such that $\mathbf{A} = \partial\mathbf{M}$ and δ is universally valid in $\mathbf{M}$. A set Γ of clauses is universally satisfiable in an algebra $\mathbf{A}$ if there exists a structure $\mathbf{M}$, such that $\mathbf{A} = \partial\mathbf{M}$ and all elements of Γ are universally valid in $\mathbf{M}$.

Suppose a set Γ of clauses is universally satisfiable in an algebra $\mathbf{A}$. Then there exists a structure $\mathbf{M}$, such that $\mathbf{A} = \partial\mathbf{M}$ and all clauses of Γ are universally valid in $\mathbf{M}$. Since all clauses derivable by resolution or hyperresolution from the clauses of Γ have to be universally valid in $\mathbf{M}$, the clause $\bot$ can not be derivable from Γ. Consequently the method of resolution and its modifications are sound with respect to the satisfiability in an algebra.

The following proposition summarises what we have found so far about the soundness of the resolution:

Proposition. *If a set of clauses is satisfiable, or universally valid in a structure, or satisfiable in an algebra, then $\bot$ is not derivable by resolution or hyperresolution from it.*

Is the method of resolution complete with respect to the unsatisfiability in algebra? Unfortunately, the answer is "no" and it is easy to see why.

Example. Consider the set $\Gamma = \{\mathtt{p}(\mathtt{a}), \neg\mathtt{p}(\mathtt{b})\}$ where $\mathtt{a}$ and $\mathtt{b}$ are different constant symbols. Let $\mathbf{A}$ be an algebra, such that $\mathtt{a}^{\mathbf{A}} = \mathtt{b}^{\mathbf{A}}$. Obviously, Γ is not satisfiable in $\mathbf{A}$, but no resolvents can be produced from Γ, hence $\bot$ is not derivable.

This simple example can help us to understand what we need in order to make the resolution complete with respect to the unsatisfiability in an algebra. The symbols $\mathtt{a}$ and $\mathtt{b}$ have equal values in $\mathbf{A}$, so if we are reasoning with respect to the algebra $\mathbf{A}$, we should be able to unify `a` with `b` in order to produce the resolvent $\perp$ from `p(a)` and $\neg$`p(b)`.

Definition. Given an algebra $\mathbf{A}$,
(1) if τ is a ground term, $\tau^{\mathbf{A}}$ is the value of τ in $\mathbf{A}$;
(2) if δ is a ground clause, $\delta^{\mathbf{A}}$ is the result of replacement in δ of any term τ with $\ulcorner\tau^{\mathbf{A}}\urcorner$.

It is not difficult to see that $\delta^{\mathbf{A}}$ contains no functional symbols for any clause δ. The clauses without functional symbols will be called *relational*.
For any ground clause δ and structure $\mathbf{M}$, $\delta^{\mathbf{M}} = (\delta^{\partial\mathbf{M}})^{\mathbf{M}}$.

Proposition. *Given an algebra* $\mathbf{A}$*, consider the combination of the following two rules:*

1. *For any assignment* v *in* $\mathbf{A}$*, we produce* $\delta[v]^{\mathbf{A}}$ *from* δ*.*
2. *We produce resolvents from any ground relational clauses* δ *and* ε*.*

$\perp$ *is derivable from a set* Γ *of clauses by these two rules if and only if* Γ *is unsatisfiable in* $\mathbf{A}$*.*

This proposition gives us a method which is both sound and complete with respect to the unsatisfiability in an algebra. In fact, both the soundness and the completeness will be preserved if instead of the basic resolution method we use positive hyperresolution combined with many other resolution refinements. Nevertheless, the algorithmic usefulness of such resolutive procedure is limited.

For one thing, in order to be able to produce $\delta[v]^{\mathbf{A}}$ from δ, the algebra $\mathbf{A}$ has to be constructivisable (computable).[21] And even if this is so, unless the universe of $\mathbf{A}$ is finite, infinitely many clauses $\delta[v]^{\mathbf{A}}$ will exist, so there will be infinitely many resolvents, hence we will be able to only semidecide the problem of the unsatisfiability in an algebra. We will not be able to use resolution in order to prove the satisfiability in $\mathbf{A}$ and, even less so, in order to build a model $\mathbf{M}$ with the property $\partial\mathbf{M} = \mathbf{A}$.

Remark. Let me mention that the problem of the satisfiability of a set of clauses in an algebra is much easier than the problem of the satisfiability in an algebra of a set of arbitrary formulae. For example, consider the algebra whose universe is the set of the natural numbers and the functional symbols are interpreted with various computable functions. While the problem of the unsatisfiability in this algebra of any computably enumerable set of clauses is semidecidable, the problem of the unsatisfiability

[21] That is the interpretations of all functional and predicate symbols are computable with respect to a some enumeration of the universe of $\mathbf{A}$.

of even one arbitrary formula is not semidecidable (for most algebras of this kind).

Resolution which is Complete for Finite Satisfiability

If a set Γ of clauses is satisfiable in an algebra $\mathbf{A}$, then $\perp$ is not derivable by resolution from Γ. On the other hand, if $\perp$ is not derivable from Γ, the set Γ is universally valid in some but not necessarily satisfiable in $\mathbf{A}$. We have already seen that the main reason for this is the fact that the usual unification algorithm is, in a sense, incomplete with respect to $\mathbf{A}$.

Suppose our goal is to find a finite model of a set of clauses. In this case we are not interested in the satisfiability in a particular algebra $\mathbf{A}$. Instead, we would like to have a resolutive method which is complete with respect to sufficiently large class of algebras — large enough to include at least one finite algebra. If such method is unable to derive $\perp$ from a set Γ of clauses, we can be sure that Γ is satisfiable in at least one finite algebra.

In order to invent such a resolutive method, we have to investigate the following question: why the usual method of resolution is not complete with respect to the satisfiability in finite algebras?

Let us look at some examples.

Example. (1) $\Gamma_1 = \{\mathtt{p(a)}, \neg\mathtt{p(b)}\}$ where $\mathtt{a}$ and $\mathtt{b}$ are different constant symbols. No resolvents can be derived from Γ_1. Notice that Γ_1 is finitely satisfiable. In fact, Γ_1 is satisfiable in any algebra $\mathbf{A}$ where $\mathtt{a}^{\mathbf{A}} \neq \mathtt{b}^{\mathbf{A}}$.

(2) $\Gamma_2 = \{\mathtt{p(f(x,y))}, \neg\mathtt{p(g(x,x))}\}$ where $\mathtt{f}$ and $\mathtt{g}$ are different functional symbols. No resolvents can be derived from Γ_2. Notice that Γ_2 is finitely satisfiable. Let $\mathbf{B}$ be an algebra with two elements 0 and 1 in its universe. Let $\mathtt{f}^{\mathbf{B}}$ be a function whose value is always 0 and $\mathtt{g}^{\mathbf{B}}$ be a function whose value is always 1. Then Γ_2 is satisfiable in $\mathbf{B}$. Moreover, Γ_2 is satisfiable in the cartesian product of $\mathbf{B}$ with any other algebra.

(3) $\Gamma_3 = \{\mathtt{p(x,f(x))}, \neg\mathtt{p(x,x)}\}$. No resolvents can be derived from Γ_3. Notice that Γ_3 is finitely satisfiable. Let $\mathbf{B}$ be an algebra with two elements 0 and 1 in its universe and $\mathtt{f}^{\mathbf{B}}(n) = 1 - n$ for any $n \in \{0,1\}$. Then Γ_3 is satisfiable in $\mathbf{B}$. Moreover, Γ_3 is satisfiable in the cartesian product of $\mathbf{B}$ with any other algebra.

It turns out, the incompleteness of the method of resolution is mostly due to one particular step in the unification algorithm,[22] namely the step

[22] The algorithm of Robinson [26].

leading from the unification of

$$\mathtt{f}(\tau_1, \tau_2, \ldots, \tau_n) \sim \mathtt{f}(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{1}$$

to the simultaneous unification of

$$\left| \begin{array}{l} \tau_1 \sim \sigma_1 \\ \tau_2 \sim \sigma_2 \\ \ldots \\ \tau_n \sim \sigma_n \end{array} \right. \tag{2}$$

If the system (2) is true in an algebra, then the identity (1) also is true. The opposite, however, is not so. Identities like (1) are equivalent to the system (2) only in algebras where the functional symbols are interpreted by injective functions. One such algebra is the *Herbrand algebra* (i.e. the algebraic fragment of a Herbrand structure).[23] Notice, however, that it is almost impossible to interpret the functional symbols with injective functions in algebras with finite universe.[24]

In order to make the unification algorithm complete with respect to a relatively large class of algebras, instead of terms we have to use different objects called *termoids*.

§7. BETA-TERMOIDS

One Simple Language

Several kinds of termoids can be defined. Arguably, the simplest kind are the beta-termoids.

Suppose we are working in a language with only one unary functional symbol `f`. Any ground term in this language has the form $\mathtt{f}(\mathtt{f}(\ldots(\mathtt{f}(\mathtt{c}))\ldots))$ for some constant symbol `c`. For brevity, we are going to use the notation $\mathtt{f}^k(\mathtt{c})$ for the term with k functional symbols `f` and constant `c`.

Definition. (1) *Beta-termoid* is an expression of the form $n + \mathtt{f}^k(\mathtt{c})$ where $\mathtt{f}^k(\mathtt{c})$ is an arbitrary term and n is a natural number.

[23]Since in Herbrand algebra the identity (1) is equivalent to the system (2), the method of resolution is both sound and complete with respect to the satisfiability in this algebra. This observation gives us another perspective for the well known result that a set of clauses is satisfiable if and only if it is universally valid in some Herbrand structure.

[24]If there is at least one functional symbol with more than one argument, then this certainly is impossible.

(2) *Value* of the beta-termoid $n + \mathtt{f}^k(\mathtt{c})$ in the algebra $\mathbf{A}$ is any $\alpha \in |\mathbf{A}|$, such that the following identity is true in $\mathbf{A}$:

$$\mathtt{f}^n(\ulcorner\alpha\urcorner) \sim \mathtt{f}^{n+k}(\mathtt{c})$$

(3) $\tau[\![v]\!]^{\mathcal{PM}}$ is the set of all values of τ in the structure $\mathbf{M}$ with assignment v.

Notice that a termoid can have many different values in an algebra. This one of the most noticeable differences between terms and termoids.

Definition. (1) Informally, beta-clausoid is a clause in which beta-termoids are used instead of terms.

(2) A beta-clausoid is universally valid in a structure if it is true with all possible values of its beta-termoids.

Notice that since we want to use resolution in order to prove finite satisfiability, we need a resolutive method which is complete but not necessarily sound. We want to be sure that if $\perp$ is not derivable, then the clausoids have a model of certain kind. On the other hand, we don't have to require the resolvent of clausoids δ and ε to be universally valid in any structure where δ and ε are universally valid.

Beta-termoidal Substitutions

Given an algebra $\mathbf{A}$, suppose that α is value of $n + \mathtt{f}^k(\mathtt{c})$ in $\mathbf{A}$ and β is value of $m + \mathtt{f}^l(\ulcorner\alpha\urcorner)$ in $\mathbf{A}$. Then the following two identities will be true in $\mathbf{A}$:

$$\mathtt{f}^m(\ulcorner\beta\urcorner) \sim \mathtt{f}^{m+l}(\ulcorner\alpha\urcorner)$$
$$\mathtt{f}^n(\ulcorner\alpha\urcorner) \sim \mathtt{f}^{n+k}(\mathtt{c})$$

Let $j = \max\{m, n - l\}$. If we apply $\mathtt{f}^{j+l-n}$ to the first identity and $\mathtt{f}^{j-m}$ to the second one, we can conclude that the following identities are true in $\mathbf{A}$:

$$\mathtt{f}^j(\ulcorner\beta\urcorner) \sim \mathtt{f}^{j+l}(\ulcorner\alpha\urcorner)$$
$$\mathtt{f}^{j+l}(\ulcorner\alpha\urcorner) \sim \mathtt{f}^{j+k+l}(\mathtt{c})$$

Therefore, the identity

$$\mathtt{f}^j(\ulcorner\beta\urcorner) \sim \mathtt{f}^{j+k+l}(\mathtt{c})$$

also is true in $\mathbf{A}$. This means that β is value of $j + \mathtt{f}^{k+l}(\mathtt{c})$ in $\mathbf{A}$.

This observation motivates the following definition:

Definition. (1) Beta-termoidal *substitution* is a function mapping each variable to a beta-termoid.

(2) Given a beta-termoid $\tau = m + \mathtt{f}^l(\mathtt{x})$ and a beta-termoidal substitution s, if $s\mathtt{x} = n + \mathtt{f}^k(\mathtt{y})$ and $j = \max\{m, n - l\}$, then the result of the application of s to τ, written $\tau[\![s]\!]^{[\![X]\!]}$, is the beta-termoid $j + \mathtt{f}^{k+l}(\mathtt{y})$.

(3) Given a beta-clausoid δ and a beta-termoidal substitution s, the result of the application to s to δ, written $\delta[\![s]\!]^{[\![X]\!]}$, is the clausoid obtained from δ by replacing each beta-termoid τ with $\tau[\![s]\!]^{[\![X]\!]}$.

Proposition. *Given a beta-termoid* $\tau = m + \mathtt{f}^l(\mathtt{x})$*, a beta-termoidal substitution* s *and a structure* $\mathbf{M}$*, if* α *is a value in* $\mathbf{M}$ *of* $s\mathtt{x}$ *and* β *is a value in* $\mathbf{M}$ *of* $m + \mathtt{f}^l(\ulcorner\alpha\urcorner)$*, then* β *is a value in* $\mathbf{M}$ *of* $\tau[\![s]\!]^{[\![X]\!]}$*.*

In order to prove the completeness of the resolution we don't need the opposite direction of this proposition. In fact, for some kinds of termoids the application of a termoidal substitution is very rough, so $\tau[\![s]\!]^{[\![X]\!]}$ is going to have much more values than what we can expect from something which is the result of the application of a substitution to a termoid.

Beta-termoidal Unification

In order to define a resolutive method with beta-termoids, we have to define some kind of unification between termoids.

Definition. (1) A *termoidal identity* is an expression of the form $\tau \sim \sigma$, where both τ and σ are beta-termoids or both are beta-clausoids. $\tau \sim \sigma$ is a *termal identity*, if both τ and σ are terms or both are clauses.

(2) A *termoidal system* is a set of termoidal identities. A termoidal system is *finite* if it is a finite set of identities. The notion *termal system* is defined analogously.

(3) An identity $0 + \mathtt{x} \sim \tau$ is *solving* for $\mathtt{x}$ if there are no occurrences of $\mathtt{x}$ in τ. Such an identity is solving for a system Θ, if it belongs to Θ and there are no occurrences of $\mathtt{x}$ in the other identities of Θ. In this case we also say that Θ is *solved* with respect to $\mathtt{x}$.

(4) The assignment v in a structure $\mathbf{M}$ is a *solution* of the termoidal identity $\tau \sim \sigma$, if $\tau[\![v]\!]^{\mathcal{P}\mathbf{M}} \cap \sigma[\![v]\!]^{\mathcal{P}\mathbf{M}} \neq \varnothing$. Notice the peculiarity of this definition — v is a solution of $\tau \sim \sigma$ if at least one value of τ with assignment v is also a value of σ.

(5) The assignment v is a solution of the system Θ if v is a solution of each of the identities of Θ.

(6) A system is *termally consistent* if it has a solution in Herbrand algebra (i.e. the algebraic fragment of a Herbrand structure). A system is *termally inconsistent* if it is not termally consistent.

(7) Two systems are *termally equivalent* if they have same solutions in Herbrand algebra.

We are going to define two kinds of special transformations of a beta-termoidal system. We are going to call them *special solving transformations*.

First special solving transformation. If the system contains an identity of the form $n + \tau \sim m + \sigma$ where $n \neq m$, we replace it with the identity $\max\{n, m\} + \tau \sim \max\{n, m\} + \sigma$.

Or, if it contains an identity $n + \mathtt{f}(\tau) \sim n + \mathtt{f}(\sigma)$, we replace it with $(n + 1) + \tau \sim (n + 1) + \sigma$.

Or, if it contains an identity $n + \tau \sim n + \mathtt{x}$ where $\mathtt{x}$ is a variable and τ is not a variable, we replace it with $0 + \mathtt{x} \sim n{+}\tau$.

Or, if it contains an identity $n + \mathtt{x} \sim n + \tau$ where $n \neq 0$, we replace it with $0 + \mathtt{x} \sim n{+}\tau$.

Or, if it contains an identity $\mathtt{p}(\tau_1, \ldots, \tau_n) \sim \mathtt{p}(\sigma_1, \ldots, \sigma_n)$ where $\mathtt{p}$ is an n-ary predicate symbol, we replace it with the following identities: $\tau_1 \sim \sigma_1, \tau_2 \sim \sigma_2, \ldots, \tau_n \sim \sigma_n$.

Or, if it contains an identity $\neg\varphi \sim \neg\psi$, we replace it with $\varphi \sim \psi$.

Or, if it contains an identity $\varphi' \vee \varphi'' \sim \psi' \vee \psi''$, we replace it with the following two identities: $\varphi' \sim \psi'$ and $\varphi'' \sim \psi''$.

Second special solving transformation. If the system contains a solving identity $0 + \mathtt{x} \sim \tau$, which, however, is not solving for the system, let s be the substitution, such that $s\mathtt{x} = \tau$ and $s\xi = \xi$ for any $\xi \neq \mathtt{x}$. Then we replace each identity $\tau' \sim \tau''$ in the system (except $0 + \mathtt{x} \sim \tau$ itself) with the identity $\tau'[\![s]\!]^{[\![X]\!]} \sim \tau''[\![s]\!]^{[\![X]\!]}$.

Proposition. *If we apply a special solving transformation to a system Θ, the result being Θ', any solution of Θ in an algebra is a solution of Θ' as well.*

<u>Proof.</u> For most kinds of the solving transformations this is obvious. We only have to see that any solution of $n + \mathtt{f}(\tau) \sim n + \mathtt{f}(\sigma)$ is a solution of $(n + 1) + \tau \sim (n + 1) + \sigma$.

Suppose the assignment v in $\mathbf{M}$ is a solution of the identity

$$n + \mathtt{f}(\tau) \sim n + \mathtt{f}(\sigma)$$

By definition, v is a solution, if $(n + \mathtt{f}(\tau))[\![v]\!]^{\mathcal{P}\mathbf{M}} \cap (n + \mathtt{f}(\sigma))[\![v]\!]^{\mathcal{P}\mathbf{M}} \neq \varnothing$, so there exists $\mu \in |\mathbf{M}|$, such that $\mu \in (n + \mathtt{f}(\tau))[\![v]\!]^{\mathcal{P}\mathbf{M}}$ and

$\mu \in (n + \mathtt{f}(\sigma))[\![v]\!]^{\mathcal{P}\mathbf{M}}$, hence $(\mathtt{f}^{\mathbf{M}})^{n+1}(\tau[v]^{\mathbf{M}}) = (\mathtt{f}^{\mathbf{M}})^{n}\mu = (\mathtt{f}^{\mathbf{M}})^{n+1}(\sigma[v]^{\mathbf{M}})$, so v is a solution also of the identity $(n+1)+\tau \sim (n+1)+\sigma$. ∎

Proposition. *If it is impossible to apply any special solving transformation to a system, then either the system is solved, or it is termally inconsistent.*

Proof. By inspection of the various special solving transformations. ∎

Proposition. *Let Θ be a beta-termoidal system and Θ' is obtained from Θ by removing each subexpression of the form "$n+$". Then Θ and Θ' are termally equivalent. In particular, Θ is termally consistent if and only if Θ' is termally consistent.*

Proof. Notice that in Herbrand algebra any termoid $n+\tau$ has just one value, namely the value of the term τ. ∎

Corollary. *Suppose we apply a special solving transformation to a system Θ, the result being Θ'. If Θ' is termally inconsistent, then so is Θ.*

Proof. We inspect the various special solving transformations and use the previous proposition in order to reason about terms instead of termoids. ∎

Proposition. *It is impossible to apply special solving transformations to a finite beta-termoidal system infinitely many times.*

Proof. It is impossible to apply the second solving transformation to a system infinitely many times, since with each application the system becomes solved with respect to one additional variable and if the system is solved with respect to some variable, it remains such forever.

Suppose we have passed the last application of second solving transformation. After that, it will be impossible to apply only the first solving transformation to a system infinitely many times, since with each application the identities in the system become "simpler". ∎

Given a finite system of beta-termoidal identities, consider the following procedure. We begin applying special solving transformations to the system (in arbitrary order). After finitely many steps we are going to reach a system, such that no special solving transformations can be applied to it. If the resulting system is termally unsolvable,[25] then the initial system is

[25] This is so if and only if the resulting system contains an identity which is termally unsolvable in an obvious way. For example $n + \mathtt{c} \sim n + \mathtt{f}(\tau)$, or $n + \mathtt{f}(\tau) \sim n + \mathtt{c}$ or $n + \mathtt{c} \sim n + \mathtt{d}$ ($\mathtt{c} \neq \mathtt{d}$), or $\mathtt{p}(\dots) \sim \mathtt{q}(\dots)$ ($\mathtt{p} \neq \mathtt{q}$).

termally unsolvable. Otherwise, the resulting system is solved, so it has the form $\{0 + \mathtt{x}_1 \sim \tau_1, \ldots, 0 + \mathtt{x}_n \sim \tau_n\}$ for some variables $\mathtt{x}_1, \ldots, \mathtt{x}_n$ and beta-termoids $\tau_1, \ldots, \tau_n$. In this case any solution of the initial system is a solution of the resulting system and any solution of the resulting system is an instance of the substitution:

$$s\xi = \begin{cases} \tau_i, & \text{if } \xi = \mathtt{x}_i, \\ \xi, & \text{otherwise.} \end{cases}$$

Definition. If we start with a system of the form $\{\tau_0 \sim \tau_1, \tau_0 \sim \tau_2, \ldots, \tau_0 \sim \tau_n\}$, any substitution obtained in way just described is called *unifier* of the termoids $\tau_0, \tau_1, \ldots, \tau_n$.

§8. FINITE MODELS BY TERMOIDAL RESOLUTION

Resolution with Termoids

Given the definitions of termoid, clausoids, termoidal substitution and unification of termoids, it is not difficult to define the notions resolvent and positive hyperresolvent of clausoids. In result, we obtain a resolutive method with termoids instead of terms. This resolutive method has the following properties: it is sound in Herbrand structures and it is sound with respect to satisfiability. However, in general this resolutive method is not sound with respect to universal validity in a structure. On the other hand this resolutive method is complete with respect to satisfiability in large class of algebras.

Proposition. *Let* Γ *be a set of universally valid in a Herbrand structure clausoids. Then any clausoid derived from* Γ *by resolution or positive resolution is universally valid in this structure.*

Proof. In any Herbrand structure the termoids are equivalent to terms. ∎

Corollary. *If a set of clausoids is universally satisfiable, then* $\perp$ *is not derivable by resolution or positive hyperresolution.*

Proof. Herbrand's theorem implies that if a set of clausoids is universally satisfiable, then its clausoids are universally valid in some Herbrand structure. ∎

Notice that the termoidal resolution is sound with respect to validity in Herbrand structures but not with respect to validity in arbitrary structures. With some kinds of termoids there exist clausoids δ' and δ'' such that both are universally valid in a structure $\mathbf{M}$ but at the same time some resolvent of δ' and δ'' is false in $\mathbf{M}$.[26]

Now, let us turn our attention to the completeness of the resolution with termoids. Since we aim to use this resolution in order to build finite models, the completeness of the resolution is more important property for us than the soundness. This is so because if the resolution is complete with respect to the satisfiability in a particular finite algebra and $\perp$ is not derivable from a set Γ, then it will follow that Γ is satisfiable in this algebra.

Recall that the unification of two termoids τ' and τ'' has the following property: either τ' and τ'' are not unifiable, in which case the identity $\tau' \sim \tau''$ is termally unsolvable, or τ' and τ'' are unifiable, in which case any solution of the identity $\tau' \sim \tau''$ (in any algebra) is an instance of the unifier. We shall see that it is possible to use this property in order to prove that the resolution with termoids is complete with respect to the satisfiability in any algebra where all identities belonging to a particular set of termally unsolvable identities are unsolvable.

This set of termally unsolvable identities will be finite in the case when there are only finitely many resolvents that can be produced from the set of clausoids. Therefore, the following theorem is true:

Theorem. *If the set of clausoids* Γ *is such that there are only finitely many clausoids that can be derived by positive hyperresolution from* Γ *and* $\perp$ *is not derivable, then there exists a finite set of termally unsolvable identities, such that* Γ *is universally satisfiable in any algebra where none of these identities is solvable.*

It can be shown that for any finite set of termally unsolvable termoidal identities there exists a finite algebra where none of these identities is solvable. Therefore, we obtain the following corollary:

Corollary. *Let the set of clausoids* Γ *be such that there are only finitely many clausoids that can be derived by positive hyperresolution from* Γ *and* $\perp$ *is not derivable. Then the elements of* Γ *are universally valid in some finite structure.*

Example. Let Γ be the set of the following two clausoids: $\mathtt{p}(0+\mathtt{c})$ and $\neg\mathtt{p}(0+\mathtt{d})$. If the structure $\mathbf{M}$ is such that $\mathtt{c}^{\mathbf{M}} = \mathtt{d}^{\mathbf{M}}$, these two clausoids can not be simultaneously valid in $\mathbf{M}$. Therefore, Γ is not universally satisfiable in any algebra $\mathbf{A}$, such that $\mathtt{c}^{\mathbf{A}} = \mathtt{d}^{\mathbf{A}}$. Nevertheless, since the

[26] This situation is impossible with the simplest kind of termoids, the beta-termoids, we are considering now. That is why I am unable to give an example here.

termoids $0+\mathtt{c}$ and $0+\mathtt{d}$ are not unifiable, it is impossible to derive $\perp$ from Γ by resolution.

The appropriate finite set of termally unsolvable identities is $\{0+\mathtt{c} \sim 0+\mathtt{d}\}$. Notice that Γ is universally satisfiable in any algebra where the identity $0+\mathtt{c} \sim 0+\mathtt{d}$ is unsolvable, that is in any algebra $\mathbf{A}$ where $\mathtt{c}^{\mathbf{A}} \neq \mathtt{d}^{\mathbf{A}}$. Obviously, there are finite algebras with this property.

The Example by Baaz, Revisited

Consider again the example by Baaz:

$$\begin{gathered} \mathtt{p}(\mathtt{x},\mathtt{x}) \\ \neg\mathtt{p}(\mathtt{f}(\mathtt{x}),\mathtt{f}(\mathtt{y})) \vee \mathtt{p}(\mathtt{x},\mathtt{y}) \\ \neg\mathtt{p}(\mathtt{c},\mathtt{f}(\mathtt{x})) \end{gathered}$$

The set of these three clauses is not universally satisfiable in any finite algebra, yet, no new positive hyperresolvent can be produced from them.

If we convert these clauses to clausoids we will obtain the following set (for convenience, the variables are marked with indices):

$$\mathtt{p}(0+\mathtt{x}_1, 0+\mathtt{x}_1) \tag{1}$$
$$\neg\mathtt{p}(0+\mathtt{f}(\mathtt{x}_2), 0+\mathtt{f}(\mathtt{y}_2)) \vee \mathtt{p}(0+\mathtt{x}_2, 0+\mathtt{y}_2) \tag{2}$$
$$\neg\mathtt{p}(0+\mathtt{c}, 0+\mathtt{f}(\mathtt{x}_3)) \tag{3}$$

In order to produce a hyperresolvent from (2) and (1), we start the unification process with the system

$$\left|\ \mathtt{p}(0+\mathtt{f}(\mathtt{x}_2), 0+\mathtt{f}(\mathtt{y}_2)) \sim \mathtt{p}(0+\mathtt{x}_1, 0+\mathtt{x}_1)\right.$$

By applying the first special solving transformation to this system we obtain

$$\left|\ \begin{aligned} &0+\mathtt{f}(\mathtt{x}_2) \sim 0+\mathtt{x}_1 \\ &0+\mathtt{f}(\mathtt{y}_2) \sim 0+\mathtt{x}_1 \end{aligned}\right.$$

then

$$\left|\ \begin{aligned} &0+\mathtt{x}_1 \sim 0+\mathtt{f}(\mathtt{x}_2) \\ &0+\mathtt{f}(\mathtt{y}_2) \sim 0+\mathtt{x}_1 \end{aligned}\right.$$

then by substitution (second solving transformation)

$$\left|\ \begin{aligned} &0+\mathtt{x}_1 \sim 0+\mathtt{f}(\mathtt{x}_2) \\ &0+\mathtt{f}(\mathtt{y}_2) \sim 0+\mathtt{f}(\mathtt{x}_2) \end{aligned}\right.$$

then by the first special solving transformation about the second identity

$$\left|\begin{array}{l} 0+\mathtt{x}_1 \sim 0+\mathtt{f}(\mathtt{x}_2) \\ 1+\mathtt{y}_2 \sim 1+\mathtt{x}_2 \end{array}\right.$$

and finally we obtain the following solved system:

$$\left|\begin{array}{l} 0+\mathtt{x}_1 \sim 0+\mathtt{f}(\mathtt{x}_2) \\ 0+\mathtt{y}_2 \sim 1+\mathtt{x}_2 \end{array}\right.$$

The application of the corresponding termoidal substitution to the clausoids (2) and (1) gives the following clausoids:

$$\neg\mathtt{p}(0+\mathtt{f}(\mathtt{x}_2), 0+\mathtt{f}(\mathtt{x}_2)) \vee \mathtt{p}(0+\mathtt{x}_2, 1+\mathtt{x}_2)$$
$$\mathtt{p}(0+\mathtt{f}(\mathtt{x}_2), 0+\mathtt{f}(\mathtt{x}_2))$$

In result, we obtain the following positive hyperresolvent from (2) and (1):

$$\mathtt{p}(0+\mathtt{x}_2, 1+\mathtt{x}_2) \tag{4}$$

Analogously, from (2) and (4) we obtain hyperresolvent

$$\mathtt{p}(0+\mathtt{x}_2, 2+\mathtt{x}_2) \tag{5}$$

then from (2) and (5)

$$\mathtt{p}(0+\mathtt{x}_2, 3+\mathtt{x}_2) \tag{6}$$

and so on.

We see that while no new hyperresolvents can be produced from the initial clauses, if we use resolution with termoids we are able to produce infinitely many hyperresolvents. It can be shown that if the termoidal resolution produces finitely many clausoids, then the termal resolution is going to produce finitely many clauses as well. Although the example of Baaz shows that there are cases when the termal resolution produces finitely many clauses and yet, the set is not finitely satisfiable, if the termoidal resolution produces finitely many clausoids, the set is going to be finitely satisfiable for sure.

No hyperresolvent can be produced from the clausoids $\neg\mathtt{p}(0+\mathtt{c}, 0+\mathtt{f}(\mathtt{x}_3))$ and $\mathtt{p}(0+\mathtt{x}_2, n+\mathtt{x}_2)$ for any natural n. This suggests that the initial set of clausoids will be universally satisfiable in any algebra where the identity $\mathtt{p}(0+\mathtt{c}, 0+\mathtt{f}(\mathtt{x}_3)) \sim \mathtt{p}(0+\mathtt{x}_2, n+\mathtt{x}_2)$ has no solutions for any natural n. This is so if and only if the identity $n+\mathtt{c} \sim n+\mathtt{f}(\mathtt{x})$ has no solutions for any n. It can be shown, however, that in any finite algebra there exists a natural number n, such that the identity $n+\mathtt{c} \sim n+\mathtt{f}(\mathtt{x})$ has a solution.

Alpha-, Beta-, Gamma-, Delta-, Epsilon-,...

There isn't one universal sort of termoids that can be used for anything. In this work several kinds of termoids are going to be defined.

Alpha-termoids are simply terms in disguise. Since the termal unification is correct only in Herbrand algebra, alpha-termoids can not be used in order to build finite models.

Beta-termoids. We won't talk more about them. Beta-termoids are special case of epsilon-termoids.

Gamma-termoids are expressions like $\mathtt{f}_2^{-1}(\mathtt{f}(\mathtt{g}(\mathtt{c}), \mathtt{g}_1^{-1}(\mathtt{g}(\mathtt{d})), \mathtt{c}))$.

The interpretation of the functional symbols is traditional — given a structure $\mathbf{M}$, μ is value in $\mathbf{M}$ of gamma-termoid $\mathtt{f}(\tau_1, \tau_2, \tau_3)$ if $\mu = \mathtt{f}^{\mathbf{M}}(\nu_1, \nu_2, \nu_3)$ for some values ν_1, ν_2, ν_3 of τ_1, τ_2, τ_3. The symbols like $\mathtt{f}_2^{-1}$ are interpreted as inverse functions (the subscript 2 means that we are inversing on the second argument of $\mathtt{f}$) — μ is a value of gamma-termoid $\mathtt{f}_2^{-1}(\tau)$ if $\mathtt{f}^{\mathbf{M}}(\nu', \mu, \nu'')$ is value of τ for some ν' and ν''.

Gamma-termoids permit useful unification. For example we can transform the identity $\mathtt{f}(\tau_1, \ldots, \tau_n) \sim \mathtt{f}(\sigma_1, \ldots, \sigma_n)$ to the system

$$\left|\begin{array}{l} \tau_1 \sim \mathtt{f}_1^{-1}(\mathtt{f}(\sigma_1, \ldots, \sigma_n)) \\ \tau_2 \sim \mathtt{f}_2^{-1}\mathtt{f}(\sigma_1, \ldots, \sigma_n)) \\ \ldots \\ \tau_n \sim \mathtt{f}_n^{-1}\mathtt{f}(\sigma_1, \ldots, \sigma_n)) \end{array}\right.$$

Notice that any solution of the identity (in any algebra) is a solution of the system as well.

Delta-termoids are expressions like $1 + \mathtt{f}(\mathtt{g}(\mathtt{c}), 3 + \mathtt{g}(\mathtt{d}), \mathtt{c})$. The intended meaning of the numbers is similar to what we saw in beta-termoids but the exact definition is more difficult to state.

Given a structure $\mathbf{M}$, each gamma-termoid with n free variables defines a multi-valued function $\mathtt{t} : |\mathbf{M}|^n \to \mathcal{P}|\mathbf{M}|$. Then we can say that μ is value of delta-termoid $n + \tau$ if $\mu \in \mathtt{t}(\nu, \nu_1, \nu_2, \ldots, \nu_k)$ for some $\nu, \nu_1, \nu_2, \ldots, \nu_k$ such that ν is a value of τ and the multi-valued function $\mathtt{t}$ is defined by a gamma-termoid whose "height" is smaller than or equal to n.

Delta-termoids permit useful unification. For example we can transform the identity $\mathtt{f}(\tau_1, \ldots, \tau_n) \sim n + \mathtt{f}(\sigma_1, \ldots, \sigma_n)$ to the system

$$\left|\begin{array}{l} \tau_1 \sim (n+1) + \sigma_1 \\ \tau_2 \sim (n+1) + \sigma_2 \\ \ldots \\ \tau_n \sim (n+1) + \sigma_n \end{array}\right.$$

Notice that any solution of the identity (in any algebra) is a solution of the system as well.

Epsilon-termoids. Like beta-termoids, epsilon-termoids are expressions of the form $n+\tau$ where τ is a term. The interpretation of the epsilon-termoids is like that of the delta-termoids with the following difference — in order to find the values of $n+\tau$ we apply multivalued functions defined by gamma-termoids not only to τ, but also to the components of τ. Each epsilon-termoid is equivalent to a delta-termoid of special kind. For example the epsilon-termoid

$$3 + \mathtt{f}(\mathtt{g}(\mathtt{h}(\mathtt{c}), \mathtt{g}(\mathtt{c}, \mathtt{h}(\mathtt{d}))))$$

is equivalent to the delta-termoid

$$3 + \mathtt{f}(4 + \mathtt{g}(5 + \mathtt{h}(6 + \mathtt{c}), 5 + \mathtt{g}(6 + \mathtt{c}, 6 + \mathtt{h}(7 + \mathtt{d}))))$$

Notice how the deeper the subterm is, the greater its number is.

In this work it will be shown that epsilon-termoids can be used in order to prove that the class VED has the finite model property.

§9. OUTLINE OF THE FURTHER SECTIONS

When a new interesting kind of termoids is found, it is not enough to give a simple definition. We have to define how to make substitutions, how to unify termoids, how to make resolvents and we have to prove that the resulting resolutive procedure is complete. All these things are relatively long, so it will be beneficial if we manage to make most of them in a more general way rather than separately for each kind of termoids. This means we have to investigate what properties termoids have to have so that the termoidal resolution is useful. After that we have to develop the theory of termoids axiomatically, or rather algebraically. Finally, we have to show that the termoids we are interested of satisfy the specified axioms.

In §§10–13 we begin with the well known theory of terms using a new terminology. The main reason I decided to write about well known facts in a new way is to prepare the reader with the terminology we have to use while developing the theory of termoids. I think it is going to be easier if we first apply this new terminology on well known objects in order to get used to it. An additional benefit of this approach is that while the theory of termoids is being developed it will be easier to compare termoids with terms in order to feel better in what aspects termoids are identical with terms and in what aspects they are different.

In §14 we proceed with the algebraic theory of termoids. As an example, in §15 we show that terms can be considered special kind of termoids — the so called alpha-termoids. The results from this section are not going to be used later so the reader may want to skip it. §16 contains several important propositions about connections between the various manipulations of termoids and the corresponding manipulations of terms. In §§17–18 a rather general unification procedure for termoids is specified.

In §§19–23 the theory of termoidal resolution is developed. §19 is preliminary. In §20 the formal definition of clausoid is given and in §21 the theory of SLD resolution with termoids is developed. In result a proof is given that the machinery of Prolog has the finite model property.

In §§22–23 the theory of the positive hyperresolution with termoids is developed. The theory I have presented here does not make use of any orderings. Since anything in these sections is based on the abstract algebraic notion of termoid from §14 rather than on one particular kind of termoids, if we were to prove the completeness of the resolution with orderings, we would have to work with orderings in rather abstract way and I think this would be unnecessarily complex. Nevertheless, I think the completeness of the termoidal resolution with orderings can be proved in more or less the same way as the completeness of the usual termal resolution.

In §§24–26 the definitions of gamma-, delta- and epsilon-termoids are given and in §§27-28 it is shown that delta- and epsilon-termoids have useful unification. Although gamma-termoids have useful unification as well, no proof of this will be given.

In §§29-30 the finite model property of the class VED is proved.

Algebras and Terms

§10. STRUCTURES

A) The structures we are going to use in this work will be many-sorted. This is not going to cause any significant difficulties and has the following advantages:

- In some cases we won't have to differenciate between functional symbols and predicate symbols. The only difference between these symbols is that they have different result sorts.
- We will need structures where the values of the formulae are not simply true or false but some other objects. This implies that our structures have to have at least two carriers — one where the values of the terms reside, and second for the values of the formulae. Since it won't be too different whether our structures have two sorts or many sorts, it makes sense to permit many sorts.
- The use of sorts leads to significant reduction of the number of resolvents. [13, p. 229] There are cases when many-sorted resolution generates only finitely many clauses while the sort-less resolution generates infinitely many. Consequently, many-sorted resolution is more useful as a tool for model generation than the sort-less resolution.
- The class of formulae for which a finite model theorem holds is richer in a many-sorted framework than in the one-sorted case. [25]

B) Throughout this work we will fix a set `Sort`. Its elements will be called *sorts*. The special sort $\mathtt{Log} \in \mathtt{Sort}$ is called the *logical sort*. The elements of $\mathtt{Sort} \setminus \{\mathtt{Log}\}$ will be called *algebraic sorts*.

We assume that three fixed disjoint sets of "symbols" are given. The elements of the first set are called *functional symbols* and the elements of the second set are called *predicate symbols*. The third set is $\{\vee, \wedge, \neg, \bot, \top\}$; its elements are called *logical symbols*.

In addition we assume there is a fixed function assigning a *type* to each functional, predicate and logical symbol. The type of any functional symbol is an element of $(\mathtt{Sort} \setminus \{\mathtt{Log}\})^n \times (\mathtt{Sort} \setminus \{\mathtt{Log}\})$ for some n and the type of any predicate symbol is an element of $(\mathtt{Sort} \setminus \{\mathtt{Log}\})^n \times \{\mathtt{Log}\}$. The type of $\vee$ and $\wedge$ is $\langle\langle \mathtt{Log}, \mathtt{Log}\rangle, \mathtt{Log}\rangle$, the type of $\neg$ is $\langle\langle \mathtt{Log}\rangle, \mathtt{Log}\rangle$ and the type of $\bot$ and $\top$ is $\langle\langle\rangle, \mathtt{Log}\rangle$.

Operation symbol is a functional, predicate or logical symbol. *Constant symbol* is a nullary functional symbol.

When an operation symbol $\mathtt{d}$ has type $\langle\langle \kappa_1, \ldots, \kappa_k\rangle, \lambda\rangle$ we say $\langle \kappa_1, \ldots, \kappa_k\rangle$ is the *argument type* of $\mathtt{d}$ and λ is its *result sort*. We also say $\mathtt{d}$ is *nullary*, *unary*, *binary*, *ternary*, *n-ary*, etc, if the length of its argument type is 0, 1, 2, 3, n, respectively.

All operation symbols are assumed to be different from all other formal symbols we are going to use — parentheses, comma, etc.

C) Let I be a set. *I-indexed object* $\xi = \{\xi_i\}_{i \in I}$ is a function with domain I. The image of $i \in I$ is denoted by ξ_i. All such images are called *components* of ξ.

An I-indexed object $X = \{X_i\}_{i \in I}$ is called *I-indexed set*, if all X_i are sets. An I-indexed object $f = \{f_i\}_{i \in I}$ is called *I-indexed function*, if all f_i are functions.

It will be convenient to say that ξ is an element of the I-indexed set $X = \{X_i\}_{i \in I}$ and write $\xi \in X$ when ξ is an element of X_i for some i.

The I-indexed set X is a *subset* of the I-indexed set Y if $X_i \subseteq Y_i$ for any $i \in I$.

Let f be an I-indexed function. The domain of f is the I-indexed set $\operatorname{Dom} f = \{\operatorname{Dom} f_i\}_{i \in I}$; the codomain of f, written $\operatorname{Cod} f$, is defined analogously.

If f is an I-indexed function and $\xi \in \operatorname{Dom} f$, then define $f\xi = \{f_i \xi_i\}_{i \in I}$. Obviously $f\xi \in \operatorname{Cod} f$.

If $f = \{f_i\}_{i \in I}$ and $g = \{g_i\}_{i \in I}$, then $f \circ g = \{f_i \circ g_i\}_{i \in I}$.

If X is a subset of the domain of the I-indexed function f, then the *restriction* $\{f_i \restriction X_i\}_{i \in I}$ of f to X will be denoted $f \restriction X$.

For any two I-indexed sets $X = \{X_i\}_{i \in I}$ and $Y = \{Y_i\}_{i \in I}$, let $X \cap Y$, $X \cup Y$ and $X \times Y$ be the I-indexed sets $\{X_i \cap Y_i\}_{i \in I}$, $\{X_i \cup Y_i\}_{i \in I}$ and $\{X_i \times Y_i\}_{i \in I}$, respectively.

For any I-indexed set $X = \{X_i\}_{i \in I}$, let $\mathcal{P}X$ be the I-indexed set $\{\mathcal{P}X_i\}_{i \in I}$.

Given an I-indexed function $f = \{f_i\}_{i\in I}$ from X to Y, let $f^{\mathcal{P}} : \mathcal{P}X \to \mathcal{P}Y$ be the I-indexed function $\{{f_i}^{\mathcal{P}}\}_{i\in I}$.

It can be shown that the I-indexed sets together with the I-indexed functions form a category $\mathfrak{Set}^I$.

D) **Definition.** Denote by Σ the set of all operation symbols. A *structure* $\mathbf{M}$ is an ordered tuple $\langle\{A_\kappa\}_{\kappa\in\mathtt{Sort}}, \{j_{\mathtt{d}}\}_{\mathtt{d}\in\Sigma}\rangle$, where $\{A_\kappa\}_{\kappa\in\mathtt{Sort}}$ is a $\mathtt{Sort}$-indexed family of sets and $\{j_{\mathtt{d}}\}_{\mathtt{d}\in\Sigma}$ is a Σ-indexed family of functions, such that if $\mathtt{d} \in \Sigma$ has type $\langle\langle\kappa_1, \kappa_2, \dots, \kappa_n\rangle, \lambda\rangle$, then $j_{\mathtt{d}}$ is a function from $A_{\kappa_1} \times A_{\kappa_2} \times \cdots \times A_{\kappa_n}$ to A_λ.

The $\mathtt{Sort}$-indexed family $\{A_\kappa\}_{\kappa\in\mathtt{Sort}}$, written $|\mathbf{M}|$, is called *universe* of $\mathbf{M}$. The sets A_κ, written $\mathbf{M}_\kappa$, are called the *carriers* of $\mathbf{M}$. The carriers of algebraic sorts are *algebraic carriers* and $\mathbf{M}_{\mathtt{Log}}$ is the *logical carrier*. If $\mathtt{d}$ is an operation symbol, the function $j_{\mathtt{d}}$, written $\mathtt{d}^{\mathbf{M}}$, is called the *interpretation* of $\mathtt{d}$ in $\mathbf{M}$. The interpretations of all operation symbols are called *fundamental operations* of $\mathbf{M}$.

Notice that we do not require from the carriers to be non-empty sets.

E) **Definition.** If $\mathbf{M}$ and $\mathbf{K}$ are two structures, a *homomorphism* h from $\mathbf{M}$ to $\mathbf{K}$ is a $\mathtt{Sort}$-indexed family of functions from the carriers of $\mathbf{M}$ to the corresponding carriers of $\mathbf{K}$, such that for every operation symbol $\mathtt{d}$ with type $\langle\langle\kappa_1, \dots, \kappa_n\rangle, \lambda\rangle$ and for any $\langle\alpha_1, \dots, \alpha_n\rangle \in \mathbf{M}_{\kappa_1} \times \cdots \times \mathbf{M}_{\kappa_n}$ we have

$$h_\lambda(\mathtt{d}^{\mathbf{M}}\langle\alpha_1, \dots, \alpha_n\rangle) = \mathtt{d}^{\mathbf{K}}\langle h_{\kappa_1}\alpha_1, \dots, h_{\kappa_n}\alpha_n\rangle$$

When it is clear from the context that $\alpha \in \mathbf{M}_\kappa$, we will permit ourselves to write $h\alpha$ instead of $h_\kappa\alpha$.

F) Structures and homomorphisms between them form a category $\mathfrak{Str}$. If $\kappa \in \mathtt{Sort}$, the map $\mathbf{M} \mapsto \mathbf{M}_\kappa$ is a functor from $\mathfrak{Str}$ to $\mathfrak{Set}$. The map $\mathbf{M} \mapsto |\mathbf{M}|$ is a functor from $\mathfrak{Str}$ to $\mathfrak{Set}^{\mathtt{Sort}}$.

G) As usually, a homomorphism $h : \mathbf{M} \to \mathbf{K}$ is called an *isomorphism*, if all its components are bijective.

H) **Definition.** (1) A structure is *normal* if all its carriers are non-empty sets.

(2) The structure $\mathbf{M}$ is a *logical structure* if $\mathbf{M}$ is normal, $\mathbf{M}_{\mathtt{Log}} = \{0, 1\}$ and $\mathbf{M}_{\mathtt{Log}}$ together with the interpretation of the logical operations forms the standard two-element Boolean algebra (1 is true and 0 is false).

(3) The structure $\mathbf{M}$ is a *logical variant* of $\mathbf{K}$ if $\mathbf{M}$ is a logical structure, $\mathbf{M}$ and $\mathbf{K}$ have identical algebraic carriers and $\mathtt{f}^{\mathbf{M}} = \mathtt{f}^{\mathbf{K}}$ for any functional symbol $\mathtt{f}$.

I) **Proposition.** *If* $h : \mathbf{M} \to \mathbf{K}$ *is a homomorphism between logical structures, then* $h_{\mathtt{Log}}0 = 0$ *and* $h_{\mathtt{Log}}1 = 1$.

Proof. In all logical structures, $\bot$ and $\top$ are interpreted as 0 and 1, respectively.[27] All homomorphisms from $\mathbf{M}$ to $\mathbf{K}$ map the interpretations of $\bot$ and $\top$ in $\mathbf{M}$ to the corresponding interpretations in $\mathbf{K}$. ∎

J) **Proposition.** *Let* $h : \mathbf{M} \to \mathbf{K}$ *be a homomorphism. There exist unique structure* $\mathbf{N}$ *and homomorphisms* $h' : \mathbf{M} \to \mathbf{N}$ *and* $h'' : \mathbf{N} \to \mathbf{K}$*, such that*

1. *The algebraic carriers of* $\mathbf{N}$ *and the interpretations of the functional symbols are the same as in* $\mathbf{M}$.
2. *The logical carrier of* $\mathbf{N}$ *and the interpretations of the logical symbols are the same as in* $\mathbf{K}$.
3. *The homomorphism* h' *is identity over the algebraic carriers and the same as* h *over the logical carrier. The homomorphism* h'' *is identity over* $\mathtt{Log}$ *and the same as* h *over the algebraic carriers.*
4. $h = h'' \circ h'$

Proof. **(!)** The uniqueness of h' and h'' immediately follows from the third condition. The proposition specifies everything about $\mathbf{N}$ except the interpretation of the predicate symbols. Choose an arbitrary predicate symbol $\mathtt{p}$. The homomorphism h' is identity for the algebraic sorts, so for any element ν of an algebraic carrier of $\mathbf{N}$ we have $h''\nu = (h'' \circ h')\nu = h\nu$. The homomorphism h'' is identity over $\mathtt{Log}$, whence for arbitrary elements $\nu_1, \ldots, \nu_n$ of the algebraic carriers of $\mathbf{N}$ we have $\mathtt{p}^{\mathbf{N}}\langle \nu_1 \ldots, \nu_n\rangle = h''\mathtt{p}^{\mathbf{N}}\langle \nu_1 \ldots, \nu_n\rangle = \mathtt{p}^{\mathbf{K}}\langle h''\nu_1 \ldots, h''\nu_n\rangle = \mathtt{p}^{\mathbf{K}}\langle h\nu_1 \ldots, h\nu_n\rangle = h\mathtt{p}^{\mathbf{M}}\langle \nu_1 \ldots, \nu_n\rangle$.

(∃) Obviously we can define $\mathbf{N}$, h' and h'' as it has been specified in the condition of the proposition and during the proof of the uniqueness. A simple check shows that h' and h'' are homomorphisms. Condition 4. follows from condition 3. ∎

K) **Corollary.** *Let* $\mathbf{M}$ *be a normal structure,* $\mathbf{K}$ *be a logical structure and* $h : \mathbf{M} \to \mathbf{K}$ *be a homomorphism. Then there exists unique logical variant* $\mathbf{N}$ *of* $\mathbf{M}$ *and a homomorphism* $g : \mathbf{N} \to \mathbf{K}$*, such that* g *and* h *are identical over the algebraic carriers.*

Proof. **(∃)** Let $\mathbf{N}$, h' and h'' be as in (J). The algebraic carriers of $\mathbf{N}$ are the same as the algebraic carriers of $\mathbf{M}$ and $\mathbf{M}$ is normal, hence $\mathbf{N}$ is

[27] Strictly speaking, the interpretations of $\bot$ and $\top$ are nullary functions mapping $\langle\rangle$ to 0 and 1, respectively.

normal. Moreover, the logical carrier of $\mathbf{N}$ and the interpretations of the logical symbols are the same as in $\mathbf{K}$, whence $\mathbf{N}$ is logical. The algebraic carriers and the interpretations of the functional symbols in $\mathbf{N}$ are the same as in $\mathbf{M}$, whence $\mathbf{N}$ is a logical variant of $\mathbf{M}$. We also know h'' and h are identical over the algebraic carriers, so we can define $g = h''$.

(!) We are going to apply (J) again. The algebraic carriers of $\mathbf{N}$ and the interpretations of the functional symbols are the same as in $\mathbf{M}$ because $\mathbf{N}$ is a logical variant of $\mathbf{M}$. The logical carrier of $\mathbf{N}$ and the interpretation of the logical symbols are the same as in $\mathbf{K}$ because both structures are logical. Let $h' : \mathbf{M} \to \mathbf{N}$ be the `Sort`-indexed function that is identity over the algebraic carriers and same as h over the logical carrier. Obviously h' is a homomorphism. Define $h'' = g$, then h'' will be the same as h over the algebraic carriers. But it also is a homomorphism between logical structures, so (I) implies h'' is identity over the logical carrier. This and the definition of h' imply $h = h'' \circ h'$, hence the uniqueness of $\mathbf{N}$ and g follows from (J). ∎

L) **Definition.** Given a structure $\mathbf{M}$, by $\mathcal{P}\mathbf{M}$ we will denote the structure whose carriers are the power sets of the carriers of $\mathbf{M}$. More formally, $(\mathcal{P}\mathbf{M})_\kappa = \mathcal{P}(\mathbf{M}_\kappa)$ for any κ and the operation symbols are interpreted in the following way:

$$\mathtt{d}^{\mathcal{P}\mathbf{M}}\langle\alpha_1,\ldots,\alpha_n\rangle = \{\mathtt{d}^{\mathbf{M}}\langle\beta_1,\ldots,\beta_n\rangle : \beta_1 \in \alpha_1,\ldots,\beta_n \in \alpha_n\}$$

M) **Definition.** Given a homomorphism $h : \mathbf{M} \to \mathbf{K}$, by $h^{\mathcal{P}}$ we will denote the homomorphism $h^{\mathcal{P}} : \mathcal{P}\mathbf{M} \to \mathcal{P}\mathbf{K}$, such that $h^{\mathcal{P}}$ is the `Sort`-indexed function defined in (2C): $h^{\mathcal{P}}\alpha = \{h\beta : \beta \in \alpha\}$.

We have to prove that $h^{\mathcal{P}}$ is indeed a homomorphism.

Proof. By definition, $h^{\mathcal{P}}$ is a `Sort`-indexed function mapping $|\mathbf{M}^{\mathcal{P}}|$ to $|\mathbf{K}^{\mathcal{P}}|$. Choose an arbitrary operation symbol $\mathtt{d}$. Then

$$\begin{aligned}
h^{\mathcal{P}}(\mathtt{d}^{\mathcal{P}\mathbf{M}}\langle\alpha_1,\ldots,\alpha_n\rangle &= h^{\mathcal{P}}(\{\mathtt{d}^{\mathbf{M}}\langle\beta_1,\ldots,\beta_n\rangle : \beta_1 \in \alpha_1,\ldots,\beta_n \in \alpha_n\})\\
&= \{h(\mathtt{d}^{\mathbf{M}}\langle\beta_1,\ldots,\beta_n\rangle) : \beta_1 \in \alpha_1,\ldots,\beta_n \in \alpha_n\}\\
&= \{\mathtt{d}^{\mathbf{K}}\langle h\beta_1,\ldots,h\beta_n\rangle : \beta_1 \in \alpha_1,\ldots,\beta_n \in \alpha_n\}\\
= \{\mathtt{d}^{\mathbf{K}}\langle\gamma_1,\ldots,\gamma_n\rangle &: \exists\beta_1 \in \alpha_1(\gamma_1 = h\beta_1),\ldots,\exists\beta_n \in \alpha_n(\gamma_n = h\beta_n)\}\\
&= \{\mathtt{d}^{\mathbf{K}}\langle\gamma_1,\ldots,\gamma_n\rangle : \gamma_1 \in h^{\mathcal{P}}\alpha_1,\ldots,\gamma_n \in h^{\mathcal{P}}\alpha_n\}\\
&= \mathtt{d}^{\mathcal{P}\mathbf{K}}\langle h^{\mathcal{P}}\alpha_1,\ldots,h^{\mathcal{P}}\alpha_n\rangle
\end{aligned}$$

∎

Notice that $\mathcal{P}$ is an endofunctor in the category $\mathfrak{Str}$ of all structures. In particular, $(g \circ h)^{\mathcal{P}} = g^{\mathcal{P}} \circ h^{\mathcal{P}}$ and $(\mathrm{id}_{\mathbf{A}})^{\mathcal{P}} = \mathrm{id}_{\mathcal{P}\mathbf{A}}$.

N) **Proposition.** *Given a structure* $\mathbf{M}$, *let* $\{\}_{\mathbf{M}} : |\mathbf{M}| \to |\mathcal{P}\mathbf{M}|$ *be the* `Sort`*-indexed function, such that* $(\{\}_{\mathbf{M}})_\kappa \mu = \{\mu\}$ *for any sort* κ. *Then* $\{\}_{\mathbf{M}}$ *is an injective homomorphism from* $\mathbf{M}$ *to* $\mathcal{P}\mathbf{M}$.

Proof. The definition of $\{\}_{\mathbf{M}}$ implies that all components of $\{\}_{\mathbf{M}}$ are injective functions, so we only have to prove that $\{\}_{\mathbf{M}}$ is a homomorphism. Let `d` be an arbitrary operation symbol with type $\langle\langle\kappa_1, \dots, \kappa_n\rangle, \lambda\rangle$. Then for any $\mu_1 \in \mathbf{M}_{\kappa_1}, \dots, \mu_n \in \mathbf{M}_{\kappa_n}$ we have $(\{\}_{\mathbf{M}})_\lambda(d^{\mathbf{M}}\langle\mu_1, \dots, \mu_n\rangle) = \{d^{\mathbf{M}}\langle\mu_1, \dots, \mu_n\rangle\} = d^{\mathcal{P}\mathbf{M}}\langle\{\mu_1\}, \dots, \{\mu_n\}\rangle = d^{\mathcal{P}\mathbf{M}}\langle(\{\}_{\mathbf{M}})_{\kappa_1}\mu_1, \dots, (\{\}_{\mathbf{M}})_{\kappa_n}\mu_n\rangle$. ■

§11. TERMS AND FORMULAE

A) We need a way to include arbitrary mathematical objects in syntactic objects (terms, formulae, etc.). In order to achieve this, we assume that for any `Sort`-indexed set X and sort κ we have a function called "$\mathrm{nam}_{X,\kappa}$" mapping the elements of X_κ to symbols. If $\mathtt{y} \in X_\kappa$, then $\mathrm{nam}_{X,\kappa}(\mathtt{y})$ is called the *name* of `y`. We assume only the following properties of the functions $\mathrm{nam}_{X,\kappa}$:

1. $\mathrm{nam}_{X,\kappa}(\mathtt{y})$ is defined whenever $\mathtt{y} \in X_\kappa$ and for all sorts κ.
2. The symbols $\mathrm{nam}_{X,\kappa}(\mathtt{y})$ are different from all operation symbols and from any other formal symbols we are going to use — parentheses, comma, etc.
3. If $\kappa \neq \lambda$ or $\mathtt{y} \neq \mathtt{z}$, then $\mathrm{nam}_{X,\kappa}(\mathtt{y}) \neq \mathrm{nam}_{X,\lambda}(\mathtt{z})$.
4. If $X_\kappa = Y_\kappa$, then $\mathrm{nam}_{X,\kappa}(\mathtt{y}) = \mathrm{nam}_{Y,\kappa}(\mathtt{y})$ for any $\mathtt{y} \in X_\kappa$.

With a suitable definition of what constitutes a "symbol" or "syntactic object" we can simply assume that $\mathrm{nam}_{X,\kappa}(\mathtt{y}) = \langle\kappa, \mathtt{y}\rangle$. On the other hand, if we have been given a predefined set of symbols, then the proof that a mapping $\mathrm{nam}_{X,\kappa}$ exists may require the use of the axiom of choice.[28] In any case, obviously we have to have sufficiently many symbols.

B) **Definition.** (1) Let X be a `Sort`-indexed set. *Termal expressions* over X are defined inductively:

- If $\mathtt{y} \in X_\kappa$, then $\mathrm{nam}_{X,\kappa}(\mathtt{y})$ is a termal expression of sort κ over X.
- If `d` is an operation symbol of type $\langle\langle\kappa_1, \dots, \kappa_n\rangle, \lambda\rangle$ and $\tau_1, \dots, \tau_n$ are termal expressions over X of sorts $\kappa_1, \dots, \kappa_n$, respectively, then the string $\mathtt{d}(\tau_1, \dots, \tau_n)$ is a termal expression of sort λ over X.

[28] If we don't want to limit the sets X somehow (for instance, by requiring that all possible sets X form a set), the proof that the function $\mathrm{nam}_{X,\kappa}$ exists may depend on the Morse-Kelley set theory and the axiom of global choice.

(2) *Term* over X is a termal expression over X of an algebraic sort. *Formula* over X is a termal expression over X of logical sort that contains no names of logical sort. *Atomic formula* is a formula without logical symbols.[29] *Literal* is an atomic formula or negation of an atomic formula.

C) **Notation.** In the following sections I will prefer to write $\ulcorner \mathtt{y} \urcorner$ for $\mathrm{nam}_{X,\kappa}(\mathtt{y})$; this convention creates an ambiguity which seldom causes a problem. When $\mathtt{c}$ is a nullary operation symbol I will prefer to write simply $\mathtt{c}$ for $\mathtt{c}()$. Occasionally, when Δ is a binary operation symbol I will prefer to write $\tau'\Delta\tau''$ for $\Delta(\tau',\tau'')$ and when Δ is an unary operation symbol I will prefer to write $\Delta\tau$ for $\Delta(\tau)$. In particular, I will write $\varphi \vee \psi$, $\varphi \wedge \psi$, $\neg\varphi$, $\bot$ and $\top$ for $\vee(\varphi,\psi)$, $\wedge(\varphi,\psi)$, $\neg(\varphi)$, $\bot()$ and $\top()$, respectively. I will assume left grouping for the parentheses, for example I will write $\varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \varphi_4 \vee \dots \vee \varphi_n$ instead of $(\dots(((\varphi_1 \vee \varphi_2) \vee \varphi_3) \vee \varphi_4) \vee \dots) \vee \varphi_n$.

D) **Example.** Suppose we have two algebraic sorts `Int` and `Real` and two binary functional symbols "+" and "/" of types $\langle\langle \mathtt{Int}, \mathtt{Int}\rangle, \mathtt{Int}\rangle$ and $\langle\langle \mathtt{Real}, \mathtt{Int}\rangle, \mathtt{Real}\rangle$, respectively. Let the `Sort`-indexed set X be such that $42 \in X_{\mathtt{Int}}$ and $3.14 \in X_{\mathtt{Real}}$. Then $/(\ulcorner 3.14\urcorner, +(\ulcorner 42\urcorner, \ulcorner 42\urcorner))$ is a termal expression over X of sort `Real`. We may write this termal expression more conveniently as $\ulcorner 3.14\urcorner/(\ulcorner 42\urcorner + \ulcorner 42\urcorner)$.

E) **Definition.** (1) Given a `Sort`-indexed set X, the *termal structure* over X, written $[X]$, is defined as follows: for any sort κ, the carrier $[X]_\kappa$ is the set of all termal expressions of sort κ over X and the fundamental operations of $[X]$ satisfy

$$\mathtt{d}^{[X]}\langle\tau_1,\dots,\tau_n\rangle = \mathtt{d}(\tau_1,\dots,\tau_n)$$

where on the right side of the equality sign stays a formal expression, $\mathtt{d}$ is an operation symbol with argument type $\langle\kappa_1,\dots,\kappa_n\rangle$ and $\tau_1,\dots,\tau_n$ are termal expressions of sorts $\kappa_1,\dots,\kappa_n$, respectively.

(2) Suppose $f : X \to Y$ is an arbitrary `Sort`-indexed function. Then $[f] : [X] \to [Y]$ is the `Sort`-indexed function replacing all occurrences of names $\ulcorner \mathtt{z} \urcorner$ in its argument with $\ulcorner f\mathtt{z} \urcorner$.

More formally, $[f] : [X] \to [Y]$ is the `Sort`-indexed function replacing all occurrences of $\mathrm{nam}_{X,\lambda}(\mathtt{z})$ in the termal expressions of sort κ over X (for any sort λ and $\mathtt{z} \in X_\lambda$) with $\mathrm{nam}_{Y,\lambda}(f_\lambda \mathtt{z})$.

(3) It is convenient to use postfix notation for this function, thus $\alpha[f]$ means to apply $[f]$ to α.

[29] Consequently, neither $\bot$, nor $\top$ is an atomic formula.

F) **Remark.** If for all sorts κ there is at least one nullary operation symbol with result sort κ or the set X_κ is non-empty, then for all sorts κ there is at least one termal expression of sort κ, whence in this case the structure $[X]$ is normal.

Notice that this is only a sufficient, but not a necessary condition. For example, suppose we have only two algebraic sorts `Int` and `Real`, a nullary functional symbol "`1`" of type $\langle\langle\rangle, \mathtt{Int}\rangle$, a binary functional symbol "$-$" of type $\langle\langle\mathtt{Int}, \mathtt{Int}\rangle, \mathtt{Int}\rangle$ and a binary functional symbol "/" of type $\langle\langle\mathtt{Int}, \mathtt{Int}\rangle, \mathtt{Real}\rangle$. Then `1/1` is a term of sort `Real`, so $[X]$ is a normal structure even if $X_{\mathtt{Real}} = \varnothing$.

G) **Proposition.** *For any* `Sort`*-indexed function* $f : X \to Y$*,* $[f]$ *is a homomorphism from* $[X]$ *to* $[Y]$*.*

Proof. First we show by induction that $\tau \in [X]_\kappa$ implies[30] $\tau[f]_\kappa \in [Y]_\kappa$. If $\tau = \mathrm{nam}_{X,\kappa}(\mathtt{z})$, then by the definition of $[f]$, $\tau[f]_\kappa = \mathrm{nam}_{Y,\kappa}(f\mathtt{z}) \in [Y]_\kappa$. Otherwise, $\tau = \mathtt{d}(\tau_1, \ldots, \tau_n)$, where $\tau_i \in [X]_{\lambda_i}$ for some sorts λ_i. By induction hypothesis, $\tau_i[f] \in [Y]_{\lambda_i}$, whence $\tau[f] = \mathtt{d}(\tau_1[f], \ldots, \tau_n[f]) \in [Y]_\kappa$.

It only remains to notice that for any operation symbol `d` of type $\langle\langle\kappa_1, \ldots, \kappa_n\rangle, \lambda\rangle$ and arbitrary termal expressions $\tau_1, \ldots, \tau_n$ of sorts $\kappa_1, \ldots, \kappa_n$, $(\mathtt{d}^{[X]}\langle\tau_1, \ldots, \tau_n\rangle)[f]_\lambda = \mathtt{d}(\tau_1, \ldots, \tau_n)[f]_\lambda = \mathtt{d}(\tau_1[f]_{\kappa_1}, \ldots, \tau_n[f]_{\kappa_n}) = \mathtt{d}^{[Y]}\langle\tau_1[f]_{\kappa_1}, \ldots, \tau_n[f]_{\kappa_n}\rangle$. ∎

H) **Remark.** Definition (E2) implies that $[f \circ g] = [f] \circ [g]$ and $[\mathrm{id}_X] = \mathrm{id}_{[X]}$. Therefore, from proposition (G) we can conclude that the map $[\,.\,]$ is a functor from $\mathfrak{Set}^{\mathtt{Sort}}$ to $\mathfrak{Str}$.

I) **Proposition.** (1) *Given a* `Sort`*-indexed function* $f : X \to Y$ *and* $\xi \in X$*,* $\ulcorner\xi\urcorner[f] = \ulcorner f\xi\urcorner$*.*

(2) *Given a* `Sort`*-indexed function* $f : X \to Y$*,* $[f] \circ \mathrm{nam}_X = \mathrm{nam}_Y \circ f$*.*

Proof. **(1)** follows immediately from definition (E2).
(2) is a reformulation of (1). ∎

J) **Definition.** Given a structure $\mathbf{M}$, the *value* of a termal expression τ over $|\mathbf{M}|$ in $\mathbf{M}$, written $\tau^{\mathbf{M}}$, is defined recursively:[31]

1. If $\mu \in \mathbf{M}_\kappa$, the value of the name of μ is μ. More formally, $(\mathrm{nam}_{|\mathbf{M}|,\kappa}(\mu))^{\mathbf{M}} = \mu$.
2. If $\tau = \mathtt{d}(\tau_1, \ldots, \tau_n)$, then $\tau^{\mathbf{M}} = \mathtt{d}^{\mathbf{M}}\langle\tau_1^{\mathbf{M}}, \ldots, \tau_n^{\mathbf{M}}\rangle$.

[30]Recall we are using postfix notation for $[f]$.

[31]Recall $|\mathbf{M}|$ is the `Sort`-indexed set of the carriers of $\mathbf{M}$. The termal expressions over $|\mathbf{M}|$ may contain names of the elements of the carriers of $\mathbf{M}$.

K) **Definition.** Let $\mathrm{val}_{\mathbf{M}}$ be the `Sort`-indexed function, such that $\mathrm{val}_{\mathbf{M},\kappa}$ maps the termal expressions over $|\mathbf{M}|$ of sort κ to their corresponding values in $\mathbf{M}$.

L) **Proposition.** *For any structure* $\mathbf{M}$*, the evaluating function* $\mathrm{val}_{\mathbf{M}}$ *is a homomorphism from* $[|\mathbf{M}|]$ *to* $\mathbf{M}$.

Proof. By definition, the carriers of $[|\mathbf{M}|]$ consist of the termal expressions over $|\mathbf{M}|$, hence $\mathrm{val}_{\mathbf{M}}$ is a `Sort`-indexed function from the universe of $[|\mathbf{M}|]$ to the universe of $\mathbf{M}$. For any operation symbol $\mathtt{d}$, $\mathrm{val}_{\mathbf{M}}(\mathtt{d}^{[|\mathbf{M}|]}\langle\tau_1,\ldots,\tau_n\rangle) = \mathrm{val}_{\mathbf{M}}(\mathtt{d}(\tau_1,\ldots,\tau_n)) = \mathtt{d}^{\mathbf{M}}\langle\mathrm{val}_{\mathbf{M}}\,\tau_1,\ldots,\mathrm{val}_{\mathbf{M}}\,\tau_n\rangle$. ∎

M) **Notation.** Given an arbitrary structure $\mathbf{M}$ and a `Sort`-indexed function $v : X \to |\mathbf{M}|$, let $[v]^{\mathbf{M}} = \mathrm{val}_{\mathbf{M}} \circ [v]$. It is convenient to use postfix notation for the homomorphism $[v]^{\mathbf{M}}$. Notice that $\tau([v]^{\mathbf{M}}) = (\tau[v])^{\mathbf{M}}$ so we can write $\tau[v]^{\mathbf{M}}$ without ambiguity and call $\tau[v]^{\mathbf{M}}$ *value* of the termal expression τ in the structure $\mathbf{M}$ with *assignment* v.

Occasionally, we are going to informally call functions like v *assignment functions*.

N) **Proposition.** (1) *Given a homomorphism* $h : \mathbf{M} \to \mathbf{K}$ *and a termal expression* τ *over* $|\mathbf{M}|$, $h(\tau^{\mathbf{M}}) = \tau[h]^{\mathbf{K}}$.

(2) *Given a homomorphism* $h : \mathbf{M} \to \mathbf{K}$, $h \circ \mathrm{val}_{\mathbf{M}} = \mathrm{val}_{\mathbf{K}} \circ [h]$.

Proof. **(1)** By induction on the termal expression τ. If $\tau = \ulcorner\mu\urcorner$, where $\mu \in \mathbf{M}_\kappa$, then $h_\kappa(\tau^{\mathbf{M}}) = h_\kappa(\mu) = (\ulcorner h_\kappa(\mu)\urcorner)^{\mathbf{K}} = \ulcorner\mu\urcorner[h]^{\mathbf{K}} = \tau[h]^{\mathbf{K}}$. If $\tau = \mathtt{d}(\tau_1,\ldots,\tau_n)$, by induction hypothesis, $h(\tau_i^{\mathbf{M}}) = \tau_i[h]^{\mathbf{K}}$, whence $h(\tau^{\mathbf{M}}) = h(\mathtt{d}^{\mathbf{M}}\langle\tau_1^{\mathbf{M}},\ldots,\tau_n^{\mathbf{M}}\rangle) = \mathtt{d}^{\mathbf{K}}\langle h(\tau_1^{\mathbf{M}}),\ldots,h(\tau_n^{\mathbf{M}})\rangle = \mathtt{d}^{\mathbf{K}}\langle\tau_1[h]^{\mathbf{K}},\ldots,\tau_n[h]^{\mathbf{K}}\rangle = (\mathtt{d}^{[|\mathbf{M}|]}\langle\tau_1,\ldots,\tau_n\rangle)[h]^{\mathbf{K}} = \tau[h]^{\mathbf{K}}$.

(2) is a reformulation of (1). ∎

O) **Proposition.** *Given a structure* $\mathbf{M}$ *and a* `Sort`*-indexed function* $v : X \to |\mathbf{M}|$, $\mathrm{val}_{\mathbf{M}} \circ [v]$ *is the unique homomorphism from* $[X]$ *to* $\mathbf{M}$*, mapping the name of any element* ξ *of* X *to* $v\xi$.

Proof. By definition $\mathrm{val}_{\mathbf{M}} \circ [v]$ is a homomorphism from $[X]$ to $\mathbf{M}$, mapping the name of any element ξ of X to $v\xi$.

Suppose both h and g are homomorphisms from $[X]$ to $\mathbf{M}$, mapping the name of any element ξ of X to $v\xi$. We will prove $h\tau = g\tau$ by induction on the termal expression τ. If $\xi \in X$ and $\tau = \ulcorner\xi\urcorner$, then both h and g map τ to $v\xi$. If $\tau = \mathtt{d}(\tau_1,\ldots,\tau_n)$, then by induction hypothesis $h\tau_i = g\tau_i$, whence $h\tau = h\mathtt{d}(\tau_1,\ldots,\tau_n) = h(\mathtt{d}^{[X]}\langle\tau_1,\ldots,\tau_n\rangle) = \mathtt{d}^{\mathbf{M}}\langle h\tau_1,\ldots,h\tau_n\rangle = \mathtt{d}^{\mathbf{M}}\langle g\tau_1,\ldots,g\tau_n\rangle = g(\mathtt{d}^{[X]}\langle\tau_1,\ldots,\tau_n\rangle) = g\mathtt{d}(\tau_1,\ldots,\tau_n) = g\tau$ ∎

P) **Corollary.** *Given homomorphisms* $h, g : [X] \to \mathbf{M}$*, if* $h\ulcorner\xi\urcorner = g\ulcorner\xi\urcorner$ *for any* $\xi \in X$*, then* $h = g$*. More formally, if* $h_\kappa(\mathrm{nam}_{X,\kappa}\,\xi) = g_\kappa(\mathrm{nam}_{X,\kappa}\,\xi)$ *for any sort* κ *and* $\xi \in X_\kappa$*, then* $h = g$*.*

Proof. Define the $\mathtt{Sort}$-indexed function $v : X \to |\mathbf{M}|$ by $v_\kappa\xi = h_\kappa\xi = g_\kappa\xi$. Then (O) will imply there is unique homomorphism mapping the name of any $\xi \in X$ to $v\xi$. Since both h and g are such homomorphisms, $h = g$. ■

Q) **Proposition.** (1) *Given a* $\mathtt{Sort}$*-indexed assignment function* $v : X \to |\mathbf{M}|$*, a homomorphism* $h : \mathbf{M} \to \mathbf{K}$ *and a termal expression* τ *over* X*,* $h(\tau[v]^{\mathbf{M}}) = \tau[h \circ v]^{\mathbf{K}}$*.*

(2) *In addition, if the structures* $\mathbf{M}$ *and* $\mathbf{K}$ *are logical and* φ *is a formula over* X*,* $\varphi[v]^{\mathbf{M}} = \varphi[h \circ v]^{\mathbf{K}}$*.*

Proof. **(1)** Simply apply (N) for $\sigma = \tau[v]$: $h(\tau[v]^{\mathbf{M}}) = h(\sigma^{\mathbf{M}}) = \sigma[h]^{\mathbf{K}} = \tau[v][h]^{\mathbf{K}} = \tau[h \circ v]^{\mathbf{K}}$.

(2) follows from (1) and (10l). ■

R) **Definition.** (1) A (termal) *substitution* is a $\mathtt{Sort}$-indexed function whose codomain is the universe of a termal structure, i.e. a function of the form $s : X \to |[Y]|$.

(2) Given a substitution $s : X \to |[Y]|$, $\tau[s]^{[Y]}$ is called *application* of the substitution s to the termal expression τ.

Given a substitution $s : X \to |[Y]|$, from (P) it follows that $[s]^{[Y]} : [X] \to [Y]$ is the only homomorphism mapping any name $\ulcorner\xi\urcorner$ to $s\xi$.

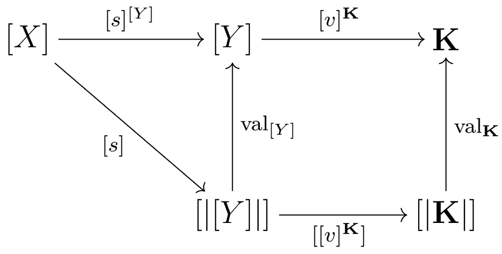
The following proposition shows that this definition of application of substitution is equivalent with the more common one.

S) **Proposition.** *Let* $s : X \to |[Y]|$ *be a substitution. Then* $[s]^{[Y]}$ *is the* $\mathtt{Sort}$*-indexed function replacing all occurrences of names* $\ulcorner\mathtt{z}\urcorner$*,* $\mathtt{z} \in X$ *in the termal expressions of the carriers of* $[X]$ *with* $s\mathtt{z}$*.*

Proof. By induction on the termal expression τ. If τ is the name $\ulcorner\mathtt{z}\urcorner$, then by definition $\tau[s]^{[Y]} = \ulcorner\mathtt{z}\urcorner[s]^{[Y]} = \ulcorner s\mathtt{z}\urcorner^{[Y]} = s\mathtt{z}$. If $\tau = \mathtt{d}(\tau_1, \ldots, \tau_n)$ and $\tau'_1 = \tau_1[s]^{[Y]}, \ldots, \tau'_n = \tau_n[s]^{[Y]}$, then by induction hypothesis $\tau'_1, \ldots, \tau'_n$ are the result of the replacement of all occurrences of names $\ulcorner\mathtt{z}\urcorner$ in $\tau_1, \ldots, \tau_n$ with $s\mathtt{z}$. Consequently, $\tau[s]^{[Y]} = \mathtt{d}(\tau_1, \ldots, \tau_n)[s]^{[Y]} = \mathtt{d}^{[X]}\langle\tau_1, \ldots, \tau_n\rangle[s]^{[Y]} = \mathtt{d}^{[Y]}\langle\tau_1[s]^{[Y]}, \ldots, \tau_n[s]^{[Y]}\rangle = \mathtt{d}(\tau'_1, \ldots, \tau'_n)$. ■

T) **Lemma** ***(of the substitutions).*** *Given a substitution* $s : X \to |[Y]|$ *and an assignment function* $v : Y \to |\mathbf{K}|$*, define a new assignment function* $w : X \to |\mathbf{K}|$*, such that* $w\xi = (s\xi)[v]^{\mathbf{K}}$*. Then for any termal expression* τ*,* $\tau[w]^{\mathbf{K}} = (\tau[s]^{[Y]})[v]^{\mathbf{K}}$*. In other words,* $[v]^{\mathbf{K}} \circ [s]^{[Y]} = [[v]^{\mathbf{K}} \circ s]^{\mathbf{K}}$*.*

Proof. The definition $w = [v]^{\mathbf{K}} \circ s$ implies $[w] = [[v]^{\mathbf{K}}] \circ [s]$, so we only have to prove that $[v]^{\mathbf{K}} \circ ([s]^{[Y]}) = \mathrm{val}_{\mathbf{K}} \circ [[v]^{\mathbf{K}}] \circ [s]$. This follows from the commutativity of the following diagram



The commutativity of the triangle follows from (M) and the commutativity of the square follows from (N), where $\mathbf{M}$ is $[Y]$ and h is $[v]^{\mathbf{K}}$. ■

One difference between the substitutions defined in (R1) and the common notion of substitution is that our substitutions may replace names of logical type with formulae. The following corollary shows that we can use such substitutions in order to prove that if we replace a subformula $\varphi \vee \psi$ in a formula with $\psi \vee \varphi$, we will obtain an equivalent formula.

U) **Corollary.** *Let* $\mathbf{K}$ *be a structure,* $v : X \to |\mathbf{K}|$ *be an assignment function and* τ_1 *and* τ_2 *be termal expressions over* X*, such that* τ_2 *is obtained from* τ_1 *by replacing a termal expression* σ_1 *with termal expression* σ_2*. If* $\sigma_1[v]^{\mathbf{K}} = \sigma_2[v]^{\mathbf{K}}$*, then* $\tau_1[v]^{\mathbf{K}} = \tau_2[v]^{\mathbf{K}}$*.*

Proof. Take some $\mathsf{z} \notin X$, let Y is obtained from X by adding to it z as sort κ and let τ be the termal expression obtained from τ_1 by replacing the same occurrence of σ_1 with $\ulcorner \mathsf{z} \urcorner$. Define substitutions $s_1, s_2 : Y \to [X]$:

$$s_1\xi = \begin{cases} \sigma_1, & \text{if } \xi = \mathsf{z}, \\ \xi, & \text{otherwise.} \end{cases} \qquad s_2\xi = \begin{cases} \sigma_2, & \text{if } \xi = \mathsf{z}, \\ \xi, & \text{otherwise.} \end{cases}$$

Then $\tau[s_1]^{[X]} = \tau_1$ and $\tau[s_2]^{[X]} = \tau_2$. In addition, define new assignment functions $w_1, w_2 : Y \to |\mathbf{K}|$ such that $w_i\xi = (s_i\xi)[v]^{\mathbf{K}}$. By definition, w_1 and w_2 are identical for $\xi \neq \mathsf{z}$ and for $\xi = \mathsf{z}$ we have $w_1\xi = \sigma_1[v]^{\mathbf{K}} = \sigma_2[v]^{\mathbf{K}} = w_2\xi$. Consequently, $w_1 = w_2$. Now from the lemma we obtain $\tau_1[v]^{\mathbf{K}} = (\tau[s_1]^{[X]})[v]^{\mathbf{K}} = \tau[w_1]^{\mathbf{K}} = \tau[w_2]^{\mathbf{K}} = (\tau[s_2]^{[X]})[v]^{\mathbf{K}} = \tau_2[v]^{\mathbf{K}}$. ■

Recall that the naming morphism $\mathrm{nam}_X : X \to |[X]|$ maps each $\xi \in X$ to the name $\ulcorner \xi \urcorner$. This morphism is a substitution and its application does not change the termal expression. Notice also that for any structure $\mathbf{M}$, the value of the name $\ulcorner \mu \urcorner$ is μ. The following proposition states formally these two simple facts.

V) **Proposition.** (1) *Given a termal expression* τ *over* X*, we have* $\tau[\mathrm{nam}_X]^{[X]} = \tau$*. Equivalently,* $\mathrm{val}_{[X]} \circ [\mathrm{nam}_X] = \mathrm{id}_{[X]}$.

$$[X] \xrightarrow{[\mathrm{nam}_X]} [|[X]|] \xrightarrow{\mathrm{val}_{[X]}} [X]$$

(2) *Given a structure* $\mathbf{M}$ *and* $\mu \in |\mathbf{M}|$*, we have* $(\ulcorner\mu\urcorner)^{\mathbf{M}} = \mu$*. Equivalently,* $\mathrm{val}_{\mathbf{M}} \circ \mathrm{nam}_{|\mathbf{M}|} = \mathrm{id}_{|\mathbf{M}|}$.

$$|\mathbf{M}| \xrightarrow{\mathrm{nam}_{|\mathbf{M}|}} |[|\mathbf{M}|]| \xrightarrow{\mathrm{val}_{\mathbf{M}}} |\mathbf{M}|$$

Proof. **(1)** For any $\xi \in X$, immediately from definitions (E2) and (J) it follows that $(\mathrm{nam}_X\, \xi)[\mathrm{nam}_X]^{[X]} = (\mathrm{nam}_{|[X]|}(\mathrm{nam}_X\, \xi))^{[X]} = \mathrm{nam}_X\, \xi$. Consequently the homomorphism $[\mathrm{nam}_X]^{[X]}$ preserves all names, so (P) implies this homomorphism is the identity homomorphism.

(2) immediately follows from definition (J). ■

W) **Remark.** We have defined two functors $[\,.\,] : \mathfrak{Set}^{\mathsf{Sort}} \to \mathfrak{Str}$ and $|\,.\,| : \mathfrak{Str} \to \mathfrak{Set}^{\mathsf{Sort}}$ in (H) and (10F), respectively. Proposition (I) implies that nam_X is a natural transformation from the identity functor of $\mathfrak{Set}^{\mathsf{Sort}}$ to the functor composition $|[\,.\,]|$. Proposition (N) implies that $\mathrm{val}_{\mathbf{M}}$ is a natural transformation from the functor composition $[|\,.\,|]$ to the identity functor of $\mathfrak{Str}$. Proposition (V) implies that the functors $[\,.\,] : \mathfrak{Set}^{\mathsf{Sort}} \to \mathfrak{Str}$ and $|\,.\,| : \mathfrak{Str} \to \mathfrak{Set}^{\mathsf{Sort}}$ form an adjunction where $[\,.\,]$ is the left adjoint and $|\,.\,|$ is the right adjoint. The naming morphism nam_X is the unit of this adjunction and the evaluation morphism $\mathrm{val}_{\mathbf{M}}$ is its counit.

§12. ALGEBRAS. ALGEBRAIC FRAGMENT OF A STRUCTURE

The traditional definition of algebra or algebraic structure is a structure for a language without predicate symbols. However, in this work we will benefit if we work always with only one fixed language with fixed sets of functional and predicate symbols. In order to be able to talk about algebraic structures in a language with predicate symbols we need a canonical way to supply any algebra with a logical carrier and with an interpretation of the predicate and the logical symbols.

There are two natural ways to do this. The simplest one seems to be the following. Let the logical carrier be some set with only one element. Since the logical carrier has unique element, there is only one possible interpretation for the predicate and the logical symbols. This observation leads to the following definition:

A) **Definition.** A structure $\mathbf{A}$ is a *terminal algebraic structure* if $\mathbf{A}_{\mathtt{Log}} = \{0\}$.

B) The main usefulness of the terminal algebraic structures is due to their simplicity. In particular, when defining a terminal algebraic structure we don't have to specify the interpretation of the predicate and the logical symbols. Indeed, since the logical carrier contains unique element, there can be only one possible interpretation of the predicate and the logical symbols. Similarly, when defining a homomorphism between terminal algebraic structures we don't have to specify the mapping of the elements of the logical carrier (as there is only one element in it). Unfortunately, for our purposes a different and more complex definition will be more useful.

C) **Definition.** (1) A *relational termal expression (or term, or formula)* is a termal expression (resp. term, formula) with no functional symbols and no names of logical sort.[32]

(2) The structure $\mathbf{A}$ is an *initial algebraic structure* or simply *algebra*, if the carrier $\mathbf{A}_{\mathtt{Log}}$ is the set of all relational formulae over $|\mathbf{A}|$, the predicate symbols satisfy $\mathtt{p}^{\mathbf{A}}\langle \alpha_1, \dots, \alpha_n\rangle = \mathtt{p}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner)$ and the logical symbols satisfy $\mathtt{d}^{\mathbf{A}}\langle \varphi_1, \dots, \varphi_n\rangle = \mathtt{d}(\varphi_1, \dots, \varphi_n)$.

(3) The *algebraic fragment* of a structure $\mathbf{M}$ is the unique algebra $\partial\mathbf{M}$, such that $\mathbf{M}$ and $\partial\mathbf{M}$ have same algebraic carriers and the functional symbols in $\partial\mathbf{M}$ are interpreted the same way as in $\mathbf{M}$.[33]

(4) Given a homomorphism $h : \mathbf{M} \to \mathbf{K}$, we can define a homomorphism $\partial h : \partial\mathbf{M} \to \partial\mathbf{K}$ and call it the *algebraic fragment of* h. Let ∂h maps the algebraic carriers the same way as h and if $\varphi \in (\partial\mathbf{M})_{\mathtt{Log}}$, let $(\partial h)\varphi = \varphi[h]$.

(5) $\mathbf{A}$ is a *normal algebra* if $\mathbf{A}$ is an algebra and $\mathbf{A}$ is a normal structure.

We have to prove that ∂h is a homomorphism.

Immediately from the definition, it follows that if $h : \mathbf{M} \to \mathbf{K}$ is a homomorphism, then ∂h maps the carriers of $\partial\mathbf{M}$ to the corresponding carriers of $\partial\mathbf{K}$. For all functional symbols $\mathtt{f}$ we have $\partial h(\mathtt{f}^{\partial\mathbf{M}}\langle \mu_1, \dots, \mu_n\rangle) = h(\mathtt{f}^{\mathbf{M}}\langle \mu_1, \dots, \mu_n\rangle) = \mathtt{f}^{\mathbf{K}}\langle h\mu_1, \dots, h\mu_n\rangle = \mathtt{f}^{\partial\mathbf{K}}\langle (\partial h)\mu_1, \dots, (\partial h)\mu_n\rangle$. For all predicate symbols $\mathtt{p}$ we have $\partial h(\mathtt{p}^{\partial\mathbf{M}}\langle \mu_1, \dots, \mu_n\rangle) = (\mathtt{p}^{\partial\mathbf{M}}\langle \mu_1, \dots, \mu_n\rangle)[h] = (\mathtt{p}(\ulcorner\mu_1\urcorner, \dots, \ulcorner\mu_n\urcorner))[h] = \mathtt{p}(\ulcorner h\mu_1\urcorner, \dots, \ulcorner h\mu_n\urcorner) = \mathtt{p}^{\partial\mathbf{K}}\langle h\mu_1, \dots, h\mu_n\rangle = \mathtt{p}^{\partial\mathbf{K}}\langle (\partial h)\mu_1, \dots, (\partial h)\mu_n\rangle$. And for all logical symbols $\mathtt{d}$ we have $\partial h(\mathtt{d}^{\partial\mathbf{M}}\langle \varphi_1, \dots, \varphi_n\rangle) = (\mathtt{d}^{\partial\mathbf{M}}\langle \varphi_1, \dots, \varphi_n\rangle)[h] = \mathtt{d}^{\partial\mathbf{K}}\langle \varphi_1[h], \dots, \varphi_n[h]\rangle) = \mathtt{d}^{\partial\mathbf{K}}\langle (\partial h)\varphi_1, \dots, (\partial h)\varphi_n\rangle)$.

[32] This definition implies that all relational terms are names.

[33] Definition (C2) determines uniquely the interpretation of the predicate and the logical symbols in an algebra.

D) **Example.** Suppose we have a single algebraic sort $\mathtt{Nat}$, a binary functional symbol "+" of type $\langle\langle \mathtt{Nat}, \mathtt{Nat}\rangle, \mathtt{Nat}\rangle$ and a binary predicate symbol $\mathtt{p}$ of type $\langle\langle \mathtt{Nat}, \mathtt{Nat}\rangle, \mathtt{Log}\rangle$. Let the algebra $\mathbf{A}$ be such that the carrier $\mathbf{A}_{\mathtt{Nat}}$ be the set of the natural numbers and the functional symbol "+" is interpreted as the function addition. Then $\mathtt{p}(\ulcorner 42\urcorner, \ulcorner 2\urcorner + \ulcorner 2\urcorner)$ is an atomic formula over $|\mathbf{A}|$ and its value in $\mathbf{A}$ is the relational formula $\mathtt{p}(\ulcorner 42\urcorner, \ulcorner 4\urcorner)$.

E) **Definition.** (1) The structures $\mathbf{M}$ and $\mathbf{K}$ are *variants* (one of another), if they have identical carriers and the functional symbols are interpreted the same way.

(2) The homomorphisms $h' : \mathbf{M}' \to \mathbf{K}'$ and $h'' : \mathbf{M}'' \to \mathbf{K}''$ are *variants* (one of another), if $\mathbf{M}'$ and $\mathbf{M}''$ are variants, $\mathbf{K}'$ and $\mathbf{K}''$ are variants and h' and h'' map the algebraic carriers the same way.

F) Obviously, the relations defined in (E) are reflexive, symmetric and transitive. For any structure $\mathbf{M}$, $\mathbf{M}$ and $\partial\mathbf{M}$ are variants and for any homomorphism h, h and ∂h are variants.

G) **Lemma.** *Given an algebra* $\mathbf{A}$*, a structure* $\mathbf{K}$ *and a homomorphism* $h : \mathbf{A} \to \mathbf{K}$*, if* φ *is an element of the logical carrier of* $\mathbf{A}$*, then* $h\varphi = \varphi[h]^{\mathbf{K}}$*.*

Proof. According to the definition of algebra (C2), φ is a relational formula. We will prove the Lemma by induction on φ.

Let $\varphi = \mathtt{p}(\tau_1, \dots, \tau_n)$ for some predicate symbol $\mathtt{p}$ and terms $\tau_1, \dots, \tau_n$. But φ is relational so these terms are not permitted to contain functional symbols, hence $\tau_1 = \ulcorner\alpha_1\urcorner, \dots, \tau_n = \ulcorner\alpha_n\urcorner$ for some elements $\alpha_1, \dots, \alpha_n$ of the algebraic carriers of $\mathbf{A}$. Consequently, $h\varphi = h\mathtt{p}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner) = h(\mathtt{p}^{\mathbf{A}}\langle\alpha_1, \dots, \alpha_n\rangle) = \mathtt{p}^{\mathbf{K}}\langle h\alpha_1, \dots, h\alpha_n\rangle = (\mathtt{p}(\ulcorner h\alpha_1\urcorner, \dots, \ulcorner h\alpha_n\urcorner))^{\mathbf{K}} = (\mathtt{p}(\ulcorner\alpha_1\urcorner[h], \dots, \ulcorner\alpha_n\urcorner[h]))^{\mathbf{K}} = \mathtt{p}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner)[h]^{\mathbf{K}} = \varphi[h]^{\mathbf{K}}$.

If $\varphi = \mathtt{d}(\psi_1, \dots, \psi_n)$ for some logical symbol $\mathtt{d}$ and formulae $\psi_1, \dots, \psi_n$, then by induction hypothesis $h\psi_i = \psi_i[h]^{\mathbf{K}}$, whence $h\varphi = h\mathtt{d}(\psi_1, \dots, \psi_n) = h(\mathtt{d}^{\mathbf{A}}\langle\psi_1, \dots, \psi_n\rangle) = \mathtt{d}^{\mathbf{K}}\langle h\psi_1, \dots, h\psi_n\rangle = \mathtt{d}^{\mathbf{K}}\langle\psi_1[h]^{\mathbf{K}}, \dots, \psi_n[h]^{\mathbf{K}}\rangle = (\mathtt{d}(\psi_1[h], \dots, \psi_n[h]))^{\mathbf{K}} = \mathtt{d}(\psi_1, \dots, \psi_n)[h]^{\mathbf{K}} = \varphi[h]^{\mathbf{K}}$. ∎

H) **Proposition.** (1) *If two algebras are variants, they are equal.*

(2) *Given an algebra* $\mathbf{A}$*, a structure* $\mathbf{K}$ *and homomorphisms* h *and* g *from* $\mathbf{A}$ *to* $\mathbf{K}$*, if* h *and* g *are variants, they are equal.*

Proof. **(1)** Suppose the algebras $\mathbf{A}$ and $\mathbf{B}$ are variants. Then $\mathbf{A}$ and $\mathbf{B}$ have the same algebraic carriers and the functional symbols are interpreted the same way. Their logical carriers are the sets of all relational formula over $|\mathbf{A}|$ and $|\mathbf{B}|$, respectively. Since formulae may contain no names of logical sort, the logical carriers of $\mathbf{A}$ and $\mathbf{B}$ are identical. It remains to notice that the definition of algebra (C2) implies that the predicate and the logical symbols are interpreted identically in $\mathbf{A}$ and in $\mathbf{B}$.

(2) By definition of variants, $h\alpha = g\alpha$ if α belongs to some of the algebraic carriers of $\mathbf{A}$. Take an arbitrary element φ of the logical carrier of $\mathbf{A}$. From (G) it follows that $h\varphi = \varphi[h]^{\mathbf{K}}$ and $g\varphi = \varphi[g]^{\mathbf{K}}$. The homomorphisms $[h]$ and $[g]$ map the names of the algebraic sorts the same way and there are no names of the logical sort in φ, whence $\varphi[h] = \varphi[g]$. ∎

I) **Corollary.** (1) *The algebraic fragment of a structure* $\mathbf{M}$ *is the only algebra which is a variant of* $\mathbf{M}$.

(2) *The algebraic fragment of a homomorphism* $h : \mathbf{M} \to \mathbf{K}$ *is the only homomorphism from* $\partial\mathbf{M}$ *to* $\partial\mathbf{K}$ *which is a variant of* h.

Proof. **(1)** If the algebra $\mathbf{A}$ is a variant of $\mathbf{M}$, then $\mathbf{A}$ and $\partial\mathbf{M}$ are variants. Therefore, from (H1) we conclude that $\mathbf{A} = \partial\mathbf{M}$.

(2) If $g : \partial\mathbf{M} \to \partial\mathbf{K}$ is a variant of h, then g and ∂h are variants. Therefore, from (H2) we conclude that $g = \partial h$. ∎

J) **Corollary.** (1) *Any algebra is the algebraic fragment of itself.*

(2) *Any homomorphism between algebras is the algebraic fragment of itself.*

Proof. **(1)** follows from (I1) because any algebra is a variant of its algebraic fragment.

(2) follows from (I2) because any homomorphism $h : \mathbf{A} \to \mathbf{B}$ is a variant of $\partial h : \partial\mathbf{A} \to \partial\mathbf{B}$ and if $\mathbf{A}$ and $\mathbf{B}$ are algebras then (1) implies $\mathbf{A} = \partial\mathbf{A}$ and $\mathbf{B} = \partial\mathbf{B}$. ∎

The reader is kindly asked to remember the following proposition because I shell not give references to it when I use it.

K) **Proposition.** (1) $\partial\mathbf{A} = \mathbf{A}$, *if* $\mathbf{A}$ *is algebra.*

(2) $\partial h = h$, *if* h *is a homomorphism between algebras.*

(3) $\mathbf{M}$ *and* $\mathbf{K}$ *are variants if and only if* $\partial\mathbf{M} = \partial\mathbf{K}$.

(4) h *and* g *are variants if and only if* $\partial h = \partial g$.

(5) $\partial(\mathrm{id}_{\mathbf{M}}) = \mathrm{id}_{\partial\mathbf{M}}$ *for any structure* $\mathbf{M}$.

(6) $\partial(h \circ g) = (\partial h) \circ (\partial g)$, *if the domain of* h *is identical with the codomain of* g.

(7) $\partial(\partial\mathbf{M}) = \partial\mathbf{M}$, $\partial(\partial h) = \partial h$

(8) ∂ *is an idempotent endofunctor in* $\mathfrak{Str}$.

Proof. **(1)** and **(2)** are reformulations of (J).

(3) First notice that by definition (C3), $\mathbf{M}$ and $\partial\mathbf{M}$ have identical algebraic carriers and the functional symbols are interpreted the same way.

Analogous property is also valid for $\mathbf{K}$ and $\partial\mathbf{K}$. Consequently $\mathbf{M}$ and $\mathbf{K}$ are variants if and only if $\partial\mathbf{M}$ and $\partial\mathbf{K}$ are variants. According to (H1), this is so if and only if $\partial\mathbf{M} = \partial\mathbf{K}$.

(4) If the domains and the codomains of h and g are not variants, then neither f and g are variants, nor $\partial h = \partial g$. Suppose that the domains of h and g are variants and the codomains of h and g are variants as well. Then by definition, h and g are variants if and only if h and g map the algebraic carriers identically, if and only if ∂h and ∂g map the algebraic carriers identically (because by definition (C4), h and ∂h map the algebraic carriers identically and g and ∂g similarly do), if and only if $\partial f = \partial g$ (because of H2).

(5) By definition (C4), $\partial(\mathrm{id}_{\mathbf{M}})$ maps the algebraic carriers identically to $\mathrm{id}_{\partial\mathbf{M}}$, so the required follows from (H2).

(6) By definition (C4), $\partial(h \circ g)$ and $\partial h \circ \partial g$ map the algebraic carriers identically to $h \circ g$, so the required follows from (H2) as well.

(7) follows from (K1) and (K2) because $\partial\mathbf{M}$ is an algebra and ∂h is a homomorphism between algebras.

(8) follows from (5), (6) and (7). ∎

L) **Definition.** Given an arbitrary structure $\mathbf{M}$, let $\int_{\mathbf{M}}$ be the `Sort`-indexed function from $|\partial\mathbf{M}|$ to $|\mathbf{M}|$, such that $\int_{\mathbf{M}}$ is identity over the algebraic carriers and if $\varphi \in (\partial\mathbf{M})_{\mathtt{Log}}$, then $\int_{\mathbf{M}}\varphi = \varphi^{\mathbf{M}}$.

The definition is correct because (C2) implies all elements of $(\partial\mathbf{M})_{\mathtt{Log}}$ are formulae over $|\partial\mathbf{M}|$ without names of logical sort. However, $\partial\mathbf{M}$ and $\mathbf{M}$ have same algebraic carriers, hence all elements of $(\partial\mathbf{M})_{\mathtt{Log}}$ are formulae over $|\mathbf{M}|$.

M) **Proposition.** (1) *$\int_{\mathbf{M}} : \partial\mathbf{M} \to \mathbf{M}$ is a homomorphism for any structure $\mathbf{M}$.*

(2) *$\partial(\int_{\mathbf{M}}) = \mathrm{id}_{\partial\mathbf{M}}$ for any structure $\mathbf{M}$.*

(3) *$\int_{\mathbf{M}}$ and $\mathrm{id}_{\mathbf{M}}$ are variants.*

(4) *$\int_{\mathbf{A}} = \mathrm{id}_{\mathbf{A}}$ for any algebra $\mathbf{A}$.*

(5) *$\int_{\mathbf{K}} \circ\ \partial h = h \circ \int_{\mathbf{M}}$ for any homomorphism $h : \mathbf{M} \to \mathbf{K}$.*

$$\begin{array}{ccc} \partial\mathbf{M} & \xrightarrow{\partial h} & \partial\mathbf{K} \\ {\scriptstyle\int_{\mathbf{M}}}\downarrow & & \downarrow{\scriptstyle\int_{\mathbf{K}}} \\ \mathbf{M} & \xrightarrow{h} & \mathbf{K} \end{array}$$

<u>Proof.</u> **(1)** The homomorphism $\int$ is identity over the algebraic carriers and $\mathbf{M}$ and $\partial\mathbf{M}$ interpret the functional symbols the same way, so if `f` is a

functional symbol, then

$$\begin{aligned}\textstyle\int_{\mathbf{M}}\mathtt{f}^{\partial\mathbf{M}}\langle\mu_1,\ldots,\mu_n\rangle &= \mathtt{f}^{\partial\mathbf{M}}\langle\mu_1,\ldots,\mu_n\rangle\\ &= \mathtt{f}^{\mathbf{M}}\langle\mu_1,\ldots,\mu_n\rangle\\ &= \mathtt{f}^{\mathbf{M}}\langle\textstyle\int_{\mathbf{M}}\mu_1,\ldots,\int_{\mathbf{M}}\mu_n\rangle\end{aligned}$$

If $\mathtt{p}$ is a predicate symbol, then

$$\begin{aligned}\textstyle\int_{\mathbf{M}}\mathtt{p}^{\partial\mathbf{M}}\langle\mu_1,\ldots,\mu_n\rangle &= \textstyle\int_{\mathbf{M}}\mathtt{p}(\ulcorner\mu_1\urcorner,\ldots,\ulcorner\mu_n\urcorner) && \text{from } (\mathsf{C}2)\\ &= (\mathtt{p}(\ulcorner\mu_1\urcorner,\ldots,\ulcorner\mu_n\urcorner))^{\mathbf{M}} && \text{from } (\mathsf{L})\\ &= \mathtt{p}^{\mathbf{M}}\langle\mu_1,\ldots,\mu_n\rangle\\ &= \mathtt{p}^{\mathbf{M}}\langle\textstyle\int_{\mathbf{M}}\mu_1,\ldots,\int_{\mathbf{M}}\mu_n\rangle && \text{from } (\mathsf{L})\end{aligned}$$

If $\mathtt{d}$ is a logical symbol, then

$$\begin{aligned}\textstyle\int_{\mathbf{M}}\mathtt{d}^{\partial\mathbf{M}}\langle\varphi_1,\ldots,\varphi_n\rangle &= \textstyle\int_{\mathbf{M}}\mathtt{d}(\varphi_1,\ldots,\varphi_n) && \text{from } (\mathsf{C}2)\\ &= (\mathtt{d}(\varphi_1,\ldots,\varphi_n))^{\mathbf{M}} && \text{from } (\mathsf{L})\\ &= \mathtt{d}^{\mathbf{M}}\langle\varphi_1^{\mathbf{M}},\ldots,\varphi_n^{\mathbf{M}}\rangle\\ &= \mathtt{d}^{\mathbf{M}}\langle\textstyle\int_{\mathbf{M}}\varphi_1,\ldots,\int_{\mathbf{M}}\varphi_n\rangle && \text{from } (\mathsf{L})\end{aligned}$$

(2) The definition of $\int_{\mathbf{M}}$ implies that $\int_{\mathbf{M}}$ and $\mathrm{id}_{\mathbf{M}}$ are variants, so $\partial(\int_{\mathbf{M}})$ and $\partial(\mathrm{id}_{\mathbf{M}})$ are variants. Now the required follows from (K5) and (H2).

(3) From (2) it follows that $\partial(\int_{\mathbf{M}}) = \mathrm{id}_{\partial\mathbf{M}} = \partial(\mathrm{id}_{\mathbf{M}})$.

(4) follows from (3), (H2) and the fact that $\partial\mathbf{A} = \mathbf{A}$.

(5) $\partial(\int_{\mathbf{K}}\circ\ \partial h) = \partial(\int_{\mathbf{K}})\circ\ \partial(\partial h) = \mathrm{id}_{\partial\mathbf{K}}\circ\partial h = \partial h = \partial h\circ\mathrm{id}_{\partial\mathbf{M}} = \partial h\circ\ \partial(\int_{\mathbf{M}}) = \partial(h\circ\int_{\mathbf{M}})$, hence $\int_{\mathbf{K}}\circ\ \partial h$ and $h\circ\int_{\mathbf{M}}$ are variants so (H2) implies the required. ∎

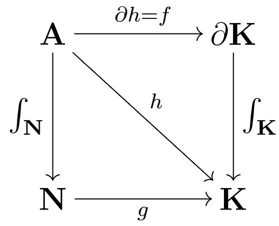
N) **Remark.** According to (M5), $\int_{\mathbf{M}}$ is a natural transformation from ∂ to the identity endofunctor of $\mathfrak{Str}$.

O) **Proposition.** (1) *For any algebra* $\mathbf{A}$*, a structure* $\mathbf{K}$ *and a homomorphism* $h : \mathbf{A}\to\mathbf{K}$ *we have* $h = \int_{\mathbf{K}}\circ\ \partial h$.

(2) *For any algebra* $\mathbf{A}$*, a structure* $\mathbf{K}$ *and a homomorphism* $f : \mathbf{A}\to\partial\mathbf{K}$*, there exists unique homomorphism* $h : \mathbf{A}\to\mathbf{K}$*, such that* $\partial h = f$.

(3) *For any normal algebra* $\mathbf{A}$*, a logical structure* $\mathbf{K}$ *and a homomorphism* $h : \mathbf{A}\to\mathbf{K}$*, there exists unique logical variant* $\mathbf{N}$ *of* $\mathbf{K}$ *and a homo-*

morphism $g : \mathbf{N} \to \mathbf{K}$*, such that* $h = g \circ \int_{\mathbf{N}}$.



Proof. **(1)** From (M5) it follows that $\int_{\mathbf{K}} \circ\ \partial h = h \circ \int_{\mathbf{A}}$ and (M4) implies $\int_{\mathbf{A}} = \mathrm{id}_{\mathbf{A}}$.

(2) Suppose $\partial h = f$. Then (1) implies $h = \int_{\mathbf{K}} \circ\ \partial h = \int_{\mathbf{K}} \circ f$. This proves the uniqueness. In order to prove the existence, let $h = \int_{\mathbf{K}} \circ f$. Then $\int_{\mathbf{K}} \circ\ \partial h = \int_{\mathbf{K}} \circ\ \partial(\int_{\mathbf{K}} \circ f) = \int_{\mathbf{K}} \circ\ \partial(\int_{\mathbf{K}}) \circ\ \partial f = \int_{\mathbf{K}} \circ \mathrm{id}_{\partial \mathbf{K}} \circ \partial f = \int_{\mathbf{K}} \circ\ \partial f = \int_{\mathbf{K}} \circ f = h$.

(3) For any logical variant $\mathbf{N}$ of $\mathbf{A}$ and a homomorphism $g : \mathbf{N} \to \mathbf{K}$, the homomorphisms g and h are identical over the algebraic carriers if and only if $\partial g = \partial h$.

This is so if and only if $h = g \circ \int_{\mathbf{N}}$. Indeed, on one hand if $\partial h = \partial g$, then (1) implies $h = \int_{\mathbf{K}} \circ \partial h = \int_{\mathbf{K}} \circ \partial g = g \circ \int_{\mathbf{N}}$, because of (M5). On the other hand, if $h = g \circ \int_{\mathbf{N}}$, then $\partial h = \partial g \circ \partial(\int_{\mathbf{N}}) = \partial g \circ \mathrm{id}_{\mathbf{A}} = \partial g$.

Therefore, from (10K) we obtain the required.[34] ■

P) **Corollary.** (1) *Given a structure* $\mathbf{M}$ *and a termal expression* τ *over* $|\mathbf{M}|$ *which doesn't contain names of logical sort,* $\tau^{\mathbf{M}} = \int_{\mathbf{M}}(\tau^{\partial \mathbf{M}})$.

(2) *Given a structure* $\mathbf{M}$ *and a term* τ *over* $|\mathbf{M}|$, $\tau^{\mathbf{M}} = \tau^{\partial \mathbf{M}}$.

(3) *Given structures* $\mathbf{M}'$ *and* $\mathbf{M}''$ *that are variants, if* τ *is a term over* $|\mathbf{M}'|$*, then* τ *also is a term over* $|\mathbf{M}''|$ *and* $\tau^{\mathbf{M}'} = \tau^{\mathbf{M}''}$.

(4) *Given a structure* $\mathbf{M}$ *and a formula* φ *over* $|\mathbf{M}|$, $\varphi^{\mathbf{M}} = (\varphi^{\partial \mathbf{M}})^{\mathbf{M}}$.

(5) *Given an algebra* $\mathbf{A}$ *and a relational formula* φ *over* $|\mathbf{A}|$, $\varphi^{\mathbf{A}} = \varphi$.

(6) *Given an algebra* $\mathbf{A}$ *and a formula* φ *over* $|\mathbf{A}|$, $(\varphi^{\mathbf{A}})^{\mathbf{A}} = \varphi^{\mathbf{A}}$.

Proof. **(1)** τ doesn't contain names of logical sort and τ is a termal expression over $|\mathbf{M}|$, so τ is a termal expression over $|\partial \mathbf{M}|$ as well. Moreover, $\int_{\mathbf{M}}$ is identity over the algebraic carriers, so $\tau[\int_{\mathbf{M}}] = \tau$. From (11N) it follows that $\int_{\mathbf{M}}(\tau^{\partial \mathbf{M}}) = \tau[\int_{\mathbf{M}}]^{\mathbf{M}} = \tau^{\mathbf{M}}$.

(2) Follows from (1) because $\int_{\mathbf{M}}$ is identity over the algebraic carriers.

(3) $\partial \mathbf{M}' = \partial \mathbf{M}''$, so from (2) it follows that $\tau^{\mathbf{M}'} = \tau^{\partial \mathbf{M}'} = \tau^{\partial \mathbf{M}''} = \tau^{\mathbf{M}''}$.

(4) Formulae do not contain names of logical sort, so from (1) it follows that $\varphi^{\mathbf{M}} = \int_{\mathbf{M}}(\varphi^{\partial \mathbf{M}}) = (\varphi^{\partial \mathbf{M}})^{\mathbf{M}}$.

[34] The structure $\mathbf{M}$ in (10K) is our algebra $\mathbf{A}$.

(5) By definition (C2), the logical carrier of $\mathbf{A}$ contains all relational formulae over $|\mathbf{A}|$, so $\varphi \in \mathbf{A}_{\mathtt{Log}}$, hence definition (L) implies $\varphi^{\mathbf{A}} = \int_{\mathbf{A}}(\varphi)$, so $\varphi^{\mathbf{A}} = \varphi$ because of (M4).

(6) $\mathbf{A} = \partial\mathbf{A}$ for any algebra $\mathbf{A}$, so from (4) it follows that $\varphi^{\mathbf{A}} = (\varphi^{\partial\mathbf{A}})^{\mathbf{A}} = (\varphi^{\mathbf{A}})^{\mathbf{A}}$.

Alternatively, we can deduce (6) as a corollary from (5). Indeed, $\varphi^{\mathbf{A}}$ belongs to the logical carrier of $\mathbf{A}$, so it is a relational formula. ∎

Q) **Observation.** (1) *When defining an algebra, it is enough to specify only the algebraic carriers and the interpretation of the functional symbols.*

(2) *When defining a homomorphism from algebra to some structure it is enough to consider only the algebraic sorts. More formally, suppose $\mathbf{A}$ is an algebra, $\mathbf{M}$ is a structure, we have defined a $\mathtt{Sort} \setminus \{\mathtt{Log}\}$-indexed function g from the algebraic carriers $\mathbf{A}_\kappa$ to the corresponding carriers $\mathbf{M}_\kappa$ and we have proved that $g_\kappa(\mathtt{f}^{\mathbf{A}}\langle\alpha_1, \ldots, \alpha_n\rangle) = \mathtt{f}^{\mathbf{M}}\langle g_\kappa\alpha_1, \ldots, g_\kappa\alpha_n\rangle$ for any functional symbol $\mathtt{f}$. Then there exists unique homomorphism $h : \mathbf{A} \to \mathbf{M}$, such that for all algebraic sorts κ we have $h_\kappa = g_\kappa$.*

Proof. **(1)** Let $\mathbf{A}$ be the unique terminal algebraic structure with specified algebraic carriers and interpretation of the functional symbols (it exists, see B). Then $\partial\mathbf{A}$ is an algebra with specified algebraic carriers and interpretation of the functional symbols; moreover it is unique because of (I1).

(2) Let $\mathbf{A}'$ and $\mathbf{M}'$ be terminal algebraic structures that are variants of $\mathbf{A}$ and $\mathbf{M}$, respectively (they exist, see B). Let $h' : \mathbf{A}' \to \mathbf{M}'$ be a homomorphism, such that for all algebraic sorts κ we have $h'_\kappa = g_\kappa$ (there is such homomorphism and only one at that, see B). Now, we can define $h = \partial(h')$. The uniqueness of h is guaranteed by (I2). ∎

R) **Definition.** (1) For any $\mathtt{Sort}$-indexed set X, let X° be the $\mathtt{Sort}$-indexed set, such that $(X^\circ)_{\mathtt{Log}} = \varnothing$ and $(X^\circ)_\kappa = X_\kappa$ for all algebraic sorts κ.

(2) For any $\mathtt{Sort}$-indexed function $f : X \to Y$, let $f^\circ : X^\circ \to Y^\circ$ be the $\mathtt{Sort}$-indexed function, such that $(f^\circ)_{\mathtt{Log}}$ is the function whose both the domain and the codomain are empty sets and $(f^\circ)_\kappa = f_\kappa$ for all algebraic sorts κ.

The following proposition is obvious. It says that $^\circ$ is an idempotent endofunctor of $\mathfrak{Set}^{\mathtt{Sort}}$.

S) **Proposition.** (1) $(\mathrm{id}_X)^\circ = \mathrm{id}_{X^\circ}$.
(2) $(f \circ g)^\circ = (f^\circ) \circ (g^\circ)$.
(3) $(X^\circ)^\circ = X^\circ$, $(f^\circ)^\circ = f^\circ$.

T) **Proposition.** (1) *If $X^\circ = Y^\circ$, then $\partial[X] = \partial[Y]$ for any* $\mathtt{Sort}$-

indexed sets X *and* Y.

(2) *If* $f^\circ = g^\circ$, *then* $\partial[f] = \partial[h]$ *for any* `Sort`-*indexed functions* f *and* g.

Proof. **(1)** If $X^\circ = Y^\circ$, then term over X and term over Y are one and the same thing, so $[X]$ and $[Y]$ are variants, i.e. the corresponding algebraic carriers are identical.

(2) If $f^\circ = g^\circ$, then from (1) it follows that the domains of $[f]$ and $[g]$ are variants and the codomains also are variants. Moreover, f and g map the algebraic names the same way, so $[f]$ and $[g]$ map the terms from the domain of $[f]$ and $[g]$ identically. Consequently $[f]$ and $[g]$ are variants. ■

U) **Proposition.** (1) *Given* `Sort`-*indexed functions* $f, g : X^\circ \to Y$, *if the algebraic components of* f *and* g *are equal, then* $f = g$.

(2) *Given* `Sort`-*indexed functions* $f, g : X^\circ \to Y$, *if* $f^\circ = g^\circ$, *then* $f = g$.

Proof. Obviously $f^\circ = g^\circ$ if and only if the algebraic component of f and g are equal. On the other hand the logical components of f and g are necessarily equal because their domain is the empty set. ■

§13. SATISFIABILITY IN AN ALGEBRA

A) **Definition.** (1) Given a logical structure $\mathbf{M}$, a formula φ over $|\mathbf{M}|$ is *true* in $\mathbf{M}$ if $\varphi^{\mathbf{M}} = 1$.

(2) A formula φ over X is *universally valid* in a logical structure $\mathbf{M}$, if for any assignment function $v : X \to |\mathbf{M}|$, $\varphi[v]$ is true in $\mathbf{M}$.

(3) A formula is *tautology* if it is universally valid in all logical structures.

B) **Definition.** Let $\mathbf{A}$ be an algebra. Two termal expressions τ_1 and τ_2 over $|\mathbf{A}|$ with no names of logical sort are *equivalent* in $\mathbf{A}$, if for any logical variant $\mathbf{M}$ of $\mathbf{A}$, $\tau_1^{\mathbf{M}} = \tau_2^{\mathbf{M}}$.

C) **Example.** Suppose we have an unary functional symbol `f` of type $\langle\langle\kappa\rangle, \kappa\rangle$, a constant symbol `c` of type $\langle\langle\rangle, \kappa\rangle$ and a predicate symbol `p` of type $\langle\langle\kappa\rangle, \mathtt{Log}\rangle$. Take an arbitrary algebra $\mathbf{A}$. Then the formulae `p(f(c))` and $\mathtt{p}(\ulcorner \mathtt{f}^{\mathbf{A}}\langle \mathtt{c}^{\mathbf{A}}\rangle\urcorner)$ are equivalent in $\mathbf{A}$.

Notice that $(\mathtt{p(f(c))})^{\mathbf{A}} = \mathtt{p}(\ulcorner \mathtt{f}^{\mathbf{A}}\langle \mathtt{c}^{\mathbf{A}}\rangle\urcorner)$.

D) **Proposition.** *Given logical structures* $\mathbf{M}$ *and* $\mathbf{K}$ *and a homomorphism* $h : \mathbf{M} \to \mathbf{K}$, *if a formula is universally valid in* $\mathbf{K}$, *then it is universally valid in* $\mathbf{M}$.

Proof. Let φ be a formula over X, such that φ is universally valid in $\mathbf{K}$. Let $v : X \to |\mathbf{M}|$ be an arbitrary assignment function. Then (11Q2) implies

$\varphi[v]^{\mathbf{M}} = \varphi[h \circ v]^{\mathbf{K}}$. On the other hand, $\varphi[h \circ v]^{\mathbf{K}} = 1$ because φ is universally valid in $\mathbf{K}$. Consequently, $\varphi[v]^{\mathbf{M}} = 1$. ∎

E) **Proposition.** *For any algebra* $\mathbf{A}$ *and formula* φ *over* $|\mathbf{A}|$, φ *and* $\varphi^{\mathbf{A}}$ *are equivalent in* $\mathbf{A}$.

Proof. Take an arbitrary logical variant $\mathbf{K}$ of $\mathbf{A}$. From (12P4) and $\partial\mathbf{K} = \mathbf{A}$ it follows that $\varphi^{\mathbf{K}} = (\varphi^{\mathbf{A}})^{\mathbf{K}}$. ∎

F) **Definition.** (1) Given an algebra $\mathbf{A}$, a formula over $|\mathbf{A}|$ is *satisfiable* in $\mathbf{A}$ if it is true in some logical variant of $\mathbf{A}$.

(2) A a formula is *universally satisfiable* if it is universally valid in some logical structure.

(3) A formula is *universally satisfiable in an algebra* $\mathbf{A}$ if it is universally valid in some logical variant of $\mathbf{A}$.

(4) Given an algebra $\mathbf{A}$, a set of formulae over $|\mathbf{A}|$ is *satisfiable* in $\mathbf{A}$, if there exists a logical variant $\mathbf{M}$ of $\mathbf{A}$, such that all formulae from the set are true in $\mathbf{M}$.

(5) A set of formulae is *universally satisfiable* if there exists a logical structure $\mathbf{M}$, such that all formulae from the set are universally valid in $\mathbf{M}$.

(6) A set of formulae is *universally satisfiable in an algebra* $\mathbf{A}$, if there exists a logical variant $\mathbf{M}$ of $\mathbf{A}$, such that all formulae from the set are universally valid in $\mathbf{M}$.

Any universally satisfiable set of formulae is universally valid in some Herbrand structure. The following proposition states a mild generalisation of this fact.

G) **Proposition.** *Suppose* $[X]$ *is normal. A set of formulae*[35] *is universally satisfiable if and only if it is universally satisfiable in* $\partial[X]$.

Proof. ($\Leftarrow$) By definition.

($\Rightarrow$) Suppose the given set of formulae is universally valid in some logical structure $\mathbf{K}$. Take an arbitrary assignment function $v : X \to |\mathbf{K}|$. Then $[v]^{\mathbf{K}}$ is a homomorphism from $[X]$ to $\mathbf{K}$, so $\partial([v]^{\mathbf{K}})$ is a homomorphism from $\partial[X]$ to $\partial\mathbf{K}$, hence $\int_{\mathbf{K}} \circ\, \partial([v]^{\mathbf{K}})$ is a homomorphism from $\partial[X]$ to $\mathbf{K}$. Now from (12O3) we obtain a logical variant $\mathbf{N}$ of $\partial[X]$ and a homomorphism $g : \mathbf{N} \to \mathbf{K}$, so (D) implies the given set of formulae is universally valid

[35] Not necessarily formulae over X.

in **N**.

$$\begin{array}{ccc} \partial[X] & \xrightarrow{\partial([v]^{\mathbf{K}})} & \partial\mathbf{K} \\ \Big\downarrow{\scriptstyle \int_{\mathbf{N}}} & & \Big\downarrow{\scriptstyle \int_{\mathbf{K}}} \\ \mathbf{N} & \xrightarrow{\quad g \quad} & \mathbf{K} \end{array}$$

■

H) **Proposition.** *Let Λ be a family of satisfiable in the algebra* **A** *sets which contain only literals and for any $\Gamma', \Gamma'' \in \Lambda$, either $\Gamma' \subseteq \Gamma''$, or $\Gamma'' \subseteq \Gamma'$. Then the union of the elements of Λ also is satisfiable in* **A**.

Proof. Let Γ be the union of the elements of Λ. Let **M** be a logical variant of **A** interpreting the predicate symbols in the following way: $\mathtt{p}^{\mathbf{M}}\langle \alpha_1, \dots, \alpha_n \rangle = 1$ if and only if there exist terms $\tau_1, \dots, \tau_n$, such that $\tau_1^{\mathbf{A}} = \alpha_1, \tau_2^{\mathbf{A}} = \alpha_2, \dots, \tau_n^{\mathbf{A}} = \alpha_n$ and $\mathtt{p}(\tau_1, \dots, \tau_n) \in \Gamma$. By definition, all atomic formulae in Γ are true in **M**. Let $\neg\mathtt{p}(\tau_1, \dots, \tau_n)$ be an arbitrary negated literal belonging to Γ. Then it belongs to some $\Gamma' \in \Lambda$. Suppose it is false in **M**. Then $\mathtt{p}(\tau_1, \dots, \tau_n)$ is true in **M**, hence Γ contains an atomic formula $\mathtt{p}(\tau_1', \dots, \tau_n')$, such that $\tau_1^{\mathbf{A}} = \tau_1'^{\mathbf{A}}, \dots, \tau_n^{\mathbf{A}} = \tau_n'^{\mathbf{A}}$. Then $\mathtt{p}(\tau_1', \dots, \tau_n') \in \Gamma''$ for some $\Gamma'' \in \Lambda$. Regardless of whether $\Gamma' \subseteq \Gamma''$, or $\Gamma'' \subseteq \Gamma'$, one of these sets contains both $\mathtt{p}(\tau_1', \dots, \tau_n')$ and $\neg\mathtt{p}(\tau_1, \dots, \tau_n)$ which is impossible because it is a satisfiable set. ■

I) **Definition.** (1) Given an algebra **A**, two sets Γ and Δ of formulae over $|\mathbf{A}|$ are *equivalent* in **A**, if for any logical variant **M** of **A**, such that all formulae from one of these sets are true in **M**, all formulae from the other set also are true in **M**.

(2) Two formulae φ_1 and φ_2 are *equivalent* in an algebra **A**, if $\{\varphi_1\}$ and $\{\varphi_2\}$ are equivalent in **A**.

(3) A set Γ of formulae is *equivalent* in an algebra **A** with a formula φ, if Γ is equivalent with $\{\varphi\}$ in **A**.

(4) Two sets Γ and Δ of formulae are *universally equivalent*, if for any logical structure **M**, such that all formulae from one of these sets are universally valid in **M**, all formulae from the other set also are universally valid in **M**.

(5) Let **A** be an algebra. Two sets Γ and Δ of formulae are *universally equivalent* in **A**, if for any logical variant **M** of **A**, such that all formulae from one of these sets are universally valid in **M**, all formulae from the other set also are universally valid in **M**.

(6) Two formulae φ_1 and φ_2 are *universally equivalent*, if $\{\varphi_1\}$ and $\{\varphi_2\}$

are universally equivalent. They are *universally equivalent* in an algebra $\mathbf{A}$, if $\{\varphi_1\}$ and $\{\varphi_2\}$ are universally equivalent in $\mathbf{A}$.

(7) A set Γ of formulae is universally equivalent with a formula φ, if Γ is universally equivalent with $\{\varphi\}$. The set Γ is universally equivalent with φ in an algebra $\mathbf{A}$, if Γ is universally equivalent with $\{\varphi\}$ in the algebra $\mathbf{A}$.

The following proposition is obvious.

J) **Proposition.** (1) *The equivalency, the universal equivalency and the universal equivalency in an algebra are equivalence relations, i.e. they are reflexive, symmetric and transitive relations.*

(2) *If Θ_1 and Θ_2 are equivalent in $\mathbf{A}$ (formulae or sets of formulae), then they are universally equivalent in $\mathbf{A}$.*

(3) *If $\mathbf{A}$ is an algebra and Θ_1 and Θ_2 are equivalent in $\mathbf{A}$ (formulae or sets of formulae), then Θ_1 is satisfiable in $\mathbf{A}$ if and only if Θ_2 is satisfiable.*

(4) *If Θ_1 and Θ_2 are universally equivalent (formulae or sets of formulae), then Θ_1 is universally satisfiable if and only if Θ_2 is universally satisfiable.*

(5) *If $\mathbf{A}$ is an algebra and Θ_1 and Θ_2 are universally equivalent in $\mathbf{A}$ (formulae or sets of formulae), then Θ_1 is universally satisfiable in $\mathbf{A}$ if and only if Θ_2 is universally satisfiable in $\mathbf{A}$.*

K) **Definition.** (1) The formula φ *follows* in the algebra $\mathbf{A}$ from the set of formulae Γ, if for any logical variant $\mathbf{M}$ of $\mathbf{A}$, such that all formulae from Γ are true in $\mathbf{M}$, the formula φ also is true in $\mathbf{M}$.

(2) The formula φ *universally follows* in the algebra $\mathbf{A}$ from the set of formulae Γ, if for any logical variant $\mathbf{M}$ of $\mathbf{A}$, such that all formulae from Γ are universally valid in $\mathbf{M}$, the formula φ also is universally valid in $\mathbf{M}$.

(3) The formula φ *universally follows* from the set of formulae Γ, if for any logical structure $\mathbf{M}$, such that all formulae from Γ are universally valid in $\mathbf{M}$, the formula φ also is universally valid in $\mathbf{M}$.

Obviously if φ universally follows from Γ, then φ universally follows from Γ in any algebra $\mathbf{A}$.

Algebraic Theory of Termoids

§14. TERMINATORS

Recall that the most noticeable difference between terms and termoids is that while each term over $|\mathbf{M}|$ has exactly one value, a termoid over $|\mathbf{M}|$ may have many values. For any structure $\mathbf{M}$ we have a homomorphism $\mathrm{val}_{\mathbf{M}} : [|\mathbf{M}|] \to \mathbf{M}$, such that $\mathrm{val}_{\mathbf{M}}\,\tau = \tau^{\mathbf{M}}$ for any termal expression τ. No such homomorphism exists for termoids. Instead we have a `Sort`-indexed function $\mathrm{Val}_{\mathbf{M}} : [\![|\mathbf{M}|]\!] \to \mathcal{P}\mathbf{M}$, such that $\mathrm{Val}_{\mathbf{M}}\,\tau = \tau^{\mathcal{P}\mathbf{M}}$ for any termoid τ. This `Sort`-indexed function is not even a homomorphism. It is going to be what we call a quasimorphism.

A) **Definition.** (1) Given `Sort`-indexed functions $f, g : X \to \mathcal{P}Y$, we write $f \le g$, if $f\xi \subseteq g\xi$ for any $\xi \in X$.

(2) Given `Sort`-indexed functions $f : X \to Y$ and $g : X \to \mathcal{P}Y$, we write $f \ll g$, if $f\xi \in g\xi$ for any $\xi \in X$.

B) **Proposition.** (1) *Given* `Sort`*-indexed functions* $f, g : X \to \mathcal{P}Y$ *and* $h : Z \to X$, *if* $f \le g$, *then* $f \circ h \le g \circ h$.

(2) *Given* `Sort`*-indexed functions* $f, g : X \to \mathcal{P}Y$ *and* $h : Y \to Z$, *if* $f \le g$, *then* $h^{\mathcal{P}} \circ f \le h^{\mathcal{P}} \circ g$.

Proof. **(1)**$(f \circ h)\zeta = f(h\zeta) \subseteq g(h\zeta) = (g \circ h)\zeta$ for any $\zeta \in Z$.

(2) If $\zeta \in (h^{\mathcal{P}} \circ f)\xi$ for some $\xi \in X$, then $\zeta = hv$ for some $v \in f\xi$. But $f\xi \subseteq g\xi$, so $v \in g\xi$, hence $\zeta = hv \in h^{\mathcal{P}}(g\xi) = (h^{\mathcal{P}} \circ g)\xi$. ■

C) **Definition.** A `Sort`-indexed function $h : |\mathbf{M}| \to |\mathcal{P}\mathbf{K}|$ is *quasimorphism* from $\mathbf{M}$ to $\mathcal{P}\mathbf{K}$, if for each functional symbol `f` of type $\langle\langle\kappa_1, \dots, \kappa_n\rangle, \lambda\rangle$ and for any $\alpha_1 \in \mathbf{M}_{\kappa_1}, \dots, \alpha_n \in \mathbf{M}_{\kappa_n}$ we have

$$\mathtt{f}^{\mathcal{P}\mathbf{K}}\langle h_{\kappa_1}\alpha_1, \dots, h_{\kappa_n}\alpha_n\rangle \subseteq h_\lambda(\mathtt{f}^{\mathbf{M}}\langle\alpha_1, \dots, \alpha_n\rangle)$$

In addition, for each predicate or logical symbol $\mathtt{d}$ we have

$$\mathtt{d}^{\mathcal{P}\mathbf{K}}\langle h_{\kappa_1}\alpha_1,\dots,h_{\kappa_n}\alpha_n\rangle = h_\lambda(\mathtt{d}^{\mathbf{M}}\langle\alpha_1,\dots,\alpha_n\rangle) \tag{$\sharp$}$$

D) **Corollary.** *A composition of quasimorphism with homomorphism is a quasimorphism. A composition of homomorphism with quasimorphism is a quasimorphism.*

Proof. Immediately follows from the definitions. ■

E) **Lemma.** *Given an algebra* $\mathbf{A}$ *and a structure* $\mathbf{K}$*, if the algebraic components of the quasimorphism* $h : \mathbf{A} \to \mathcal{P}\mathbf{K}$ *map to non-empty sets, then all components of* h *map to non-empty sets.*

Proof. Let $\varphi \in \mathbf{A}_{\mathtt{Log}}$. We are going to to prove that $h\varphi \neq \varnothing$ by induction on φ. If $\varphi = \mathtt{p}(\ulcorner\alpha_1\urcorner,\dots,\ulcorner\alpha_n\urcorner)$ for some predicate symbol $\mathtt{p}$, then $h\varphi = h(\mathtt{p}(\ulcorner\alpha_1\urcorner,\dots,\ulcorner\alpha_n\urcorner)) = h(\mathtt{p}^{\mathbf{A}}\langle\alpha_1,\dots,\alpha_n\rangle) = \mathtt{p}^{\mathcal{P}\mathbf{K}}\langle h\alpha_1,\dots,h\alpha_n\rangle = \{\mathtt{p}^{\mathbf{K}}\langle\beta_1,\dots,\beta_n\rangle : \beta_1 \in h\alpha_1,\dots,\beta_n \in h\alpha_n\}$. The last set is non-empty since the algebraic components of h map to non-empty sets, so $h\alpha_1,\dots,h\alpha_n$ are non-empty.

If $\varphi = \mathtt{d}(\varphi_1,\dots,\varphi_n)$ for some logical symbol $\mathtt{d}$, then $h\varphi = h(\mathtt{d}(\varphi_1,\dots,\varphi_n)) = h(\mathtt{d}^{\mathbf{A}}\langle\varphi_1,\dots,\varphi_n\rangle) = \mathtt{d}^{\mathcal{P}\mathbf{K}}\langle h\varphi_1,\dots,h\varphi_n\rangle = \{\mathtt{d}^{\mathbf{K}}\langle\beta_1,\dots,\beta_n\rangle : \beta_1 \in h\varphi_1,\dots,\beta_n \in h\varphi_n\}$. The last set is non-empty since by the induction hypothesis, $h\varphi_1,\dots,h\varphi_n$ are non-empty sets. ■

F) **Lemma.** *Given an algebra* $\mathbf{A}$ *and a structure* $\mathbf{K}$*, if the algebraic components of the quasimorphism* $h : \mathbf{A} \to \mathcal{P}\mathbf{K}$ *map to one-element sets, then all components of* h *map to one-element sets.*

Proof. Let $\varphi \in \mathbf{A}_{\mathtt{Log}}$. We are glint to to prove that $h\varphi$ is one-element set by induction on φ. If $\varphi = \mathtt{p}(\ulcorner\alpha_1\urcorner,\dots,\ulcorner\alpha_n\urcorner)$ for some predicate symbol $\mathtt{p}$, then $h\varphi = h(\mathtt{p}(\ulcorner\alpha_1\urcorner,\dots,\ulcorner\alpha_n\urcorner)) = h(\mathtt{p}^{\mathbf{A}}\langle\alpha_1,\dots,\alpha_n\rangle) = \mathtt{p}^{\mathcal{P}\mathbf{K}}\langle h\alpha_1,\dots,h\alpha_n\rangle = \{\mathtt{p}^{\mathbf{K}}\langle\beta_1,\dots,\beta_n\rangle : \beta_1 \in h\alpha_1,\dots,\beta_n \in h\alpha_n\}$. The last set contains one element since the algebraic components of h map to one-element sets, so the sets $h\alpha_1,\dots,h\alpha_n$ contain exactly one element each.

If $\varphi = \mathtt{d}(\varphi_1,\dots,\varphi_n)$ for some logical symbol $\mathtt{d}$, then $h\varphi = h(\mathtt{d}(\varphi_1,\dots,\varphi_n)) = h(\mathtt{d}^{\mathbf{A}}\langle\varphi_1,\dots,\varphi_n\rangle) = \mathtt{d}^{\mathcal{P}\mathbf{K}}\langle h\varphi_1,\dots,h\varphi_n\rangle = \{\mathtt{d}^{\mathbf{K}}\langle\beta_1,\dots,\beta_n\rangle : \beta_1 \in h\varphi_1,\dots,\beta_n \in h\varphi_n\}$. The last set contains one-element since by the induction hypothesis, the sets $h\varphi_1,\dots,h\varphi_n$ contain one-element each. ■

G) **Lemma.** *Given an algebra* $\mathbf{A}$ *and a structure* $\mathbf{K}$*, let the quasimorphisms* $h, g : \mathbf{A} \to \mathcal{P}\mathbf{K}$ *be such that for any* α *belonging to an algebraic carrier of* $\mathbf{A}$*,* $h\alpha \subseteq g\alpha$*. Then* $h \leq g$*.*

Proof. Let $\varphi \in \mathbf{A}_{\mathtt{Log}}$. We are going to to prove that $h\varphi \subseteq g\varphi$ by induction on φ. If $\varphi = \mathtt{p}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner)$ for some predicate symbol $\mathtt{p}$, then $h\varphi = h(\mathtt{p}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner)) = h(\mathtt{p}^{\mathbf{A}}\langle\alpha_1, \dots, \alpha_n\rangle) = \mathtt{p}^{\mathcal{P}\mathbf{K}}\langle h\alpha_1, \dots, h\alpha_n\rangle = \{\mathtt{p}^{\mathbf{K}}\langle\beta_1, \dots, \beta_n\rangle : \beta_1 \in h\alpha_1, \dots, \beta_n \in h\alpha_n\}$. Since $\alpha_1, \dots, \alpha_n$ belong to algebraic carriers of $\mathbf{A}$, $h\alpha_i \subseteq g\alpha_i$ for any $i \in \{1, \dots, n\}$, so the above set is a subset of $\{\mathtt{p}^{\mathbf{K}}\langle\beta_1, \dots, \beta_n\rangle : \beta_1 \in g\alpha_1, \dots, \beta_n \in g\alpha_n\} = \mathtt{p}^{\mathcal{P}\mathbf{K}}\langle g\alpha_1, \dots, g\alpha_n\rangle = g(\mathtt{p}^{\mathbf{A}}\langle\alpha_1, \dots, \alpha_n\rangle) = g(\mathtt{p}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner)) = g\varphi$.

If $\varphi = \mathtt{d}(\varphi_1, \dots, \varphi_n)$ for some logical symbol $\mathtt{d}$, then $h\varphi = h(\mathtt{d}(\varphi_1, \dots, \varphi_n)) = h(\mathtt{d}^{\mathbf{A}}\langle\varphi_1, \dots, \varphi_n\rangle) = \mathtt{d}^{\mathcal{P}\mathbf{K}}\langle h\varphi_1, \dots, h\varphi_n\rangle = \{\mathtt{d}^{\mathbf{K}}\langle\beta_1, \dots, \beta_n\rangle : \beta_1 \in h\varphi_1, \dots, \beta_n \in h\varphi_n\}$. By induction hypothesis, $h\varphi_i \subseteq g\varphi_i$ for any $i \in \{1, \dots, n\}$, so the above set is a subset of $\{\mathtt{p}^{\mathbf{K}}\langle\beta_1, \dots, \beta_n\rangle : \beta_1 \in g\varphi_1, \dots, \beta_n \in g\varphi_n\} = \mathtt{p}^{\mathcal{P}\mathbf{K}}\langle g\varphi_1, \dots, g\varphi_n\rangle = g(\mathtt{p}^{\mathbf{A}}\langle\varphi_1, \dots, \varphi_n\rangle) = g(\mathtt{p}(\varphi_1, \dots, \varphi_n)) = g\varphi$. ∎

H) **Corollary.** *Given an algebra* $\mathbf{A}$ *and a structure* $\mathbf{K}$*, let the quasimorphisms* $h, g : \mathbf{A} \to \mathcal{P}\mathbf{K}$ *be such that for any* α *belonging to an algebraic carrier of* $\mathbf{A}$*,* $h\alpha = g\alpha$*. Then* $h = g$*.*

Proof. From (G) we obtain both $h \leq g$ and $g \leq h$. ∎

I) **Definition.** We specify the axioms of termoids in an structure called "terminator". The intuitive meaning of these axioms will be explained in (K).

Terminator is a quadruple $\langle [\![.]\!], \mathrm{Val}_{\mathbf{M}}, \mathrm{Vals}_X, \mathrm{Nam}_X\rangle$, such that:

1. $[\![X]\!]$ is an algebra for any $\mathtt{Sort}$-indexed set X.
2. $[\![f]\!]$ is a homomorphism from $[\![X]\!]$ to $[\![Y]\!]$ for any $\mathtt{Sort}$-indexed function $f : X \to Y$. We are going to use postfix notation for this homomorphism, thus $\tau[\![f]\!]$ means to apply $[\![f]\!]$ to τ.
3. Given $\mathtt{Sort}$-indexed sets X and Y, $|[\![X]\!]| \cap |[\![Y]\!]| = |[\![X \cap Y]\!]|$.[36]
4. Given $\mathtt{Sort}$-indexed functions $f' : X' \to Y'$ and $f'' : X'' \to Y''$, if X' is a subset of X'', Y'' is a subset of Y' and $f'\xi = f''\xi$ for any $\xi \in X'$, then $\tau[\![f']\!] = \tau[\![f'']\!]$ for any $\tau \in |[\![X']\!]|$. In particular, this implies that $[\![f \restriction X]\!] = [\![f]\!] \restriction [\![X]\!]$ provided X is a subset of the domain of f.
5. $[\![X]\!] = [\![X^\circ]\!]$ and $[\![f]\!] = [\![f^\circ]\!]$ for any $\mathtt{Sort}$-indexed set X and function f.
6. $[\![\mathrm{id}_X]\!] = \mathrm{id}_{[\![X]\!]}$.
7. $[\![f \circ g]\!] = [\![f]\!] \circ [\![g]\!]$, if the domain of f and the codomain of g are identical.

[36]Recall that $Z' \cap Z'' = \{Z'_\kappa \cap Z''_\kappa\}_{\kappa \in \mathtt{Sort}}$ for any $\mathtt{Sort}$-indexed sets $Z' = \{Z'_\kappa\}_{\kappa \in \mathtt{Sort}}$ and $Z'' = \{Z''_\kappa\}_{\kappa \in \mathtt{Sort}}$.

8. Nam_X is a `Sort`-indexed function from X° to $|[\![X]\!]|$ for any `Sort`-indexed set X.
9. $[\![f]\!] \circ \mathrm{Nam}_X = \mathrm{Nam}_Y \circ f^\circ$ for any `Sort`-indexed function $f : X \to Y$.

$$\begin{array}{ccc} X^\circ & \xrightarrow{\mathrm{Nam}_X} & |[\![X]\!]| \\ {\scriptstyle f^\circ}\downarrow & & \downarrow{\scriptstyle [\![f]\!]} \\ Y^\circ & \xrightarrow{\mathrm{Nam}_Y} & |[\![Y]\!]| \end{array}$$

10. $\mathrm{Val}_{\mathbf{M}}$ is a quasimorphism from $[\![|\mathbf{M}|]\!]$ to $\mathcal{P}\mathbf{M}$ for any structure $\mathbf{M}$.
11. $\mathrm{Val}_{\mathbf{M}}\,\tau = \mathrm{Val}_{\partial\mathbf{M}}\,\tau$ for any τ belonging to an algebraic carrier of $[\![|\mathbf{M}|]\!]$. Notice that (I5) implies that $[\![|\mathbf{M}|]\!] = [\![|\partial\mathbf{M}|]\!]$.
12. For any structure $\mathbf{M}$ the quasimorphism $\mathrm{Val}_{\mathbf{M}}$ maps only to non-empty sets. For any `Sort`-indexed set X the quasimorphism $\mathrm{Val}_{[X]}$ maps only to one-element sets.
13. $h^{\mathcal{P}} \circ \mathrm{Val}_{\mathbf{M}} \leq \mathrm{Val}_{\mathbf{K}} \circ [\![h]\!]$ for any homomorphism $h : \mathbf{M} \to \mathbf{K}$.

$$\begin{array}{ccc} [\![|\mathbf{M}|]\!] & \xrightarrow{\mathrm{Val}_{\mathbf{M}}} & \mathcal{P}\mathbf{M} \\ {\scriptstyle [\![h]\!]}\downarrow & \geq & \downarrow{\scriptstyle h^{\mathcal{P}}} \\ [\![|\mathbf{K}|]\!] & \xrightarrow{\mathrm{Val}_{\mathbf{K}}} & \mathcal{P}\mathbf{K} \end{array}$$

14. Vals_X is a homomorphism from $[\![|[\![X]\!]|]\!]$ to $[\![X]\!]$ for any `Sort`-indexed set X.
15. Given a `Sort`-indexed set X, a structure $\mathbf{K}$, a `Sort`-indexed function $k : X \to |[\![|\mathbf{K}|]\!]|$ and a `Sort`-indexed function $f : X \to |\mathbf{K}|$, if $f \ll \mathrm{Val}_{\mathbf{K}} \circ k$,

$$\begin{array}{ccc} X & \xrightarrow{k} & |[\![|\mathbf{K}|]\!]| \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{Val}_{\mathbf{K}}} \\ |\mathbf{K}| & \in & |\mathcal{P}\mathbf{K}| \end{array}$$

then $\mathrm{Val}_{\mathbf{K}} \circ \llbracket f \rrbracket \leq \mathrm{Val}_{\mathbf{K}} \circ \mathrm{Vals}_{|\mathbf{K}|} \circ \llbracket k \rrbracket$.

$$\begin{array}{ccccc} \llbracket X \rrbracket & \xrightarrow{\llbracket k \rrbracket} & \llbracket | \llbracket |\mathbf{K}| \rrbracket | \rrbracket & \xrightarrow{\mathrm{Vals}_{|\mathbf{K}|}} & \llbracket |\mathbf{K}| \rrbracket \\ & \searrow^{\llbracket f \rrbracket} & \leq & & \downarrow \mathrm{Val}_{\mathbf{K}} \\ & & \llbracket |\mathbf{K}| \rrbracket & \xrightarrow{\mathrm{Val}_{\mathbf{K}}} & \mathcal{P}\mathbf{K} \end{array}$$

16. $\llbracket f \rrbracket \circ \mathrm{Vals}_X = \mathrm{Vals}_Y \circ \llbracket \llbracket f \rrbracket \rrbracket$ for any `Sort`-indexed function $f : X \to Y$.

$$\begin{array}{ccc} \llbracket | \llbracket X \rrbracket | \rrbracket & \xrightarrow{\mathrm{Vals}_X} & \llbracket X \rrbracket \\ \llbracket \llbracket f \rrbracket \rrbracket \downarrow & & \downarrow \llbracket f \rrbracket \\ \llbracket | \llbracket Y \rrbracket | \rrbracket & \xrightarrow{\mathrm{Vals}_Y} & \llbracket Y \rrbracket \end{array}$$

17. $\mathrm{id}_{\llbracket X \rrbracket} = \mathrm{Vals}_X \circ \llbracket \mathrm{Nam}_X \rrbracket$ for any `Sort`-indexed set X.

$$\llbracket X \rrbracket = \llbracket X^{\circ} \rrbracket \xrightarrow{\llbracket \mathrm{Nam}_X \rrbracket} \llbracket | \llbracket X \rrbracket | \rrbracket \xrightarrow{\mathrm{Vals}_X} \llbracket X \rrbracket$$

18. $(\mathrm{Val}_{\mathbf{M}} \circ \mathrm{Nam}_{|\mathbf{M}|})\mu = \{\mu\}$ for any structure $\mathbf{M}$, algebraic sort κ and $\mu \in |\mathbf{M}|_{\kappa}$.

$$|\mathbf{M}|^{\circ} \xrightarrow{\mathrm{Nam}_{|\mathbf{M}|}} |\llbracket |\mathbf{M}| \rrbracket| \xrightarrow{\mathrm{Val}_{\mathbf{M}}} |\mathcal{P}\mathbf{M}|$$

J) **Definition.** *Termoidal expression* of sort κ over the `Sort`-indexed set X is an element of $|\llbracket X \rrbracket|_{\kappa}$. If κ is an algebraic sort, then this termoidal expression is called *termoid*. If $\kappa = \mathtt{Log}$, it is *formuloid*. Notice that (I1) and (12C2) imply that φ is a formuloid over X if an only if φ is a relational formula over $|\llbracket X \rrbracket|$. *Atomic formuloid* over X is a relational atomic formula over $|\llbracket X \rrbracket|$.

K) In the informal sections I defined the "formuloid" as a formula in which termoids are used instead of terms. Such a definition supposes that termoids are strings of symbols satisfying some peculiar properties in order be able to implement correct syntactic analysis of formuloids. This is inconvenient. Therefore, it is preferable to define the formuloids to be formulae in which the arguments of the predicate symbols are names of termoids. In this way termoids do not have to be strings of symbols and we do not have to distract our reasoning with unnecessary syntactic considerations. As a side effect, the axioms describing termoids are going to become simpler.

The termal structure $[X]$ is not an algebra. The elements of the logical carrier of $[X]$ are termal expressions, so if, for example, $\mathtt{p}$ is a predicate symbol of suitable type and τ is a term, then $\mathtt{p}(\tau)$ is an atomic formula. On the other hand, in (I1) we define $[\![X]\!]$ to be an algebra. Therefore, the elements of the logical carrier of $[\![X]\!]$ are relational formulae over $|[\![X]\!]|$, so if $\mathtt{p}$ is a predicate symbol of suitable type and τ is a termoid, then $\mathtt{p}(\ulcorner\tau\urcorner)$ is an atomic formuloid.

According to (I2), (I6) and (I7), $[\![\,.\,]\!]$ is a functor from the category of $\mathtt{Sort}$-indexed sets $\mathfrak{Set}^{\mathtt{Sort}}$ to the category of the structures $\mathfrak{Str}$. This is analogous to $[X]$ also being a functor.

If τ is a termoid over X and $X \subseteq Y$, then from (I3) we can conclude that τ is a termoid over Y as well.

Let $f : X' \to Y'$ and $f'' : X'' \to Y'$ be two $\mathtt{Sort}$-indexed functions, $Z \subseteq X' \cap X''$, $f' \restriction Z = f'' \restriction Z$ and τ be a termoid over Z. Then (I4) implies that $\tau[\![f']\!] = \tau[\![f'']\!]$.

While some elements of the logical carrier of $[X]$ are not formulae, all elements of the logical carrier of $[\![X]\!]$ are formulae, so no names of logical sort are permitted. Therefore $[\![X']\!] = [\![X'']\!]$ if the algebraic components of X' and X'' are equal. This follows from (I5). Similar property is valid for the renaming morphism $[\![f]\!]$.

Just as nam_X is a natural transformation from the identity functor of $\mathfrak{Set}^{\mathtt{Sort}}$ to the functor composition $|[\,.\,]|$, so Nam_X is a natural transformation from the functor $(\,.\,)^\circ$ to the functor composition $|[\![\,.\,]\!]|$. This follows from (I8) and (I9).

We do not have analogue of the evaluating morphism $\mathrm{val}_{\mathbf{M}}$. We have only two fragments of this morphism: $\mathrm{Val}_{\mathbf{M}}$ and Vals_X.

The intuitive meaning of $\mathrm{Val}_{\mathbf{M}}\,\tau$ is $\tau^{\mathcal{P}\mathbf{M}}$. This $\mathtt{Sort}$-indexed function is not a homomorphism, but only a quasimorphism (I10). According to (I13), this quasimorphism tries to be something like a natural transformation from the functor $[\![\,.\,]\!]$ to the functor $\mathcal{P}$, but instead it is only a "quasitransformation".[37]

According to (12P2), $\tau^{\mathbf{M}} = \tau^{\partial\mathbf{M}}$ for any structure $\mathbf{M}$ and term τ over $|\mathbf{M}|$. In other words, the value of τ does not depend on the interpretation of the predicate symbols in $\mathbf{M}$. According to (I11), the same is true for the termoids.

In the termal case, we defined the application of a substitution as a special case of a value in a structure: $\tau[s]^{[X]}$ is the result of the application of the substitution $s : X \to [X]$ to τ. We can not do the same for the termoids because we can not use the quasimorphism Val in order to evaluate $\tau^{[\![X]\!]}$.

[37] Compare with (11N).

Therefore, we are going to postulate the existence of an additional Sort-indexed function Vals_X. (I14) says it is a homomorphism and (I16) says it is a natural transformation from the functor composition $[\![|[\![\,.\,]\!]|]\!]$ to the functor $[\![\,.\,]\!]$.

Since we have two independent Sort-indexed functions $\mathrm{Val}_{\mathbf{M}}$ and Vals_X, we need an axiom connecting them. Such an axiom is (I15) and it serves one purpose only: in order to prove a termoidal analogue of the Lemma of the substitutions (11T).

In the termal case the functors $[\,.\,]$ and $|\,.\,|$ form an adjunction. In the termoidal case this is not true. What remains true is stated by (I17) and (I18). In non-functional form, these axioms say that $\tau[\![\mathrm{Nam}_X]\!]^{[\![X]\!]} = \tau$ for any termoid τ and $(\ulcorner\mu\urcorner)^{\mathcal{P}\mathbf{M}} = \{\mu\}$ for any $\mu \in |\mathbf{M}|$.

L) **Definition.** The structure $\mathbf{M}$ is called *structure of terms* if $\partial\mathbf{M} = \partial[X]$ for some Sort-indexed set X.

Notice that both $[Z]$ and $\partial[Z]$ are structures of terms for any Sort-indexed set Z.

M) **Lemma.** *If $\mathbf{M}$ is a structure of terms, then $\mathrm{Val}_{\mathbf{M}}$ maps only to one-element sets.*

Proof. Suppose that $\partial\mathbf{M} = \partial[Z]$ for some Sort-indexed set Z. From (I11) it follows that the algebraic components of $\mathrm{Val}_{\mathbf{M}}$ and $\mathrm{Val}_{\partial\mathbf{M}}$ are identical and the algebraic components of $\mathrm{Val}_{[Z]}$ and $\mathrm{Val}_{\partial[Z]}$ also are identical. But $\partial\mathbf{M} = \partial[Z]$, hence the algebraic components of $\mathrm{Val}_{\mathbf{M}}$ and $\mathrm{Val}_{[Z]}$ are identical. From this and the second part of (I12) it follows that the algebraic components of $\mathrm{Val}_{\mathbf{M}}$ map only to one-element sets. It remains to see that the logical component also maps to one-element sets. This follows from (F). ∎

N) According to (M), for any structure of terms $\mathbf{M}$, the quasimorphism $\mathrm{Val}_{\mathbf{M}}$ maps only to one-element sets. Define the Sort-indexed function $\mathrm{Valt}_{\mathbf{M}} : |[\![|\mathbf{M}|]\!]| \to |\mathbf{M}|$, such that for any τ we have $\{\mathrm{Valt}_{\mathbf{M}}\,\tau\} = \mathrm{Val}_{\mathbf{M}}\,\tau$.

O) **Proposition.** (1) $\mathrm{Valt}_{\mathbf{M}} : [\![|\mathbf{M}|]\!] \to \mathbf{M}$ *is a homomorphism for any structure of terms* $\mathbf{M}$.

(2) $h \circ \mathrm{Valt}_{\mathbf{M}} = \mathrm{Valt}_{\mathbf{K}} \circ [\![h]\!]$ *for any structures of terms* $\mathbf{M}$ *and* $\mathbf{K}$ *and*

homomorphism $h : \mathbf{M} \to \mathbf{K}$.[38]

$$\begin{array}{ccc} [\![|\mathbf{M}|]\!] & \xrightarrow{\mathrm{Valt}_{\mathbf{M}}} & \mathbf{M} \\ {\scriptstyle [\![h]\!]}\downarrow & & \downarrow{\scriptstyle h} \\ [\![|\mathbf{K}|]\!] & \xrightarrow{\mathrm{Valt}_{\mathbf{K}}} & \mathbf{K} \end{array}$$

(3) $\mathrm{Valt}_{\mathbf{M}} \circ [\![\mathrm{Valt}_{\mathbf{M}}]\!] = \mathrm{Valt}_{\mathbf{M}} \circ \mathrm{Vals}_{|\mathbf{M}|}$, *for any structure of terms* $\mathbf{M}$.

$$\begin{array}{ccc} [\![|[\![|\mathbf{M}|]\!]|]\!] & \xrightarrow{\mathrm{Vals}_{|\mathbf{M}|}} & [\![|\mathbf{M}|]\!] \\ {\scriptstyle [\![\mathrm{Valt}_{\mathbf{M}}]\!]}\downarrow & & \downarrow{\scriptstyle \mathrm{Valt}_{\mathbf{M}}} \\ [\![|\mathbf{M}|]\!] & \xrightarrow[\mathrm{Valt}_{\mathbf{M}}]{} & \mathbf{M} \end{array}$$

(4) *For any structure of terms* $\mathbf{M}$*, the algebraic components of* $(\mathrm{Valt}_{\mathbf{M}} \circ \mathrm{Nam}_{|\mathbf{M}|})$ *are identities.*

$$|\mathbf{M}|^{\circ} \xrightarrow{\mathrm{Nam}_{|\mathbf{M}|}} |[\![|\mathbf{M}|]\!]| \xrightarrow{\mathrm{Valt}_{\mathbf{M}}} |\mathbf{M}|$$

<u>Proof.</u> **(1)** Let $\mathtt{d}$ be an arbitrary operation symbol. Then

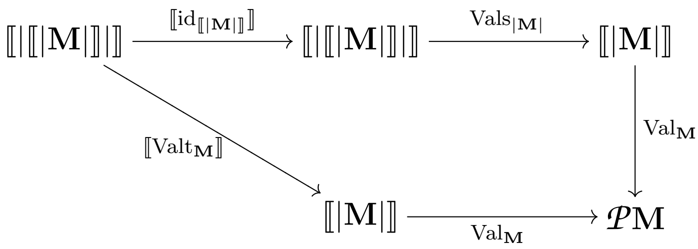
$$\begin{aligned} \{\mathrm{Valt}_{\mathbf{M}}(\mathtt{d}^{[\![|\mathbf{M}|]\!]}\langle \tau_1, \ldots, \tau_n\rangle)\} &= \mathrm{Val}_{\mathbf{M}}(\mathtt{d}^{[\![|\mathbf{M}|]\!]}\langle \tau_1, \ldots, \tau_n\rangle) && \text{from } (\mathsf{N}) \\ &\supseteq \mathtt{d}^{\mathcal{P}\mathbf{M}}\langle \mathrm{Val}_{\mathbf{M}}\,\tau_1, \ldots, \mathrm{Val}_{\mathbf{M}}\,\tau_n\rangle && \text{from } (\mathsf{C}) \\ &= \mathtt{d}^{\mathcal{P}\mathbf{M}}\langle \{\mathrm{Valt}_{\mathbf{M}}\,\tau_1\}, \ldots, \{\mathrm{Valt}_{\mathbf{M}}\,\tau_n\}\rangle && \text{from } (\mathsf{N}) \\ &= \{\mathtt{d}^{\mathbf{M}}\langle \mathrm{Valt}_{\mathbf{M}}\,\tau_1, \ldots, \mathrm{Valt}_{\mathbf{M}}\,\tau_n\rangle\} \end{aligned}$$

Consequently, $\mathrm{Valt}_{\mathbf{M}}(\mathtt{d}^{[\![|\mathbf{M}|]\!]}\langle \tau_1, \ldots, \tau_n\rangle) = \mathtt{d}^{\mathbf{M}}\langle \mathrm{Valt}_{\mathbf{M}}\,\tau_1, \ldots, \mathrm{Valt}_{\mathbf{M}}\,\tau_n\rangle$.

(2) Let τ be an arbitrary element of a carrier of $[\![|\mathbf{M}|]\!]$. Then from (N) and $(\mathsf{I}13)$ it follows that $\{h(\mathrm{Valt}_{\mathbf{M}}\,\tau)\} = h^{\mathcal{P}}\{\mathrm{Valt}_{\mathbf{M}}\,\tau\} = h^{\mathcal{P}}(\mathrm{Val}_{\mathbf{M}}\,\tau) = (h^{\mathcal{P}} \circ \mathrm{Val}_{\mathbf{M}})\tau \subseteq (\mathrm{Val}_{\mathbf{K}} \circ [\![h]\!])\tau = \mathrm{Val}_{\mathbf{K}}(\tau[\![h]\!]) = \{\mathrm{Valt}_{\mathbf{K}}(\tau[\![h]\!])\}$. Consequently, $h(\mathrm{Valt}_{\mathbf{M}}\,\tau) = \mathrm{Valt}_{\mathbf{K}}(\tau[\![h]\!])$.

[38] This equality implies that $\mathrm{Valt}_{\mathbf{M}}$ is a natural transformation from the functor composition $[\![|\,.\,|]\!]$ to the identity functor.

(3)



From definition (N) it follows that $\mathrm{Valt}_{\mathbf{M}} \ll \mathrm{Val}_{\mathbf{M}}$, hence $\mathrm{Valt}_{\mathbf{M}} \ll \mathrm{Val}_{\mathbf{M}} \circ \mathrm{id}_{\llbracket|\mathbf{M}|\rrbracket}$. From this and (I15),[39] we obtain that $\mathrm{Val}_{\mathbf{M}} \circ \llbracket \mathrm{Valt}_{\mathbf{M}} \rrbracket \leq \mathrm{Val}_{\mathbf{M}} \circ \mathrm{Vals}_{|\mathbf{M}|} \circ \llbracket \mathrm{id}_{\llbracket|\mathbf{M}|\rrbracket} \rrbracket$. But $\llbracket \mathrm{id}_{\llbracket|\mathbf{M}|\rrbracket} \rrbracket = \mathrm{id}_{\llbracket|\llbracket|\mathbf{M}|\rrbracket|\rrbracket}$, hence $\mathrm{Val}_{\mathbf{M}} \circ \llbracket \mathrm{Valt}_{\mathbf{M}} \rrbracket \leq \mathrm{Val}_{\mathbf{M}} \circ \mathrm{Vals}_{|\mathbf{M}|}$. Considering that $\mathrm{Val}_{\mathbf{M}}\, \tau = \{\mathrm{Valt}_{\mathbf{M}}\, \tau\}$ for any τ, this gives us $\mathrm{Valt}_{\mathbf{M}} \circ \llbracket \mathrm{Valt}_{\mathbf{M}} \rrbracket = \mathrm{Valt}_{\mathbf{M}} \circ \mathrm{Vals}_{|\mathbf{M}|}$.

(4) Let μ be an arbitrary element of an algebraic carrier of $\mathbf{M}$. Then from (I18) it follows that $\{(\mathrm{Valt}_{\mathbf{M}} \circ \mathrm{Nam}_{|\mathbf{M}|})\mu\} = \{\mathrm{Valt}_{\mathbf{M}}(\mathrm{Nam}_{|\mathbf{M}|}\, \mu)\} = \mathrm{Val}_{\mathbf{M}}(\mathrm{Nam}_{|\mathbf{M}|}\, \mu) = (\mathrm{Val}_{\mathbf{M}} \circ \mathrm{Nam}_{|\mathbf{M}|})\mu = \{\mu\}$. Consequently, $(\mathrm{Valt}_{\mathbf{M}} \circ \mathrm{Nam}_{|\mathbf{M}|})\mu = \mu$. ∎

P) **Notation.** By analogy with termal expressions, it will be convenient to write $\ulcorner \xi \urcorner$ for $\mathrm{Nam}_X\, \xi$, $\tau^{\mathcal{P}\mathbf{M}}$ for $\mathrm{Val}_{\mathbf{M}}\, \tau$, $\tau^{\llbracket X \rrbracket}$ for $\mathrm{Vals}_X\, \tau$ and $\tau^{\mathbf{M}}$ for $\mathrm{Valt}_{\mathbf{M}}\, \tau$. Also by analogy, for the composition of the homomorphism $\llbracket h \rrbracket$ with these homomorphisms we are going to use $\llbracket h \rrbracket^{\mathcal{P}\mathbf{M}}$, $\llbracket h \rrbracket^{\llbracket X \rrbracket}$ and $\llbracket h \rrbracket^{\mathbf{M}}$ in postfix notation.

Q) **Remark.** According to (O), termoids behave much more nicely when we evaluate them in a structure of terms. If $\mathbf{M}$ is a structure of terms and τ is a termoid over $|\mathbf{M}|$, the value $\tau^{\mathbf{M}}$ is well defined and has nice properties. In particular, the `Sort`-indexed function $\mathrm{Valt}_{\mathbf{M}}$ is a natural transformation from the functor $\llbracket\, .\, \rrbracket$ to the identity functor of the category of the structures of terms.

R) **Proposition.** *Let the* `Sort`*-indexed set X be a subset of Y.*[40] *Then:*

(1) $|\llbracket X \rrbracket| \subseteq |\llbracket Y \rrbracket|$.

(2) *Any termoidal expression of sort κ over X is a termoidal expression of sort κ over Y.*

(3) *If τ is a termoidal expression over $|\llbracket X \rrbracket|$, then τ is a termoidal expression over $|\llbracket Y \rrbracket|$ and $\tau^{\llbracket X \rrbracket} = \tau^{\llbracket Y \rrbracket}$.*

Proof. **(1)** According to (I3), $|\llbracket X \rrbracket| = |\llbracket Y \cap X \rrbracket| = |\llbracket Y \rrbracket| \cap |\llbracket X \rrbracket| \subseteq |\llbracket Y \rrbracket|$.

[39] In (I15) we use $k = \mathrm{id}_{\llbracket|\mathbf{M}|\rrbracket}$ and $f = \mathrm{Valt}_{\mathbf{M}}$

[40] I.e. $X_\kappa \subseteq Y_\kappa$ for any sort κ. See (10C).

(2) is a reformulation of (1).

(3) From (1) it follows that $|[\![X]\!]| \subseteq |[\![Y]\!]|$, hence (2) implies that τ is a termoidal expression over $|[\![Y]\!]|$. Let i be the identity inclusion map $i : X \to Y$, i.e. $i\eta = \eta$ for any $\eta \in X$. From (I4) it follows that $\sigma[\![i]\!] = \sigma$ for any $\sigma \in |[\![X]\!]|$, hence also by (I4), $\sigma[\![[\![i]\!]]\!] = \sigma$ for any $\sigma \in |[\![|[\![X]\!]|]\!]|$. But according to (I16), $(\tau^{[\![X]\!]})[\![i]\!] = (\tau[\![[\![i]\!]]\!])^{[\![Y]\!]}$, hence $\tau^{[\![X]\!]} = \tau^{[\![Y]\!]}$. ∎

S) **Definition.** (1) A *termoidal substitution* is a `Sort`-indexed function $s : X \to |[\![Y]\!]|$, where X and Y are `Sort`-indexed sets.

(2) Given a substitution $s : X \to |[\![Y]\!]|$, $\tau[\![s]\!]^{[\![Y]\!]}$ will be called *application* of the substitution s to the termoidal expression τ.

T) **Lemma** ***(of the substitutions, for terminators).*** (1) *Given a termoidal substitution $s : X \to |[\![Y]\!]|$ and assignment functions $v : Y \to |\mathbf{K}|$ and $w : X \to |\mathbf{K}|$, if $w\xi \in (s\xi)[\![v]\!]^{\mathcal{P}\mathbf{K}}$ for any $\xi \in X$, then for any termoidal expression τ over X, $\tau[\![w]\!]^{\mathcal{P}\mathbf{K}} \subseteq (\tau[\![s]\!]^{[\![Y]\!]})[\![v]\!]^{\mathcal{P}\mathbf{K}}$.*

(2) *Given a structure of terms $\mathbf{K}$, a termoidal substitution $s : X \to |[\![Y]\!]|$ and an assignment function $v : Y \to |\mathbf{K}|$, define a new termal substitution $w : X \to |\mathbf{K}|$, such that $w\xi = (s\xi)[\![v]\!]^{\mathbf{K}}$, i.e. $w = [\![v]\!]^{\mathbf{K}} \circ s$. Then for any termoidal expression τ over X, $\tau[\![w]\!]^{\mathbf{K}} = (\tau[\![s]\!]^{[\![Y]\!]})[\![v]\!]^{\mathbf{K}}$, i.e. $\tau[\![[\![v]\!]^{\mathbf{K}} \circ s]\!]^{\mathbf{K}} = (\tau[\![s]\!]^{[\![Y]\!]})[\![v]\!]^{\mathbf{K}}$.*

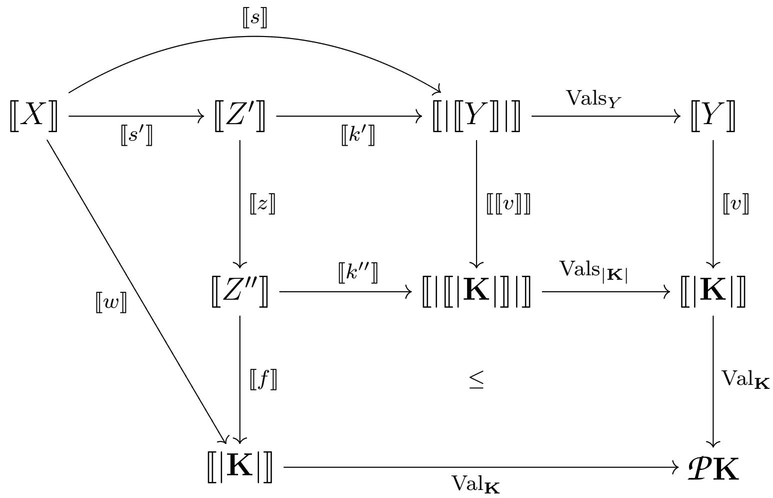
Proof. **(1)** Let $Z' = \{Z'_\kappa\}_{\kappa \in \mathtt{Sort}}$ be the `Sort`-indexed set, such that $Z'_\kappa = \{\langle \xi, s\xi \rangle : \xi \in X_\kappa\}$ for any sort κ and $Z'' = \{Z''_\kappa\}_{\kappa \in \mathtt{Sort}}$ be the `Sort`-indexed set, such that $Z''_\kappa = \{\langle \xi, (s\xi)[\![v]\!] \rangle : \xi \in X_\kappa\}$ for any sort κ. Obviously $Z' \subseteq X \times |[\![Y]\!]|$ and $Z'' \subseteq X \times |[\![|\mathbf{K}|]\!]|$.[41]

Let $s' : X \to Z'$ be the `Sort`-indexed function, such that $s'\xi = \langle \xi, s\xi \rangle$ for any $\xi \in X$, let $z : Z' \to Z''$ be the `Sort`-indexed function, such that $z\langle \xi, \tau \rangle = \langle \xi, \tau[\![v]\!] \rangle$ for any $\langle \xi, \tau \rangle \in Z'$ and let $f : Z'' \to |\mathbf{K}|$ be the `Sort`-indexed function, such that $f\langle \xi, \tau \rangle = w\xi$ for any $\langle \xi, \tau \rangle \in Z''$.

Furthermore, let $k' : Z' \to |[\![Y]\!]|$ be the right projection, i.e. $k'\langle \xi, \tau \rangle = \tau$ for any $\langle \xi, \tau \rangle \in Z'$ and let $k'' : Z'' \to |[\![|\mathbf{K}|]\!]|$ be the right projection as well, i.e. $k''\langle \xi, \tau \rangle = \tau$ for any $\langle \xi, \tau \rangle \in Z''$.

[41] Recall that for any two `Sort`-indexed sets $X = \{X_i\}_{i \in \mathtt{Sort}}$ and $Y = \{Y_i\}_{i \in \mathtt{Sort}}$, $X \times Y$ is the `Sort`-indexed set $\{X_i \times Y_i\}_{i \in \mathtt{Sort}}$.

Consider the following diagram:



The definitions of s' and k' imply $s = k' \circ s'$, hence the segment on top of the diagram commutes.

For any $\xi \in X$ we have $(f \circ z \circ s')\xi = f(z(s'\xi)) = f(z\langle \xi, s\xi \rangle) = f\langle \xi, (s\xi)[\![v]\!] \rangle = w\xi$, hence the triangle in this diagram also commutes.

The top left square commutes because the definitions of z, k' and k'' imply $[\![v]\!] \circ k' = k'' \circ z$.

The top right square commutes because (I16) implies $[\![v]\!] \circ \mathrm{Vals}_Y = \mathrm{Vals}_{|\mathbf{K}|} \circ [\![[\![v]\!]]\!]$.

It only remains to consider the rectangle in this diagram. By definition, all elements of Z'' are of the form $\langle \xi, (s\xi)[\![v]\!] \rangle$, where $\xi \in X$. On one hand, for any such element we have $f\langle \xi, (s\xi)[\![v]\!] \rangle = w\xi$. On the other hand, for the same element we have $(\mathrm{Val}_{\mathbf{K}} \circ k'')\langle \xi, (s\xi)[\![v]\!] \rangle = \mathrm{Val}_{\mathbf{K}}((s\xi)[\![v]\!]) = (s\xi)[\![v]\!]^{\mathcal{P}\mathbf{K}}$. But by the condition of the Lemma, $w\xi \in (s\xi)[\![v]\!]^{\mathcal{P}\mathbf{K}}$ for any $\xi \in X$, hence $f \ll \mathrm{Val}_{\mathbf{K}} \circ k''$. Now we are ready to apply (I15). We obtain that $\mathrm{Val}_{\mathbf{K}} \circ [\![f]\!] \leq \mathrm{Val}_{\mathbf{K}} \circ \mathrm{Vals}_{|\mathbf{K}|} \circ [\![k'']\!]$.

This inequality together with the commutativity of the rest of the diagram implies that $\mathrm{Val}_{\mathbf{K}} \circ [\![w]\!] \leq (\mathrm{Val}_{\mathbf{K}} \circ [\![v]\!]) \circ (\mathrm{Vals}_Y \circ [\![s]\!])$, hence $\tau[\![w]\!]^{\mathcal{P}\mathbf{K}} \subseteq (\tau[\![s]\!]^{[\![Y]\!]})[\![v]\!]^{\mathcal{P}\mathbf{K}}$ for any $\tau \in |[\![X]\!]|$.

(2) Since $w\xi = (s\xi)[\![v]\!]^{\mathbf{K}}$, definition (N) implies that $w\xi \in \{(s\xi)[\![v]\!]^{\mathbf{K}}\} = (s\xi)[\![v]\!]^{\mathcal{P}\mathbf{K}}$, hence we are permitted to apply (1) and from (1) we obtain that $\tau[\![w]\!]^{\mathcal{P}\mathbf{K}} \subseteq (\tau[\![s]\!]^{[\![Y]\!]})[\![v]\!]^{\mathcal{P}\mathbf{K}}$. But by definition (N), $\tau[\![w]\!]^{\mathcal{P}\mathbf{K}} = \{\tau[\![w]\!]^{\mathbf{K}}\}$ and $(\tau[\![s]\!]^{[\![Y]\!]})[\![v]\!]^{\mathcal{P}\mathbf{K}} = \{(\tau[\![s]\!]^{[\![Y]\!]})[\![v]\!]^{\mathbf{K}}\}$, hence $\tau[\![w]\!]^{\mathbf{K}} = (\tau[\![s]\!]^{[\![Y]\!]})[\![v]\!]^{\mathbf{K}}$. ∎

U) Notice that because of (I5), if $\mathbf{M}$ and $\mathbf{K}$ are algebraically equivalent, then the notions of termoid or formuloid over $|\mathbf{M}|$ are equivalent to the notions termoid or formuloid over $|\mathbf{K}|$. In particular, an object is termoid or formuloid over $|\mathbf{M}|$ if and only if it is termoid or formuloid over $|\partial\mathbf{M}|$.

V) **Proposition.** $(\int_{\mathbf{M}})^{\mathcal{P}} \circ \mathrm{Val}_{\partial\mathbf{M}} = \mathrm{Val}_{\mathbf{M}}$ *for any structure* $\mathbf{M}$.

Proof. By definition (12L) the algebraic components of $\int_{\mathbf{M}}$ are identity, so the equality is true for the algebraic components due to (I11). It only remains to apply (H). ∎

W) **Corollary.** (1) *Given a structure* $\mathbf{M}$ *and a termoid* τ *over* $|\mathbf{M}|$, $\tau^{\mathcal{P}\mathbf{M}} = \tau^{\mathcal{P}(\partial\mathbf{M})}$.

(2) *Given a structure* $\mathbf{M}$ *and a formuloid* φ *over* $|\mathbf{M}|$, $\mathrm{Val}_{\mathbf{M}}\,\varphi = \{\mathrm{val}_{\mathbf{M}}\,\psi : \psi \in \mathrm{Val}_{\partial\mathbf{M}}\,\varphi\}$. *In other words,* $\varphi^{\mathcal{P}\mathbf{M}} = \{\psi^{\mathbf{M}} : \psi \in \varphi^{\mathcal{P}(\partial\mathbf{M})}\}$.

(3) *Given a structure* $\mathbf{M}$ *of terms and a termoid* τ *over* $|\mathbf{M}|$, $\tau^{\mathbf{M}} = \tau^{\partial\mathbf{M}}$. *In other words,* $\mathrm{Valt}_{\mathbf{M}}\,\tau = \mathrm{Valt}_{\partial\mathbf{M}}\,\tau$.

(4) *Given a structure* $\mathbf{M}$ *of terms and a formuloid* φ *over* $|\mathbf{M}|$, $\varphi^{\mathbf{M}} = (\varphi^{\partial\mathbf{M}})^{\mathbf{M}}$. *In other words,* $\mathrm{Valt}_{\mathbf{M}}\,\varphi = \mathrm{val}_{\mathbf{M}}(\mathrm{Valt}_{\partial\mathbf{M}}\,\varphi)$.

Proof. $\int_{\mathbf{M}}$ is identity over the algebraic carriers so **(1)** follows from (V). Alternatively, one can obtain (1) as a corollary from (I11).

(2) follows easily from (V) as well, taking into account that for any formula ψ over $|\mathbf{M}|$, $\int_{\mathbf{M}}\psi = \psi^{\mathbf{M}} = \mathrm{val}_{\mathbf{M}}\,\psi$ (see definition 12L).

(3) follows from (1) because $\tau^{\mathcal{P}\mathbf{M}} = \{\tau^{\mathbf{M}}\}$ and $\tau^{\mathcal{P}(\partial\mathbf{M})} = \{\tau^{\partial\mathbf{M}}\}$. We only have to notice that both $\mathrm{Valt}_{\partial\mathbf{M}}$ and $\tau^{\partial\mathbf{M}}$ are well defined since $\mathbf{M}$ is a structure of terms, so $\partial\mathbf{M}$ is a structure of terms as well.

(4) follows from (2) because $\varphi^{\mathcal{P}\mathbf{M}} = \{\varphi^{\mathbf{M}}\}$ and $\varphi^{\mathcal{P}(\partial\mathbf{M})} = \{\varphi^{\partial\mathbf{M}}\}$. We only have to notice that both $\mathrm{Valt}_{\partial\mathbf{M}}$ and $\varphi^{\partial\mathbf{M}}$ are well defined since $\mathbf{M}$ is a structure of terms, so $\partial\mathbf{M}$ is a structure of terms as well. ∎

The following definition is analogous to the corresponding definition for formulae.

X) **Definition.** (1) A formuloid φ over $|\mathbf{M}|$ φ is *true* in the logical structure $\mathbf{M}$, if $\varphi^{\mathcal{P}\mathbf{M}} = \{1\}$.

(2) A formuloid φ over X is *universally valid* in a logical structure $\mathbf{M}$, if for any assignment function $v : X \to |\mathbf{M}|$, $\varphi[\![v]\!]$ is true in $\mathbf{M}$.

(3) A formuloid φ over $|\mathbf{A}|$ is *satisfiable* in an algebra $\mathbf{A}$, if it is true in some logical variant of $\mathbf{A}$.

(4) A formuloid is universally satisfiable if it is universally valid in some logical structure.

(5) A formuloid is universally satisfiable in an algebra $\mathbf{A}$, if it is universally valid in some logical variant of $\mathbf{A}$.

(6) Given an algebra $\mathbf{A}$, a set of formuloids over $|\mathbf{A}|$ is *satisfiable* in $\mathbf{A}$,

if there exists a logical variant $\mathbf{M}$ of $\mathbf{A}$, such that all formuloids from the set are true in $\mathbf{M}$.

(7) A set of formuloids is *universally satisfiable* if there exists a logical structure $\mathbf{M}$, such that all formuloids from the set are universally valid in $\mathbf{M}$.

(8) A set of formuloids is *universally satisfiable in an algebra* $\mathbf{A}$, if there exists a logical variant $\mathbf{M}$ of $\mathbf{A}$, such that all formuloids from the set are universally valid in $\mathbf{M}$.

(9) The formuloid φ *follows* in the algebra $\mathbf{A}$ from the set of formuloids Γ, if for any logical variant $\mathbf{M}$ of $\mathbf{A}$, such that all formuloids from Γ are true in $\mathbf{M}$, the formuloid φ also is true in $\mathbf{M}$.

(10) The formuloid φ *universally follows* in the algebra $\mathbf{A}$ from the set of formuloids Γ, if for any logical variant $\mathbf{M}$ of $\mathbf{A}$, such that all formuloids from Γ are universally valid in $\mathbf{M}$, the formuloid φ also is universally valid in $\mathbf{M}$.

(11) The formuloid φ *universally follows* from the set of formuloids Γ, if for any logical structure $\mathbf{M}$, such that all formuloids from Γ are universally valid in $\mathbf{M}$, the formuloid φ also is universally valid in $\mathbf{M}$.

§15. THE ALPHA-TERMINATOR

A) In this section we will see that the terms can be used in order to construct a terminator. This terminator will be called *alpha-terminator*.

Although our original intent for the alpha-terminator might be to define $[\![X]\!] = [X]$ we are not permitted to do so because axiom (14I1) requires $[\![X]\!]$ to be an algebra and $[X]$ is not an algebra. So, instead, we are going to define $[\![X]\!] = \partial[X]$. This definition implies that alpha-termoids and terms will be one and the same thing. There will be, however, two important differences between the alpha-termoidal expressions of logical sort and the termal expressions of logical sort. First, by definition the elements of the logical carrier of any algebra are formulae, so they are not permitted to contain names of logical sort (see 12C2). Consequently, the termal expressions of logical sort that are not formulae do not have an analogue among the alpha-termoidal expressions of logical sort. Second, according to (14J), alpha-formuloid over X is the same thing as relational formula over $|\partial[X]|$. Since $[X]$ and $\partial[X]$ are algebraically equivalent, alpha-formuloid over X is the same thing as relational formula over $|[X]|$. For example, $\mathtt{p}(\ulcorner\mathtt{c}\urcorner, \ulcorner\mathtt{f(x)}\urcorner)$ will be the the alpha-formuloid corresponding to the formula $\mathtt{p}(\mathtt{c}, \mathtt{f}(\mathtt{x}))$.

B) **Definition.** The *alpha-terminator* is terminator, such that:

(1) $[\![X]\!] = \partial[X]$ and $[\![f]\!] = \partial[f]$.

(2) $(\mathrm{Nam}_X)_\kappa\xi = (\mathrm{nam}_X)_\kappa\xi$ for all algebraic sorts κ.

(3) Any termoid over $|\mathbf{M}|$ is a term over $|\mathbf{M}|$, so we are permitted to define $\mathrm{Val}_\mathbf{M}\,\tau = \{\mathrm{val}_\mathbf{M}\,\tau\}$ for any termoid τ over $|\mathbf{M}|$. This uniquely determines the algebraic components of the homomorphism $\mathrm{Val}_\mathbf{M}$.

(4) For any termoid τ over $|[\![X]\!]|$ let $\mathrm{Vals}_X\,\tau = \mathrm{val}_{[X]}\,\tau$. This uniquely determines the algebraic components of the homomorphism Vals_X.

Because of (12Q2), we do not have to specify the logical components of $\mathrm{Val}_\mathbf{M}$ and Vals_X.

C) In order to avoid ambiguities, if necessary, I am going to use notation such as $[\![X]\!]_\alpha$ instead of $[\![X]\!]$, Nam^α_X instead of Nam_X, $\mathrm{Val}^\alpha_\mathbf{M}$ instead of $\mathrm{Val}_\mathbf{M}$, etc. For the same reason, if necessary I am going to use expressions such as "alpha-termoids", "alpha-formuloids", etc.

D) **Observation.** *The notations $\tau^{[\![X]\!]}$, $\tau^\mathbf{M}$ and $\tau^{\mathcal{P}\mathbf{M}}$ are unambiguous whether we are regarding τ as a term, or as an alpha-termoid.*

Proof. **($\tau^{[\![X]\!]}$)** If we regard τ as termoid, then $\tau^{[\![X]\!]} = \mathrm{Vals}_X\,\tau$ and if we regard it as term, then $\tau^{[\![X]\!]} = \tau^{\partial[X]} = \tau^{[X]} = \mathrm{val}_{[X]}\,\tau$. But according to the definition of the alpha-terminator, $\mathrm{Vals}_X\,\tau = \mathrm{val}_{[X]}\,\tau$.

($\tau^\mathbf{M}$) Unless $\mathbf{M}$ is a structure of terms, the meaning of $\tau^\mathbf{M}$ is not defined when we regard τ as a termoid.

Suppose $\mathbf{M}$ is a structure of terms and τ is a termoid (and term) over $|\mathbf{M}|$. If we regard τ as a termoid, then $\{\tau^\mathbf{M}\} = \tau^{\mathcal{P}\mathbf{M}} = \mathrm{Val}_\mathbf{M}\,\tau = \{\mathrm{val}_\mathbf{M}\,\tau\}$ and if we regard it as a term, then $\{\tau^\mathbf{M}\} = \{\mathrm{val}_\mathbf{M}\,\tau\}$.

($\tau^{\mathcal{P}\mathbf{M}}$) The notation $\tau^{\mathcal{P}\mathbf{M}}$ will not be correct when we regard τ as a term if τ contains names, because the names in τ are names for $|\mathbf{M}|$ and not names for $|\mathcal{P}\mathbf{M}|$.

Suppose τ is a termoid (and term) without names and let $\{\}_\mathbf{M} : \mathbf{M} \to \mathcal{P}\mathbf{M}$ be the homomorphism defined in (10N). If we regard τ as termoid, then $\tau^{\mathcal{P}\mathbf{M}} = \mathrm{Val}_\mathbf{M}\,\tau = \{\mathrm{val}_\mathbf{M}\,\tau\}$ and if we regard it as term, then $\tau[\{\}_\mathbf{M}] = \tau$ (because τ contains no names), so according to (11N), $\tau^{\mathcal{P}\mathbf{M}} = (\tau[\{\}_\mathbf{M}])^{\mathcal{P}\mathbf{M}} = (\mathrm{val}_{\mathcal{P}\mathbf{M}} \circ [\{\}_\mathbf{M}])\tau = (\{\}_\mathbf{M} \circ \mathrm{val}_\mathbf{M})\tau = \{\mathrm{val}_\mathbf{M}\,\tau\}$. ■

E) We haven't proved yet that the alpha-terminator is indeed a terminator, i.e. that (B) satisfies all axioms of (14I).

Proof. **(1)** $[\![X]\!] = \partial[X]$ is an algebra by definition.

(2) $[\![f]\!] = \partial[f]$ is a homomorphism from $[\![X]\!] = \partial[X]$ to $[\![Y]\!] = \partial[Y]$ by definition.

(3) and **(4)** immediately follow from the definitions and **(5)** follows from (12T).

(6) $[\![\mathrm{id}_X]\!] = \partial[\mathrm{id}_X] = \partial\mathrm{id}_{[X]} = \mathrm{id}_{\partial[X]} = \mathrm{id}_{[\![X]\!]}$.

(7) $[\![f \circ g]\!] = \partial[f \circ g] = \partial([f] \circ [g]) = \partial[f] \circ \partial[g] = [\![f]\!] \circ [\![g]\!]$.

(8) Nam_X has to be a `Sort`-indexed function from X° to $|[\![X]\!]|$. From $[\![X]\!] = \partial[X]$ it follows that (B2) defines correctly $(\mathrm{Nam}_X)_\kappa$ for all algebraic sorts κ. On the other hand, there is no need to define $(\mathrm{Nam}_X)_{\mathtt{Log}}$ because $(X^\circ)_{\mathtt{Log}} = \varnothing$.

(9) We are going to use (12U).

$$\begin{aligned}
([\![f]\!] \circ \mathrm{Nam}_X)^\circ &= ([\![f]\!])^\circ \circ (\mathrm{Nam}_X)^\circ && \text{from (12S)}\\
&= (\partial[f])^\circ \circ (\mathrm{nam}_X)^\circ && \text{from (B1) and (B2)}\\
&= ([f])^\circ \circ (\mathrm{nam}_X)^\circ &&\\
&= ([f] \circ \mathrm{nam}_X)^\circ && \text{from (12S)}\\
&= (\mathrm{nam}_Y \circ f)^\circ && \text{from (11I)}\\
&= (\mathrm{nam}_Y)^\circ \circ f^\circ && \text{from (12S)}\\
&= (\mathrm{Nam}_Y)^\circ \circ (f^\circ)^\circ && \text{from (B2) and (12S)}\\
&= (\mathrm{Nam}_Y \circ f^\circ)^\circ && \text{from (12S)}
\end{aligned}$$

(10) For any functional symbol `f` we have

$$\begin{aligned}
\mathrm{Val}_{\mathbf{M}}(\mathtt{f}^{[\![|\mathbf{M}|]\!]}\langle\tau_1, \ldots, \tau_n\rangle) &= \{\mathrm{val}_{\mathbf{M}}(\mathtt{f}^{\partial[|\mathbf{M}|]}\langle\tau_1, \ldots, \tau_n\rangle)\} && \text{from (B3)}\\
&= \{\mathrm{val}_{\mathbf{M}}(\mathtt{f}^{[|\mathbf{M}|]}\langle\tau_1, \ldots, \tau_n\rangle)\} && \text{from (12C3)}\\
&= \{\mathtt{f}^{\mathbf{M}}\langle\mathrm{val}_{\mathbf{M}}\,\tau_1, \ldots, \mathrm{val}_{\mathbf{M}}\,\tau_n\rangle\} &&\\
&= \mathtt{f}^{\mathcal{P}\mathbf{M}}\langle\{\mathrm{val}_{\mathbf{M}}\,\tau_1\}, \ldots, \{\mathrm{val}_{\mathbf{M}}\,\tau_n\}\rangle &&\\
&= \mathtt{f}^{\mathcal{P}\mathbf{M}}\langle\mathrm{Val}_{\mathbf{M}}\,\tau_1, \ldots, \mathrm{Val}_{\mathbf{M}}\,\tau_n\rangle &&
\end{aligned}$$

Because of (12Q2) we don't have to perform analogous checks for the predicate and the logical symbols.

(11) Follows from definition (B3), since $\mathrm{val}_{\mathbf{M}}\,\tau = \mathrm{val}_{\partial\mathbf{M}}\,\tau$ for any term τ (see 12P2).

(12) In this terminator $\mathrm{Val}_{\mathbf{M}}$ always maps to one-element sets, including the case when $\mathbf{M} = [X]$.

(13) We are going to use (12H2). For any structure $\mathbf{N}$, let $\{\}_{\mathbf{N}}$ be as defined in (10N).

$$\begin{aligned}
\partial(h^{\mathcal{P}} \circ \mathrm{Val}_{\mathbf{M}}) &= \partial(h^{\mathcal{P}}) \circ \partial(\mathrm{Val}_{\mathbf{M}}) && \text{from (12K8)}\\
&= \partial(h^{\mathcal{P}}) \circ \partial(\{\}_{\mathbf{M}} \circ \mathrm{val}_{\mathbf{M}}) && \text{from (B3)}\\
&= \partial(h^{\mathcal{P}} \circ \{\}_{\mathbf{M}} \circ \mathrm{val}_{\mathbf{M}}) && \text{from (12K8)}\\
&= \partial(\{\}_{\mathbf{K}} \circ h \circ \mathrm{val}_{\mathbf{M}}) &&
\end{aligned}$$

$$
\begin{aligned}
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}} \circ [h]) && \text{from (11N)}\\
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}}) \circ \partial[h] && \text{from (12K8)}\\
&= \partial(\mathrm{Val}_{\mathbf{K}}) \circ \partial(\partial[h]) && \text{from (B3) and (12K7)}\\
&= \partial(\mathrm{Val}_{\mathbf{K}} \circ \partial[h]) && \text{from (12K8)}\\
&= \partial(\mathrm{Val}_{\mathbf{K}} \circ [\![h]\!]) && \text{from (B1)}
\end{aligned}
$$

(14) For any functional symbol $\mathtt{f}$ we have

$$
\begin{aligned}
\mathrm{Vals}_X(\mathtt{f}^{[\![|[\![X]\!]|]\!]}\langle\tau_1,\dots,\tau_n\rangle) &= \mathrm{Vals}_X(\mathtt{f}^{\partial[|\partial[X]|]}\langle\tau_1,\dots,\tau_n\rangle) && \text{from (B1)}\\
&= \mathrm{Vals}_X(\mathtt{f}^{\partial[|[X]|]}\langle\tau_1,\dots,\tau_n\rangle) && \text{from (12T)}\\
&= \mathrm{Vals}_X(\mathtt{f}^{[|[X]|]}\langle\tau_1,\dots,\tau_n\rangle)\\
&= \mathrm{val}_{[X]}(\mathtt{f}^{[|[X]|]}\langle\tau_1,\dots,\tau_n\rangle) && \text{from (B4)}\\
&= \mathtt{f}^{[X]}\langle\mathrm{val}_{[X]}\,\tau_1,\dots,\mathrm{val}_{[X]}\,\tau_n\rangle\\
&= \mathtt{f}^{[X]}\langle\mathrm{Vals}_X\,\tau_1,\dots,\mathrm{Vals}_X\,\tau_n\rangle && \text{from (B4)}\\
&= \mathtt{f}^{\partial[X]}\langle\mathrm{Vals}_X\,\tau_1,\dots,\mathrm{Vals}_X\,\tau_n\rangle\\
&= \mathtt{f}^{[\![X]\!]}\langle\mathrm{Vals}_X\,\tau_1,\dots,\mathrm{Vals}_X\,\tau_n\rangle && \text{from (B1)}
\end{aligned}
$$

Because of (12Q2) we don't have to perform analogous checks for the predicate and the logical symbols.

(15) Notice that $f \ll \mathrm{Val}_{\mathbf{K}} \circ k$,

$$
\begin{array}{ccc}
X & \xrightarrow{\;k\;} & |[\![|\mathbf{K}|]\!]|\\
\downarrow f & & \downarrow \mathrm{Val}_{\mathbf{K}}\\
|\mathbf{K}| & \in & |\mathcal{P}\mathbf{K}|
\end{array}
$$

implies that $f\xi \in \mathrm{Val}_{\mathbf{K}}(k\xi) = \{\mathrm{val}_{\mathbf{K}}(k\xi)\}$ for any $\xi \in X$, hence $f\xi = \mathrm{val}_{\mathbf{K}}(k\xi)$ for any $\xi \in X$, so $f^\circ = (\mathrm{val}_{\mathbf{K}} \circ k)^\circ$, hence from (12T) we can conclude that $\partial[f] = \partial[\mathrm{val}_{\mathbf{K}} \circ k]$.

In order to prove that $\mathrm{Val}_{\mathbf{K}} \circ [\![f]\!] \leq \mathrm{Val}_{\mathbf{K}} \circ \mathrm{Vals}_{|\mathbf{K}|} \circ [\![k]\!]$,

$$
\begin{array}{ccccc}
[\![X]\!] & \xrightarrow{\;[\![k]\!]\;} & [\![|[\![|\mathbf{K}|]\!]|]\!] & \xrightarrow{\;\mathrm{Vals}_{|\mathbf{K}|}\;} & [\![|\mathbf{K}|]\!]\\
 & \searrow [\![f]\!] & \leq & & \downarrow \mathrm{Val}_{\mathbf{K}}\\
 & & [\![|\mathbf{K}|]\!] & \xrightarrow{\;\mathrm{Val}_{\mathbf{K}}\;} & \mathcal{P}\mathbf{K}
\end{array}
$$

we are going to use (12H2). Let $\{\}_{\mathbf{K}}$ be as defined in (10N). Then

$$
\begin{aligned}
\partial(\mathrm{Val}_{\mathbf{K}} \circ [\![f]\!]) &= \partial(\mathrm{Val}_{\mathbf{K}}) \circ \partial[\![f]\!] && \text{from (12K8)}\\
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}}) \circ \partial(\partial[f]) && \text{from (B3) and (B1)}\\
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}}) \circ \partial[f] && \text{from (12K7)}\\
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}}) \circ \partial[\mathrm{val}_{\mathbf{K}} \circ k] && \text{see above}\\
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}}) \circ \partial[\mathrm{val}_{\mathbf{K}}] \circ \partial[k] && \text{from (12K8) and (11H)}\\
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}} \circ [\mathrm{val}_{\mathbf{K}}]) \circ \partial[k] && \text{from (12K8)}\\
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}} \circ [\mathrm{val}_{\mathbf{K}}]) \circ \partial(\partial[k]) && \text{from (12K7)}\\
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}} \circ [\mathrm{val}_{\mathbf{K}}]) \circ \partial[\![k]\!] && \text{from (B1)}\\
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}} \circ \mathrm{val}_{[|\mathbf{K}|]}) \circ \partial[\![k]\!] && \text{from (11N)}\\
&= \partial(\{\}_{\mathbf{K}} \circ \mathrm{val}_{\mathbf{K}}) \circ \partial(\mathrm{val}_{[|\mathbf{K}|]}) \circ \partial[\![k]\!] && \text{from (12K8)}\\
&= \partial(\mathrm{Val}_{\mathbf{K}}) \circ \partial(\mathrm{Vals}_{|\mathbf{K}|}) \circ \partial[\![k]\!] && \text{from (B3) and (B4)}\\
&= \partial(\mathrm{Val}_{\mathbf{K}} \circ \mathrm{Vals}_{|\mathbf{K}|} \circ [\![k]\!]) && \text{from (12K8)}
\end{aligned}
$$

(16) Notice that $[f]^\circ = (\partial[f])^\circ$, so (12T) implies $\partial[[f]] = \partial[\partial[f]]$. We are going to use (12H2) again.

$$
\begin{aligned}
\partial([\![f]\!] \circ \mathrm{Vals}_X) &= \partial[\![f]\!] \circ \partial(\mathrm{Vals}_X) && \text{from (12K8)}\\
&= \partial(\partial[f]) \circ \partial(\mathrm{val}_{[X]}) && \text{from (B1) and (B4)}\\
&= \partial[f] \circ \partial(\mathrm{val}_{[X]}) && \text{from (12K7)}\\
&= \partial([f] \circ \mathrm{val}_{[X]}) && \text{from (12K8)}\\
&= \partial(\mathrm{val}_{[Y]} \circ [[f]]) && \text{from (11N)}\\
&= \partial(\mathrm{val}_{[Y]}) \circ \partial[[f]] && \text{from (12K8)}\\
&= \partial(\mathrm{val}_{[Y]}) \circ \partial[\partial[f]] && \\
&= \partial(\mathrm{Vals}_Y) \circ \partial[[\![f]\!]] && \text{from (B4) and (B1)}\\
&= \partial(\mathrm{Vals}_Y) \circ \partial(\partial[[\![f]\!]]) && \text{from (12K7)}\\
&= \partial(\mathrm{Vals}_Y) \circ \partial[\![[\![f]\!]]\!] && \text{from (B1)}\\
&= \partial(\mathrm{Vals}_Y \circ [\![[\![f]\!]]\!]) && \text{from (12K8)}
\end{aligned}
$$

(17) We are going to use (12H2) again.

$$
\begin{aligned}
\partial(\mathrm{Vals}_X \circ [\![\mathrm{Nam}_X]\!]) &= \partial(\mathrm{Vals}_X) \circ \partial[\![\mathrm{Nam}_X]\!] && \text{from (12K8)}\\
&= \partial(\mathrm{val}_{[X]}) \circ \partial(\partial[\mathrm{Nam}_X]) && \text{from (B4) and (B1)}\\
&= \partial(\mathrm{val}_{[X]}) \circ \partial[\mathrm{Nam}_X] && \text{from (12K7)}\\
&= \partial(\mathrm{val}_{[X]}) \circ \partial[\mathrm{nam}_X] && \text{from (B2) and (12T)}\\
&= \partial(\mathrm{val}_{[X]} \circ [\mathrm{nam}_X]) && \text{from (12K8)}
\end{aligned}
$$

$$
\begin{aligned}
&= \partial(\mathrm{id}_{[X]}) && \text{from (11V1)}\\
&= \partial(\partial(\mathrm{id}_{[X]})) && \text{from (12K7)}\\
&= \partial(\mathrm{id}_{\partial[X]}) && \text{from (12K5)}\\
&= \partial(\mathrm{id}_{[\![X]\!]}) && \text{from (B1)}
\end{aligned}
$$

(18) For any μ belonging to an algebraic carrier of $\mathbf{M}$ we have

$$
\begin{aligned}
(\mathrm{Val}_{\mathbf{M}} \circ \mathrm{Nam}_{|\mathbf{M}|})\mu &= \mathrm{Val}_{\mathbf{M}}(\mathrm{Nam}_{|\mathbf{M}|}\,\mu)\\
&= \mathrm{Val}_{\mathbf{M}}(\mathrm{nam}_{|\mathbf{M}|}\,\mu) && \text{from (B2)}\\
&= \{\mathrm{val}_{\mathbf{M}}(\mathrm{nam}_{|\mathbf{M}|}\,\mu)\} && \text{from (B3)}\\
&= \{(\mathrm{val}_{\mathbf{M}} \circ \mathrm{nam}_{|\mathbf{M}|})\mu\}\\
&= \{\mu\} && \text{from (11V2)}
\end{aligned}
$$

This completes the proof. ■

§16. THE TERMAL EMBEDDING

A) In this section it will be shown that the world of terms and formulae can be embedded isomorphically into the world of termoids and formuloids (for arbitrary terminator). Only if a termal expression contains names of logical sort, it won't have a representative among the termoidal expressions.

For example, consider again the beta-termoids defined in the introductory sections. For any term τ, the termoid $0 + \tau$ has exactly the same meaning as the term τ. Therefore, beta-termoids having the form $0 + \tau$ form an isomorphic copy of the terms.

It turns out we can define the correspondence between terms and termoids algebraically. $[\mathrm{Nam}_X]^{[\![X]\!]}$ is the injective homomorphism mapping terms into termoids. There is also a homomorphism mapping termoids in terms: $[\![\mathrm{nam}_X]\!]^{[X]}$.

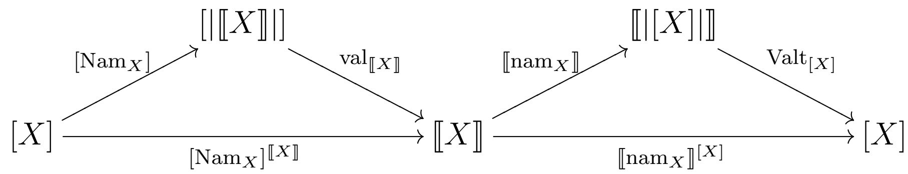
The notation $[\mathrm{Nam}_X]^{[\![X]\!]}$ reflects the fact that this homomorphism is equal to the composition $\mathrm{val}_{[\![X]\!]} \circ [\mathrm{Nam}_X]$ and similarly $[\![\mathrm{nam}_X]\!]^{[X]}$ is equal to the composition $\mathrm{Valt}_{[X]} \circ [\![\mathrm{nam}_X]\!]$. Of course, we have to use this definition in order to prove the properties of these two homomorphisms. Nevertheless, I would like to adwise the reader to disregard the meaning of this notation and to remember that $[\mathrm{Nam}_X]^{[\![X]\!]}$ and $[\![\mathrm{nam}_X]\!]^{[X]}$ are nothing more than two homomorphisms — the first mapping terms to termoids and the second mapping termoids to terms.[42]

[42]For a while I was contemplating to introduce a special notation for these two homomorphisms. Eventually I decided not to do so because I have tried to limit the use of mathematical symbolic in this work to the possible minimum. That is why, for example, I do not use the notation "$\Gamma \models \varphi$" but say in simple words "from Γ follows φ".

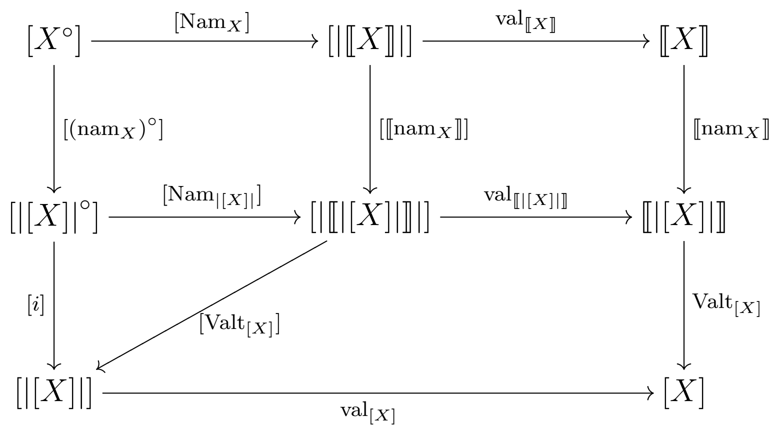
B) **Proposition.** (1) $(\tau[\mathrm{Nam}_X]^{[\![X]\!]})[\![\mathrm{nam}_X]\!]^{[X]} = \tau$ *for any termal expression* τ *over* X *without names of logical sort.*

(2) $([\![\mathrm{nam}_X]\!]^{[X]}) \circ ([\mathrm{Nam}_X]^{[\![X]\!]}) = \mathrm{id}_{[X]}$, *if* $X_{\mathtt{Log}} = \varnothing$.

(3) $[\mathrm{Nam}_X]^{[\![X]\!]} : [X] \to [\![X]\!]$ *is an injective homomorphism, if* $X_{\mathtt{Log}} = \varnothing$.



Proof. **(1)** Consider the following diagram, where i is the identity inclusion map of $|[X]|^\circ$ into $|[X]|$:



It is commutative. Indeed, the left rectangle is commutative because (14I9)[43] implies $[\![\mathrm{nam}_X]\!] \circ \mathrm{Nam}_X = \mathrm{Nam}_{[X]} \circ (\mathrm{nam}_X)^\circ$. The right rectangle is commutative because (11N)[44] implies $[\![\mathrm{nam}_X]\!] \circ \mathrm{val}_{[\![X]\!]} = \mathrm{val}_{[\![|[X]|]\!]} \circ [[\![\mathrm{nam}_X]\!]]$. The triangle is commutative because (14O4) implies $\mathrm{Valt}_{[X]} \circ \mathrm{Nam}_{[X]} = i$. And the trapezoid is commutative because (11N)[45] implies $\mathrm{Valt}_{[X]} \circ \mathrm{val}_{[\![|[X]|]\!]} = \mathrm{val}_{[X]} \circ [\mathrm{Valt}_{[X]}]$.

Since τ contains no names of logical sort, τ is a termal expression over X°, so $\tau \in [X^\circ]$. Because of the commutativity of this diagram, $(\tau[\mathrm{Nam}_X]^{[\![X]\!]})[\![\mathrm{nam}_X]\!]^{[X]} = \tau[(\mathrm{nam}_X)^\circ][i]^{[X]}$. Since τ contains no names of logical sort, $\tau[(\mathrm{nam}_X)^\circ][i]^{[X]} = \tau[i \circ (\mathrm{nam}_X)^\circ]^{[X]} = \tau[\mathrm{nam}_X]^{[X]}$. Finally, (11V1) implies that $\tau[\mathrm{nam}_X]^{[X]} = \tau$.

(2) follows from (1) and **(3)** follows from (2). ∎

[43] The function f in (14I9) is nam_X.

[44] The homomorphism h in (11N) is $[\![\mathrm{nam}_X]\!]$.

[45] The homomorphism h in (11N) is $\mathrm{Valt}_{[X]}$.

Since the alpha-termoids are simply terms, in the case of the alpha terminator we may expect both $[\![\mathrm{nam}_X]\!]^{[X]}$ and $[\mathrm{Nam}_X]^{[\![X]\!]}$ to be isomorphisms. This is indeed the case.

C) **Proposition.** *In the case of the alpha-terminator, if* $X_{\mathtt{Log}} = \varnothing$*, then:*
(1) $([\mathrm{Nam}_X]^{[\![X]\!]}) \circ ([\![\mathrm{nam}_X]\!]^{[X]}) = \mathrm{id}_{[\![X]\!]}$.
(2) *The algebraic components of* $[\mathrm{Nam}_X]^{[\![X]\!]}$ *and* $[\![\mathrm{nam}_X]\!]^{[X]}$ *are identities.*
(3) *Both* $[\![\mathrm{nam}_X]\!]^{[X]} : [\![X]\!] \to [X]$ *and* $[\mathrm{Nam}_X]^{[\![X]\!]} : [X] \to [\![X]\!]$ *are isomorphisms.*

Proof. **(1)** First, notice that

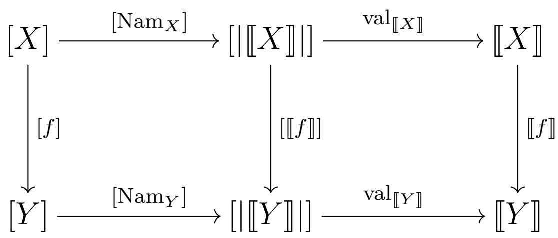
$$\begin{aligned}
\partial([\![\mathrm{nam}_X]\!]^{[X]}) &= \partial(\mathrm{Valt}_{[X]} \circ \partial[\mathrm{nam}_X]) && \text{from (15B1) and (14P)}\\
&= \partial(\mathrm{Valt}_{[X]}) \circ \partial[\mathrm{nam}_X] && \text{from (12K8) and (12K7)}\\
&= \partial(\mathrm{val}_{[X]}) \circ \partial[\mathrm{nam}_X] && \text{from (15B3) and (14N)}\\
&= \partial(\mathrm{val}_{[X]} \circ [\mathrm{nam}_X]) && \text{from (12K8)}\\
&= \partial(\mathrm{id}_{[X]}) && \text{from (11V1)}\\
&= \mathrm{id}_{\partial[X]} = \mathrm{id}_{[\![X]\!]} && \text{from (12K5)}
\end{aligned}$$

Because in the case of the alpha-terminator $[\![X]\!] = \partial[X]$ is an algebra, from (12O1) it follows that $[\![\mathrm{nam}_X]\!]^{[X]} = \int_{[X]} \circ \partial([\![\mathrm{nam}_X]\!]^{[X]}) = \int_{[X]} \circ \mathrm{id}_{[\![X]\!]} = \int_{[X]}$. The algebraic components of $\int_{[X]}$ are identities by definition (12L). From this and (B2) it follows that the algebraic components of $[\mathrm{Nam}_X]^{[\![X]\!]}$ also are identities. Consequently, the algebraic components of $([\mathrm{Nam}_X]^{[\![X]\!]}) \circ ([\![\mathrm{nam}_X]\!]^{[X]})$ are identities, hence (12H2) implies that $([\mathrm{Nam}_X]^{[\![X]\!]}) \circ ([\![\mathrm{nam}_X]\!]^{[X]}) = \mathrm{id}_{[\![X]\!]}$.
(2) has been proved during the proof of (1).
(3) follows from (1) and (B2). ■

It doesn't matter whether we apply renaming morphism $[\![f]\!]$ to a termoid and then convert it to term, or we first convert it to term and then apply $[f]$ to it. Nor it does matter whether we apply the renaming morphism $[f]$ to a term and then convert it to termoid, or we first convert it to termoid and then apply $[\![f]\!]$ to it.

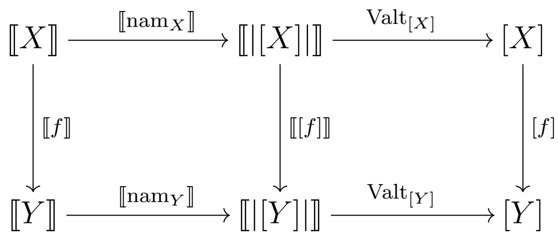
D) **Proposition.** *Given a* `Sort`*-indexed function* $f : X \to Y$*,*
(1) $([\mathrm{Nam}_Y]^{[\![Y]\!]}) \circ [f] = [\![f]\!] \circ ([\mathrm{Nam}_X]^{[\![X]\!]})$*, if* $X_{\mathtt{Log}} = \varnothing$ *and* $Y_{\mathtt{Log}} = \varnothing$*.*
(2) $([\![\mathrm{nam}_Y]\!]^{[Y]}) \circ [\![f]\!] = [f] \circ ([\![\mathrm{nam}_X]\!]^{[X]})$.

Proof. **(1)** Considering that $[\mathrm{Nam}_X]^{[\![X]\!]} = \mathrm{val}_{[\![X]\!]} \circ [\mathrm{Nam}_X]$ and $[\mathrm{Nam}_Y]^{[\![Y]\!]} = \mathrm{val}_{[\![Y]\!]} \circ [\mathrm{Nam}_Y]$, (1) follows from the commutativity of the following diagram:



The commutativity of the first rectangle follows from (14I9) and (11H) and the commutativity of the second rectangle follows from (11N).

(2) Considering that $[\![\mathrm{nam}_X]\!]^{[X]} = \mathrm{Valt}_{[X]} \circ [\![\mathrm{nam}_X]\!]$ and $[\![\mathrm{nam}_Y]\!]^{[Y]} = \mathrm{Valt}_{[Y]} \circ [\![\mathrm{nam}_Y]\!]$, (2) follows from the commutativity of the following diagram:

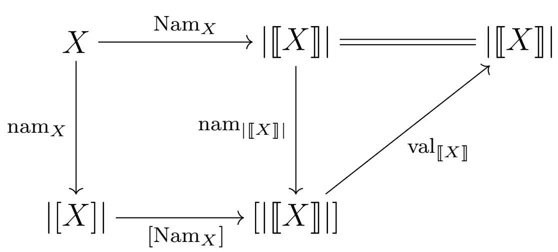


The commutativity of the first rectangle follows from (11I) and (14I7) and the commutativity of the second rectangle follows from (14O2). ∎

It doesn't matter whether we create a termal name by nam_X and then convert it to a termoid, or we create immediately a termoidal name by Nam_X. Nor it does matter whether we create a termoidal name by Nam_X and then convert it to a term, or we create immediately a termal name by nam_X.

E) **Proposition.** *If* $X_{\mathsf{Log}} = \varnothing$, *then:*
(1) $([\mathrm{Nam}_X]^{[\![X]\!]}) \circ \mathrm{nam}_X = \mathrm{Nam}_X$
(2) $([\![\mathrm{nam}_X]\!]^{[X]}) \circ \mathrm{Nam}_X = \mathrm{nam}_X$

Proof. Considering that $[\mathrm{Nam}_X]^{[\![X]\!]} = \mathrm{val}_{[\![X]\!]} \circ [\mathrm{Nam}_X]$, **(1)** follows from the commutativity of the following diagram:



The rectangle is commutative because of (11I) and the triangle is commutative because of (11V2).

(2) From (1) and (B2) it follows that $([\![\mathrm{nam}_X]\!]^{[X]}) \circ \mathrm{Nam}_X = ([\![\mathrm{nam}_X]\!]^{[X]}) \circ ([\mathrm{Nam}_X]^{[\![X]\!]}) \circ \mathrm{nam}_X = \mathrm{nam}_X$. ■

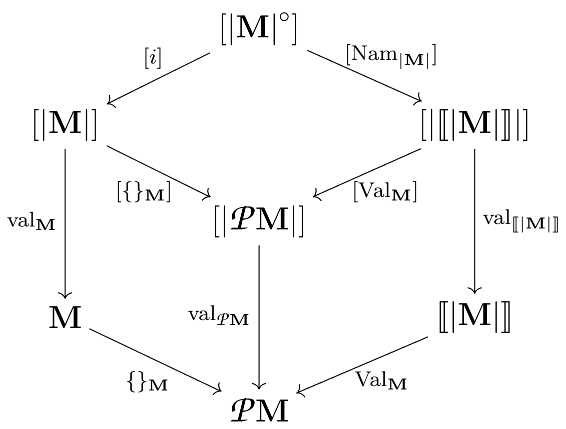
Does it matter whether we convert a term to a termoid and then evaluate it by $\mathrm{Val}_{\mathbf{M}}$, or we evaluate it immediately by $\mathrm{val}_{\mathbf{M}}$? According to the following proposition if the structure $\mathbf{M}$ is such that $\mathrm{Val}_{\mathbf{M}}$ is a homomorphism, then it doesn't matter.[46]

F) **Proposition.** *Let* $\{\}_{\mathbf{M}}$ *be as defined in* (10N)*,* i *be the identity inclusion map from* $|\mathbf{M}|^{\circ}$ *to* $|\mathbf{M}|$ *and* $\mathrm{Val}_{\mathbf{M}}$ *be not just a quasimorphism, but a homomorphism. Then:*

(1) $\mathrm{Val}_{\mathbf{M}} \circ ([\mathrm{Nam}_{|\mathbf{M}|}]^{[\![|\mathbf{M}|]\!]}) = \{\}_{\mathbf{M}} \circ \mathrm{val}_{\mathbf{M}} \circ [i]$.

(2) *If* $(\tau[\mathrm{Nam}_{|\mathbf{M}|}]^{[\![|\mathbf{M}|]\!]})^{\mathcal{P}\mathbf{M}} = \{\tau^{\mathbf{M}}\}$ *for any term or formula* τ *over* $|\mathbf{M}|$*.*

<u>Proof.</u> Considering that $[\mathrm{Nam}_{|\mathbf{M}|}]^{[\![|\mathbf{M}|]\!]} = \mathrm{val}_{[\![|\mathbf{M}|]\!]} \circ [\mathrm{Nam}_{|\mathbf{M}|}]$, **(1)** follows from the commutativity of the following diagram:



The top square is commutative because of (14I18) and the other two squares are commutative because of (11N).

(2) Neither terms nor formulae may contain names of logical sort, so τ is not only a termal expression over $|\mathbf{M}|$, but also over $|\mathbf{M}|^{\circ}$. Consequently, (2) is a reformulation of (1). ■

---

[46]I would like to give an example when $\{\tau^{\mathbf{M}}\}$ is not equal to $(\tau[\mathrm{Nam}_X]^{[\![X]\!]})^{\mathcal{P}\mathbf{M}}$. Unfortunately, in order to do this I have to use the epsilon-termoids defined in (26T).

Let the symbol $\mathtt{c}$ be of type $\langle\langle\rangle, \kappa\rangle$ and the symbols $\mathtt{f}$ and $\mathtt{g}$ be of type $\langle\langle\kappa\rangle, \kappa\rangle$. Let the structure $\mathbf{M}$ be such that $\mathbf{M}_\kappa = \{0, 1\}$, $\mathtt{c}^{\mathbf{M}} = 0$, $\mathtt{f}^{\mathbf{M}}$ be the identity function and $\mathtt{g}^{\mathbf{M}}\mu = 0$ for any μ.

Let $\tau = \mathtt{f}(\mathtt{c})$ and $\tau' = \tau[\mathrm{Nam}^{\varepsilon}_X]^{[\![X]\!]_\varepsilon} = \ulcorner 0 \urcorner + \mathtt{f}(\mathtt{c})$. Then $\tau^{\mathbf{M}} = 0$ and $(\tau')^{\mathcal{P}\mathbf{M}} = \{0, 1\}$.

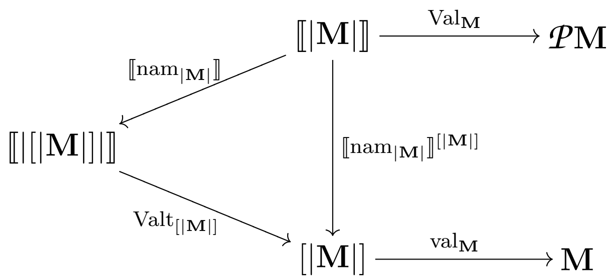
G) **Corollary.** *Let X be an arbitrary* `Sort`*-indexed set and i be the identity inclusion map from $|[X]|^{\circ}$ to $|[X]|$. Then:*
(1) $\mathrm{Valt}_{[X]} \circ ([\mathrm{Nam}_{|[X]|}]^{[\![|[X]|]\!]}) = \mathrm{val}_{[X]} \circ [i]$.
(2) $(\tau[\mathrm{Nam}_{|[X]|}]^{[\![|[X]|]\!]})^{[X]} = \tau^{[X]}$ *for any term or formula τ over $|[X]|$.*

Proof. **(1)** If $\mathbf{M}$ is a structure of terms, then $\mathrm{Val}_{\mathbf{M}} = \{\}_{\mathbf{M}} \circ \mathrm{Valt}_{\mathbf{M}}$, where $\{\}_{\mathbf{M}}$ is as defined in (10N). Therefore, from (14O) we can conclude that $\mathrm{Val}_{\mathbf{M}}$ is a homomorphism, hence (1) follows from (F1) and the definition of $\mathrm{Valt}_{[X]}$ (see 14N).

(2) Neither terms nor formulae may contain names of logical sort, so τ is not only a termal expression over $|[X]|$, but also over $|[X]|^{\circ}$. Consequently, (2) is a reformulation of (1). ∎

If we convert the beta-termoid $5 + \mathtt{f(c)}$ to term we will obtain the term $\mathtt{f(c)}$. Clearly, the value of $\mathtt{f(c)}$ is a value of $5+\mathtt{f(c)}$, however $5+\mathtt{f(c)}$ can have other values.

H) **Proposition.** (1) $\mathrm{val}_{\mathbf{M}} \circ ([\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]}) \ll \mathrm{Val}_{\mathbf{M}}$.



(2) $(\tau[\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]})^{\mathbf{M}} \in \tau^{\mathcal{P}\mathbf{M}}$ *for any termoidal expression τ over $|\mathbf{M}|$.*

Proof. **(1)** For any structure $\mathbf{N}$, let $\{\}_{\mathbf{N}}$ be as defined in (10N). Then:

$$\begin{aligned}
\{\}_{\mathbf{M}} \circ \mathrm{val}_{\mathbf{M}} \circ ([\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]}) &= \{\}_{\mathbf{M}} \circ \mathrm{val}_{\mathbf{M}} \circ \mathrm{Valt}_{[|\mathbf{M}|]} \circ [\![\mathrm{nam}_{|\mathbf{M}|}]\!] \\
&= (\mathrm{val}_{\mathbf{M}})^{\mathcal{P}} \circ \{\}_{[|\mathbf{M}|]} \circ \mathrm{Valt}_{[|\mathbf{M}|]} \circ [\![\mathrm{nam}_{|\mathbf{M}|}]\!] \\
&= (\mathrm{val}_{\mathbf{M}})^{\mathcal{P}} \circ \mathrm{Val}_{[|\mathbf{M}|]} \circ [\![\mathrm{nam}_{|\mathbf{M}|}]\!] && \text{from (14N)} \\
&\leq \mathrm{Val}_{\mathbf{M}} \circ [\![\mathrm{val}_{\mathbf{M}}]\!] \circ [\![\mathrm{nam}_{|\mathbf{M}|}]\!] && \text{from (14I13)} \\
&= \mathrm{Val}_{\mathbf{M}} \circ [\![\mathrm{val}_{\mathbf{M}} \circ \mathrm{nam}_{|\mathbf{M}|}]\!] && \text{from (14I7)} \\
&= \mathrm{Val}_{\mathbf{M}} && \text{from (11V2)}
\end{aligned}$$

(2) is a reformulation of (1). ∎

I) **Corollary.** *If $\mathbf{M}$ is a structure of terms, then*

$$\mathrm{Valt}_{\mathbf{M}} = \mathrm{val}_{\mathbf{M}} \circ ([\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]}).$$

In other words, if $\mathbf{M}$ *is a structure of terms, then for any termoidal expression* τ *over* $|\mathbf{M}|$

$$\tau^{\mathbf{M}} = (\tau[\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]})^{\mathbf{M}}.$$

Proof. See (H1) and the definition of $\mathrm{Valt}_{\mathbf{M}}$ (14N). ■

J) **Corollary.** (1) *If a formuloid* φ *over* X *is universally valid in a structure* $\mathbf{M}$*, then the formula* $\varphi[\![\mathrm{nam}_X]\!]^{[X]}$ *is universally valid in* $\mathbf{M}$.

(2) *Let* $\mathbf{M}$ *be a structure of terms. A formuloid* φ *over* X *is universally valid in* $\mathbf{M}$ *if and only if the formula* $\varphi[\![\mathrm{nam}_X]\!]^{[X]}$ *is universally valid in* $\mathbf{M}$.

(3) *If a set* Γ *of formuloids over* X *is universally satisfiable in a algebra* $\mathbf{A}$*, then then the set* $\{\varphi[\![\mathrm{nam}_X]\!]^{[X]} : \varphi \in \Gamma\}$ *of formulae over* X *is universally satisfiable in* $\mathbf{A}$.

(4) *Let* $\mathbf{A}$ *be an algebra which is a structure of terms. A set* Γ *of formuloids over* X *is universally satisfiable in* $\mathbf{A}$ *if and only if the set* $\{\varphi[\![\mathrm{nam}_X]\!]^{[X]} : \varphi \in \Gamma\}$ *of formulae over* X *is universally satisfiable in* $\mathbf{A}$.

Proof. **(1)** Let $v : X \to |\mathbf{M}|$ be an arbitrary assignment function. Then:

$$\begin{aligned}
(\varphi[\![\mathrm{nam}_X]\!]^{[X]})[v]^{\mathbf{M}} &= (([v] \circ [\![\mathrm{nam}_X]\!]^{[X]})\varphi)^{\mathbf{M}} \\
&= (([\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]} \circ [\![v]\!])\varphi)^{\mathbf{M}} && \text{from (D2)} \\
&= ((\varphi[\![v]\!])[\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]})^{\mathbf{M}} \\
&\in \varphi[\![v]\!]^{\mathcal{P}\mathbf{M}} && \text{from (H2)}
\end{aligned}$$

Since φ is universally valid in $\mathbf{M}$, $\varphi[\![v]\!]^{\mathcal{P}\mathbf{M}} = \{1\}$.

(2) According to (16I), for arbitrary formuloid φ and assignment function $v : \mathbb{X} \to |\mathbf{M}|$, the value of $\varphi[\![v]\!]$ in a structure of terms $\mathbf{M}$ is equal to the value of the formula $\varphi[\![v]\!][\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]}$ in $\mathbf{M}$. According to (16D2), $\varphi[\![v]\!][\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]} = (\varphi[\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]})[v]$. Therefore, a formuloid φ is universally valid in $\mathbf{M}$ if and only if the formula $\varphi[\![\mathrm{nam}_{|\mathbf{M}|}]\!]^{[|\mathbf{M}|]}$ is universally valid in $\mathbf{M}$.

(3) follows from (1) and **(4)** follows from (2). ■

The following Proposition is an analogue of (13G) for formuloids.

K) **Proposition.** *Suppose* $[X]$ *is normal. A set of formuloids*[47] *is universally satisfiable if and only if it is universally satisfiable in* $\partial[X]$.

Proof. If a set Γ of formuloids is universally satisfiable, then there exists a logical structure $\mathbf{M}$, such that all formula of Γ are universally valid in $\mathbf{M}$.

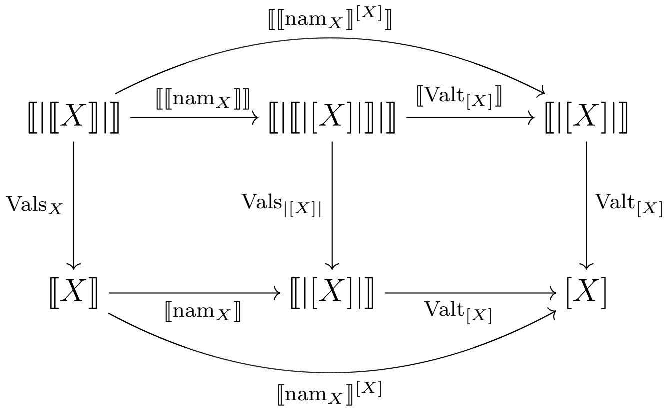
[47] Not necessarily formuloids over X.

Let $\Gamma' = \{\varphi\llbracket \mathrm{nam}_X \rrbracket^{[X]} : \varphi \in \Gamma\}$. According to (J1), Γ' is universally valid in $\mathbf{M}$, so from (13G) it follows that Γ' is universally satisfiable in $\partial[X]$. But $\partial[X]$ is an algebra which is a structure of terms, so from (J4) we can conclude that Γ is universally satisfiable in $\partial[X]$. ∎

L) **Lemma.** (1) $(\llbracket \mathrm{nam}_X \rrbracket^{[X]}) \circ \mathrm{Vals}_X = \mathrm{Valt}_{[X]} \circ \llbracket \llbracket \mathrm{nam}_X \rrbracket^{[X]} \rrbracket$ *for any* `Sort`*-indexed set* X*.*

(2) $(\tau^{\llbracket X \rrbracket})\llbracket \mathrm{nam}_X \rrbracket^{[X]} = (\tau\llbracket \llbracket \mathrm{nam}_X \rrbracket^{[X]} \rrbracket)^{[X]}$ *for any termoidal expression* τ *over* $|\llbracket X \rrbracket|$*.*

Proof. **(1)**



The left rectangle in this diagram is commutative because of (14I16) and the right rectangle is commutative because of (14O3). The segments are simply notational conventions, see (14P).

(2) is a reformulation of (1). ∎

For any termoidal substitution $s : X \to \llbracket X \rrbracket$ we have a corresponding termal substitution $s' : X \to [X]$ where $s' = \llbracket \mathrm{nam}_X \rrbracket^{[X]} \circ s$. It doesn't matter whether we apply s to a termoid and then convert it to a term, or we first convert the termoid to a term and then apply s' to it.

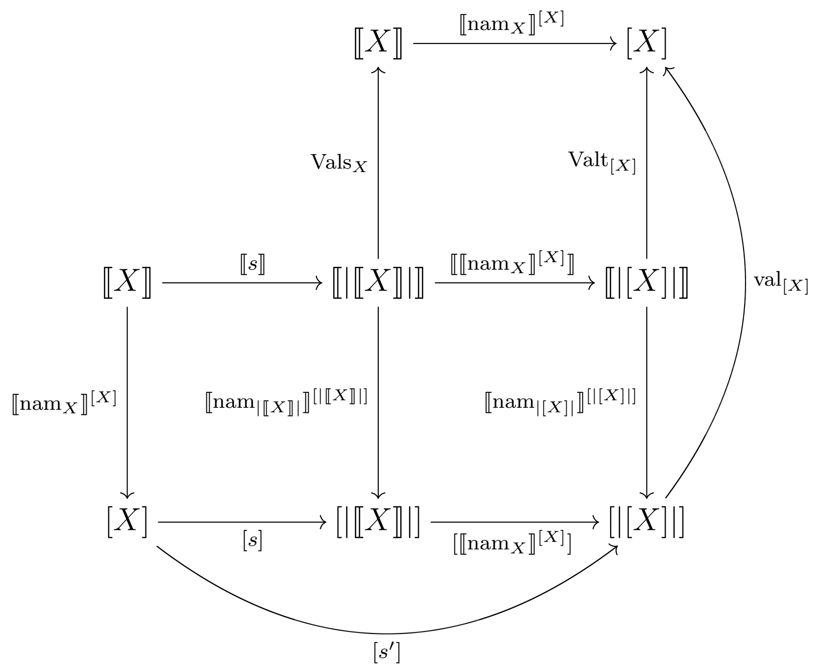
M) **Proposition.** *Given a* `Sort`*-indexed set* X *and a termoidal substitution* $s : X \to \llbracket X \rrbracket$*, define a termal substitution* $s' : X \to [X]$*, such that* $s'\xi = (s\xi)\llbracket \mathrm{nam}_X \rrbracket^{[X]}$ *for any* $\xi \in X$*. Then for any termoidal expression* τ *over* X*,*

$$(\tau\llbracket s \rrbracket^{\llbracket X \rrbracket})\llbracket \mathrm{nam}_X \rrbracket^{[X]} = \tau\llbracket s' \rrbracket^{[X]} = (\tau\llbracket \mathrm{nam}_X \rrbracket^{[X]})[s']^{[X]}$$

Proof. We have to prove that

$$\begin{aligned} \llbracket \mathrm{nam}_X \rrbracket^{[X]} \circ \mathrm{Vals}_X \circ \llbracket s \rrbracket &= \mathrm{Valt}_{[X]} \circ \llbracket \llbracket \mathrm{nam}_X \rrbracket^{[X]} \rrbracket \circ \llbracket s \rrbracket \\ &= \mathrm{val}_{[X]} \circ [s'] \circ \llbracket \mathrm{nam}_X \rrbracket^{[X]} \end{aligned}$$

Consider the following diagram:



The right segment is commutative due to (I), the top square is commutative due to (L1), the other two squares are commutative due to (D2) and the bottom segment is commutative due to the definition of s'. ■

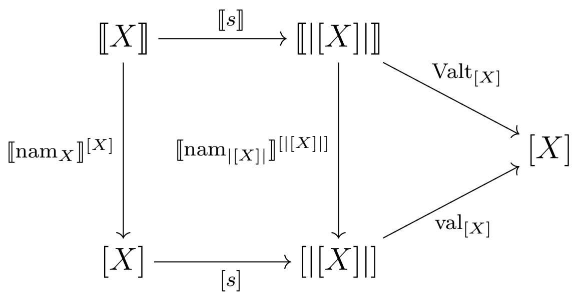
N) **Proposition.** *Let X be a* Sort*-indexed set and $s : X \to [X]$ be a termal substitution. Then for any termoidal expression τ over X,*

$$\tau[\![s]\!]^{[X]} = (\tau[\![\mathrm{nam}_X]\!]^{[X]})[s]^{[X]}$$

Proof. We have to prove that

$$\mathrm{Valt}_{[X]} \circ [\![s]\!] = \mathrm{val}_{[X]} \circ [s] \circ [\![\mathrm{nam}_X]\!]^{[X]}$$

Consider the following diagram:



The rectangle is commutative due to (D2) and the triangle is commutative due to (I). ■

§17. FINITARITY AND DEPENDENCIES

A) **Definition.** (1) A $\texttt{Sort}$-indexed set X is *finite* if for all sorts κ the components X_κ are finite sets and only finitely many of them are different from $\varnothing$.

(2) A termoidal expression τ over X is *finitary* if the $\texttt{Sort}$-indexed set X has a finite subset Y,[48] such that τ is a termoidal expression over Y.

B) **Proposition.** *For any finitary termoidal expression τ there exists a smallest $\texttt{Sort}$-indexed set X, such that τ is a termoidal expression over X. This $\texttt{Sort}$-indexed set is finite.*

Proof. Let Y be some finite $\texttt{Sort}$-indexed set such that τ is a termoidal expression over Y. Consider all $\texttt{Sort}$-indexed sets of the form $Y \cap Z$, such that τ is a termoidal expression over Z. From (14|3) it follows that τ is a termoidal expression over each of these sets. Since $Y \cap Z$ is a subset of Y and Y is a finite set, there are only finitely many such sets. Let X be the intersection of these sets. On one hand, X is a subset of all $\texttt{Sort}$-indexed sets Z, such that τ is a termoidal expression over Z. On the other hand, since we are intersecting finitely many sets, from (14|3) it follows that τ is a termoidal expression over X.

The finiteness of X follows from definition (A2). ∎

C) **Definition.** A termal (termoidal) expression τ over X *depends* on $\xi \in X$ if $\tau[f] \neq \tau$ (resp., $\tau[\![f]\!] \neq \tau$) for at least one $\texttt{Sort}$-indexed function f with domain X, such that $f\eta = \eta$ for any $\eta \neq \xi$.

D) **Proposition.** *If a termal (termoidal) expression τ over X is finitary, then there are finitely many $\xi \in X$, such that τ depends on ξ.*

Proof. According to (B), there exists a finite subset Y of X, such that τ is a termoidal expression over Y. Let $f : Y \to X$ be the identity inclusion map from Y to X and let $f' : Y \to Y$ be the identity function. According to (14|6), $\tau[\![f']\!] = \tau$ and according to (14|4), $\tau[\![f]\!] = \tau[\![f']\!]$, hence $\tau[\![f]\!] = \tau$.

Now, suppose that τ depends on some $\xi \in X$. Then there exists a $\texttt{Sort}$-indexed function g with domain X, such that $g\eta = \eta$ for any $\eta \neq \xi$ and $\tau[\![g]\!] \neq \tau$. But $\tau[\![g]\!] = \tau[\![f]\!][\![g]\!] = \tau[\![g \circ f]\!]$, so $g \circ f$ is not identity, hence $\xi \in Y$. The $\texttt{Sort}$-indexed set Y is finite, so there are finitely many such ξ. ∎

E) **Proposition.** (1) *A termal expression τ over X depends on $\xi \in X$ if and only if $\ulcorner\xi\urcorner$ occurs in τ.*

(2) *A termal (termoidal) expression τ over X depends on $\xi \in X$ if and*

[48] I.e. $Y_\kappa \subseteq X_\kappa$ for all κ.

only if τ *is not a termal (termoidal) expression over* $X \setminus \{\xi\}$.[49]

Proof. **(1,⇒)** Let $f : X \to Y$ be a `Sort`-indexed function, such that $\tau[f] \neq \tau$ and $f\eta = \eta$ for any $\eta \neq \xi$. Suppose that τ is a termal expression over $X \setminus \{\xi\}$. Then from (14|4) it follows $\tau[f] = \tau[f \restriction (X \setminus \{\xi\})] = \tau[\mathrm{id}] = \tau$, hence we obtain a contradiction.

(1,⇐) Let $\ulcorner\xi\urcorner$ occurs in τ and suppose that τ does not depend on ξ. By definition, $\tau[f] = \tau$ for any `Sort`-indexed function f with domain X, such that X is the domain of f and $f\eta = \eta$ for any $\eta \neq \xi$. Well, this can not be so. Let κ be the sort of ξ and let the `Sort`-indexed set Y be such that $X \subseteq Y$ and Y_κ contains an element ξ', such that $\xi' \notin X$. Let $f : X \to Y$ be such that $f\xi = \xi'$ and $f\eta = \eta$ for $\eta \neq \xi$. Clearly, ξ' occurs in $\tau[f]$, hence $\tau[f] \neq \tau$.

(2,⇒) follows easily from (1) when τ is a termal expression, so it remains to consider the case when τ is a termoidal expression over X, such that τ depends on $\xi \in X_\kappa$. Suppose that τ is a termoidal expression over $X \setminus \{\xi\}$. Then there exists a `Sort`-indexed function $f : X \to Y$, such that $\tau[f] \neq \tau$ and $f\eta = \eta$ for any $\eta \neq \xi$. Therefore, from (14|4) and (14|6) we can conclude that $\tau[\![f]\!] = \tau[\![f \restriction (X \setminus \{\xi\})]\!] = \tau[\![\mathrm{id}]\!] = \tau$ which is a contradiction.

(2,⇐) follows easily from (1) when τ is a termal expression, so it remains to consider the case when τ is a termoidal expression over X, such that τ is not a termoidal expression over $X \setminus \{\xi\}$. Suppose that τ does not depend on $\xi \in X$. Let ξ' and ξ'' be arbitrary different objects which do not belong to X. Let X' be like X but instead of ξ, X' contains ξ'. Similarly, let X'' be like X but instead of ξ, X'' contains ξ''. Let $f' : X \to X'$ be the `Sort`-indexed function, such that $f'\xi = \xi'$ and $f'\eta = \eta$ for $\eta \neq \xi$. Similarly, let $f'' : X \to X'$ be the `Sort`-indexed function, such that $f''\xi = \xi''$ and $f''\eta = \eta$ for $\eta \neq \xi$. Since τ does not depend on ξ, $\tau[\![f']\!] = \tau$ and $\tau[\![f'']\!] = \tau$. On the other hand $\tau[\![f']\!]$ is a termoidal expression over X' and $\tau[\![f'']\!]$ is a termoidal expression over X'', hence τ is a termoidal expression both over X' and over X''. From (14|3) it follows that τ is a termoidal expression over $X' \cap X''$. Since $X \setminus \{\xi\} = X' \cap X''$, we conclude that τ is a termoidal expression over $X \setminus \{\xi\}$, which is a contradiction. ∎

F) **Proposition.** (1) *Let* X *be a* `Sort`*-indexed set,* $s : X \to |[X]|$ *be a termal substitution and* Z *be the* `Sort`*-indexed set of all* ξ*, such that* $s\xi \neq \ulcorner\xi\urcorner$*. If* τ *is a termal expression which does not depend on any* $\xi \in Z$*, then* $\tau[s]^{[X]} = \tau$.

(2) *Let* X *be a* `Sort`*-indexed set,* $s : X \to |[\![X]\!]|$ *be a termoidal substi-*

[49] We use the notation informally. Let $\xi \in X_\kappa$. Then $X \setminus \{\xi\}$ is the `Sort`-indexed set Y, such that $Y_\kappa = X_\kappa \setminus \{\xi\}$ and $Y_\lambda = X_\lambda$ for $\lambda \neq \kappa$.

tution and Z be the `Sort`*-indexed set of all ξ, such that $s\xi \neq \ulcorner\xi\urcorner$. If τ is a finitary termoidal expression which does not depend on any $\xi \in Z$, then $\tau[\![s]\!]^{[\![X]\!]} = \tau$.*

Proof. **(1)** $s\xi = \mathrm{nam}_X\,\xi$ for any ξ, such that $\ulcorner\xi\urcorner$ occurs in τ. Consequently, $\tau[s] = \tau[\mathrm{nam}_X]$, hence $\tau[s]^{[X]} = \tau[\mathrm{nam}_X]^{[X]} = \tau$, due to (11V1).

(2) Let Y be a finite subset of the `Sort`-indexed set X, such that τ is a termoidal expression on Y. Then $s\xi = \ulcorner\xi\urcorner$ for any $\xi \in Y$, so $s\restriction Y = \mathrm{Nam}_Y$, hence

$$\begin{aligned} \tau[\![s]\!]^{[\![X]\!]} &= \tau[\![s\restriction Y]\!]^{[\![X]\!]} && \text{from (14I4)} \\ &= \tau[\![\mathrm{Nam}_Y]\!]^{[\![X]\!]} \\ &= \tau[\![\mathrm{Nam}_Y]\!]^{[\![Y]\!]} && \text{from (14R3)} \\ &= \tau && \text{from (14I17)} \end{aligned}$$

∎

G) **Definition.** The termal (termoidal) expressions $\tau_1, \ldots, \tau_n$ over X have *disjoint dependency* if there is no $\xi \in X$, such that more then one of these termal (termoidal) expressions depends on ξ.

H) **Proposition.** *Given finitary termoidal expressions τ and σ over X and* `Sort`*-indexed functions $f, g : X \to Y$, if τ and σ have disjoint dependency, then there exists a* `Sort`*-indexed function $h : X \to Y$, such that $\tau[\![f]\!] = \tau[\![h]\!]$ and $\sigma[\![g]\!] = \sigma[\![h]\!]$.*

Proof. Let $h\xi = f\xi$ if τ depends on ξ, let $h\xi = g\xi$ if σ depends on ξ and define $h\xi$ arbitrarily if neither τ, nor σ depends on ξ.[50] Suppose that $\tau[\![f]\!] \neq \tau[\![h]\!]$. Let Z be the smallest subset of X, such that τ is a termoidal expression over Z; there is such set according to (B). According to (14I4), $\tau[\![f]\!] = \tau[\![f\restriction Z]\!]$ and $\tau[\![h]\!] = \tau[\![h\restriction Z]\!]$, hence $f\restriction Z \neq h\restriction Z$, so $f\xi \neq h\xi$ for some $\xi \in Z$, hence τ does not depend on ξ, so, according to (E2), τ is a termoidal expression over $Z\backslash\{\xi\}$, which is a contradiction because Z was the smallest set with this property. Consequently, $\tau[\![f]\!] = \tau[\![h]\!]$. Analogously, we can prove that $\sigma[\![g]\!] = \sigma[\![h]\!]$. ∎

I) **Proposition.** (1) *Given a termoidal expression τ over X, if $\tau[\![\mathrm{nam}_X]\!]^{[X]}$ depends on $\xi \in X$, then τ depends on ξ as well.*

(2) *If the termoidal expressions $\tau_1, \ldots, \tau_n$ over X have disjoint dependency, then the termal expressions $\tau_1[\![\mathrm{nam}_X]\!]^{[X]}, \ldots, \tau_n[\![\mathrm{nam}_X]\!]^{[X]}$ have disjoint dependency as well.*

[50] The case when both τ and σ depend on some ξ is impossible, since τ and σ have disjoint dependency.

Proof. **(1)** Since $\tau[\![\mathrm{nam}_X]\!]^{[X]}$ depends on ξ, there exists a `Sort`-indexed function f with domain X, such that $f\eta = \eta$ for any $\eta \neq \xi$ and $\tau[\![\mathrm{nam}_X]\!]^{[X]}[f] \neq \tau[\![\mathrm{nam}_X]\!]^{[X]}$. According to (16D2), $\tau[\![\mathrm{nam}_X]\!]^{[X]}[f] = \tau[\![f]\!][\![\mathrm{nam}_X]\!]^{[X]}$, hence $\tau[\![f]\!][\![\mathrm{nam}_X]\!]^{[X]} \neq \tau[\![\mathrm{nam}_X]\!]^{[X]}$, so $\tau[\![f]\!] \neq \tau$, hence τ depends on ξ.

(2) follows from (1). ■

J) **Proposition.** *Given a termal expression τ over X, if $\tau[\mathrm{Nam}_X]^{[\![X]\!]}$ depends on $\xi \in X$, then τ depends on ξ as well.*

Proof. Suppose that τ does not depend on ξ and let $Y = X \setminus \{\xi\}$; then τ will be a termal expression over Y. Let $f : Y \to X$ be the identity inclusion map of Y in X.

According to (14I4) and (14I6), $[\![f]\!]$ is the identity inclusion map from $[\![Y]\!]$ to $[\![X]\!]$, so from (14I9) we can conclude that $\mathrm{Nam}_Y = \mathrm{Nam}_X \restriction Y$, so $\tau[\mathrm{Nam}_X] = \tau[\mathrm{Nam}_Y]$.

Since $[\![f]\!]$ is an identity inclusion map, $[[\![f]\!]]$ also is an identity inclusion map from $[|[\![Y]\!]|]$ to $[|[\![X]\!]|]$. Therefore, from (11N) we can conclude that $\tau[\mathrm{Nam}_Y]^{[\![Y]\!]} = \tau[\mathrm{Nam}_Y]^{[\![X]\!]}$, hence $\tau[\mathrm{Nam}_Y]^{[\![Y]\!]} = \tau[\mathrm{Nam}_X]^{[\![X]\!]}$, so $\tau[\mathrm{Nam}_X]^{[\![X]\!]}$ is a termoidal expression over Y, hence it does not depend on ξ which is a contradiction. ■

§18. REDUCTORS AND UNIFICATION

A) **Definition.** (1) A *termal identity* of sort κ over the `Sort`-indexed set X is an expression of the form

$$\tau \sim \sigma$$

where both t and s are termal expressions over X of sort κ.

(2) A *termal system* over X is a set of identities over X (of any sort). A termal system is *finite* if it is a finite set of identities.

(3) The notions *termoidal identity*, *termoidal system* and *finite termoidal system* are analogous, but we use termoidal expressions instead of termal expressions.

(4) An identity of the form $\ulcorner\xi\urcorner \sim \sigma$ is *solving* for ξ if σ does not depend on ξ. Such an identity is solving for a system Θ, if it belongs to Θ and the termal (termoidal) expressions of the other identities of Θ do not depend on ξ. In this case, we also say that Θ is *solved* with respect to ξ.

(5) A system is solved if all its identities are solving for it.

(6) An identity *depends* on ξ if at least one of its termal (termoidal) expressions depends on ξ. A system *depends* on ξ if at least one of its

identities depends on ξ.

(7) A system is *finitary* if there exists a finite `Sort`-indexed set X, such that all termal (termoidal) expressions in the systems are termal (termoidal) expressions over X.

(8) Given a `Sort`-indexed set X, a termal identity $\tau \sim \sigma$ over X and a structure $\mathbf{M}$, the assignment function $v : X \to |\mathbf{M}|$ is a *solution* in $\mathbf{M}$ of $\tau \sim \sigma$, if $\tau[v]^{\mathbf{M}} = \sigma[v]^{\mathbf{M}}$.

(9) Given a `Sort`-indexed set X, a termoidal identity $\tau \sim \sigma$ over X and a structure $\mathbf{M}$, the assignment function $v : X \to |\mathbf{M}|$ is a *solution* in $\mathbf{M}$ of $\tau \sim \sigma$, if $\tau[\![v]\!]^{\mathcal{P}\mathbf{M}} \cap \sigma[\![v]\!]^{\mathcal{P}\mathbf{M}} \neq \varnothing$.

(10) A `Sort`-indexed function is a *solution* in a structure $\mathbf{M}$ of the system Θ, if it is a solution in $\mathbf{M}$ of each identity of Θ.

(11) A system is *termally consistent*, if it has a solution in an algebra which is a structure of terms.[51] A system is *termally inconsistent*, if it is not termally consistent.

(12) A system Θ is *termally equivalent* to a system Φ if Θ and Φ have same solutions in any algebra which is a structure of terms.

(13) A system Θ is *reducible* to a system Φ if Θ and Φ are termally equivalent and all solutions of Θ in any structure are solutions of Φ as well.

B) Notice that if $\mathbf{M}$ is a structure of terms, then $\tau[\![v]\!]^{\mathcal{P}\mathbf{M}} = \{\tau[\![v]\!]^{\mathbf{M}}\}$ for any τ and v, hence an assignment function $v : X \to |\mathbf{M}|$ is a solution in $\mathbf{M}$ of the termoidal identity $\tau \sim \sigma$, if and only if $\tau[\![v]\!]^{\mathbf{M}} = \sigma[\![v]\!]^{\mathbf{M}}$.

C) **Definition.** (1) Given a `Sort`-indexed set X, a termal substitution $s : X \to [X]$ and a structure $\mathbf{M}$, the assignment function $v : X \to |\mathbf{M}|$ is an *instance* of s in $\mathbf{M}$, if $v = [w]^{\mathbf{M}} \circ s$ for some assignment function $w : X \to |\mathbf{M}|$.

(2) Given a `Sort`-indexed set X, a termoidal substitution $s : X \to [\![X]\!]$ and a structure $\mathbf{M}$, the assignment function $v : X \to |\mathbf{M}|$ is an *instance* of s in $\mathbf{M}$, if $v \ll [\![w]\!]^{\mathcal{P}\mathbf{M}} \circ s$ for some assignment function $w : X \to |\mathbf{M}|$.[52]

D) **Proposition.** *Let* $\Theta = \{\ulcorner\xi_1\urcorner \sim \sigma_1, \ulcorner\xi_2\urcorner \sim \sigma_2, \dots, \ulcorner\xi_n\urcorner \sim \sigma_n\}$ *be a solved termal (termoidal) system over* X *and* s *be the substitution*

$$s\xi = \begin{cases} \sigma_i, & \text{if } \xi = \xi_i \text{ for some } i \in \{1, \dots, n\}, \\ \ulcorner\xi\urcorner, & \text{otherwise.} \end{cases}$$

Then:

[51] For the definition of "structure of terms" see (14L). Each algebra which is a structure of terms is the algebraic fragment of a termal structure $[X]$.

[52] For the definition of "$\ll$" see (14A).

(1) *All instances of s in an algebra which is a structure of terms are solutions of Θ.*

(2) *If v is an arbitrary solution of Θ, then $v = [v]^{\mathbf{M}} \circ s$ (in the termal case) or $v \ll [\![v]\!]^{\mathcal{P}\mathbf{M}} \circ s$ (in the termoidal case).*

(3) *All solutions of Θ are instances of s.*

Proof. **(1,terms)** Let $\mathbf{A}$ be an algebra which is a structure of terms and v be an instance of s in $\mathbf{A}$. Then $v = [w]^{\mathbf{A}} \circ s$ for some assignment function $w : X \to |\mathbf{A}|$. Consequently,

$$\begin{aligned}
\ulcorner\xi_i\urcorner[v]^{\mathbf{A}} &= \ulcorner\xi_i\urcorner\,[[w]^{\mathbf{A}} \circ s]^{\mathbf{A}} \\
&= (\ulcorner\xi_i\urcorner\,[s]^{[X]})[w]^{\mathbf{A}} && \text{from (11T)} \\
&= \sigma_i[w]^{\mathbf{A}} && \text{because } s\xi_i = \sigma_i \\
&= (\sigma_i[s]^{[X]})[w]^{\mathbf{A}} && \text{from (17F)} \\
&= \sigma_i[[w]^{\mathbf{A}} \circ s]^{\mathbf{A}} && \text{from (11T)} \\
&= \sigma_i[v]^{\mathbf{A}}
\end{aligned}$$

(1,termoids) Let $\mathbf{A}$ be an algebra which is a structure of terms and v be an instance of s in $\mathbf{A}$. Then $v \ll [\![w]\!]^{\mathcal{P}\mathbf{A}} \circ s$ for some assignment function $w : X \to |\mathbf{A}|$. From this and (14N) it follows $v = [\![w]\!]^{\mathbf{A}} \circ s$. Consequently,

$$\begin{aligned}
\ulcorner\xi_i\urcorner[\![v]\!]^{\mathbf{A}} &= \ulcorner\xi_i\urcorner\,[\![[\![w]\!]^{\mathbf{A}} \circ s]\!]^{\mathbf{A}} \\
&= (\ulcorner\xi_i\urcorner\,[\![s]\!]^{[\![X]\!]})[\![w]\!]^{\mathbf{A}} && \text{from (14T2)} \\
&= \sigma_i[\![w]\!]^{\mathbf{A}} && \text{because } s\xi_i = \sigma_i \\
&= (\sigma_i[\![s]\!]^{[\![X]\!]})[\![w]\!]^{\mathbf{A}} && \text{from (17F)} \\
&= \sigma_i[\![[\![w]\!]^{\mathbf{A}} \circ s]\!]^{\mathbf{A}} && \text{from (14T2)} \\
&= \sigma_i[\![v]\!]^{\mathbf{A}}
\end{aligned}$$

(2,terms) $\ulcorner\xi_i\urcorner[v]^{\mathbf{M}} = \sigma_i[v]^{\mathbf{M}}$, because v is a solution of Θ. Consequently, from (11V2) and (11I) it follows $v\xi_i = (\ulcorner v\xi_i\urcorner)^{\mathbf{M}} = \ulcorner\xi_i\urcorner[v]^{\mathbf{M}} = \sigma_i[v]^{\mathbf{M}} = (s\xi_i)[v]^{\mathbf{M}} = ([v]^{\mathbf{M}} \circ s)\xi_i$.

On the other hand, if $\xi \notin \{\xi_1, \ldots, \xi_n\}$, then $s\xi = \ulcorner\xi\urcorner$, hence also from (11V2) and (11I) it follows $v\xi = (\ulcorner v\xi\urcorner)^{\mathbf{M}} = \ulcorner\xi\urcorner[v]^{\mathbf{M}} = (s\xi)[v]^{\mathbf{M}} = ([v]^{\mathbf{M}} \circ s)\xi$.

(2,termoids) From (14I9) and (14I18) it follows $\ulcorner\xi_i\urcorner[\![v]\!]^{\mathcal{P}\mathbf{M}} = (\ulcorner v\xi_i\urcorner)^{\mathcal{P}\mathbf{M}} = \{v\xi_i\}$. But $\ulcorner\xi_i\urcorner[\![v]\!]^{\mathcal{P}\mathbf{M}} \cap \sigma_i[\![v]\!]^{\mathcal{P}\mathbf{M}} \neq \varnothing$, because v is a solution of Θ. Consequently, $v\xi_i \in \sigma_i[\![v]\!]^{\mathcal{P}\mathbf{M}} = (s\xi_i)[\![v]\!]^{\mathcal{P}\mathbf{M}} = ([\![v]\!]^{\mathcal{P}\mathbf{M}} \circ s)\xi_i$.

On the other hand, if $\xi \notin \{\xi_1, \ldots, \xi_n\}$, then $s\xi = \ulcorner\xi\urcorner$, hence from (14I18) and (14I9) it follows $v\xi \in \{v\xi\} = (\ulcorner v\xi\urcorner)^{\mathcal{P}\mathbf{M}} = \ulcorner\xi\urcorner[\![v]\!]^{\mathcal{P}\mathbf{M}} = (s\xi)[\![v]\!]^{\mathcal{P}\mathbf{M}} = ([\![v]\!]^{\mathcal{P}\mathbf{M}} \circ s)\xi$.

(3) immediately follows from (2) and the definition of "instance" (C). ∎

E) **Proposition.** *Let τ, σ' and σ'' be termal (termoidal) expressions over the* Sort*-indexed set X, $\xi \in X$, ξ and τ are of the same sort and σ' and σ'' are of the same sort. Let s be the substitution*

$$s\eta = \begin{cases} \tau, & \text{if } \eta = \xi, \\ \ulcorner \eta \urcorner, & \text{otherwise.} \end{cases}$$

Then the system $\{\ulcorner \xi \urcorner \sim \tau, \sigma' \sim \sigma''\}$ is reducible to the system $\{\ulcorner \xi \urcorner \sim \tau, \sigma'[s]^{[X]} \sim \sigma''[s]^{[X]}\}$ (resp., to the system $\{\ulcorner \xi \urcorner \sim \tau, \sigma'[\![s]\!]^{[\![X]\!]} \sim \sigma''[\![s]\!]^{[\![X]\!]}\}$).

Proof. **(terms)** First, let us notice that if $v : X \to |\mathbf{M}|$ is an arbitrary solution of the identity $\ulcorner \xi \urcorner \sim \tau$ in a structure $\mathbf{M}$, then from (D1) it follows that $v = [v]^{\mathbf{M}} \circ s$, hence from (11T) we obtain $\sigma[v]^{\mathbf{M}} = \sigma[s]^{[X]}[v]^{\mathbf{M}}$ for any termoidal expression σ.

In particular, $\sigma'[v]^{\mathbf{M}} = \sigma'[s]^{[X]}[v]^{\mathbf{M}}$ and $\sigma''[v]^{\mathbf{M}} = \sigma''[s]^{[X]}[v]^{\mathbf{M}}$, so any solution of $\{\ulcorner \xi \urcorner \sim \tau, \sigma' \sim \sigma''\}$ is a solution of $\{\ulcorner \xi \urcorner \sim \tau, \sigma'[s]^{[X]} \sim \sigma''[s]^{[X]}\}$, and vice versa.

(termoids) First, let us notice that if $v : X \to |\mathbf{M}|$ is an arbitrary solution of the identity $\ulcorner \xi \urcorner \sim \tau$ in a structure $\mathbf{M}$, then from (D1) it follows $v \ll [\![v]\!]^{\mathcal{P}\mathbf{M}} \circ s$, hence from (14T1) we obtain $\sigma[\![v]\!]^{\mathcal{P}\mathbf{M}} \subseteq \sigma[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{M}}$ for any termoidal expression σ.

In particular, if v is a solution of $\{\ulcorner \xi \urcorner \sim \tau, \sigma' \sim \sigma''\}$, then $\sigma'[\![v]\!]^{\mathcal{P}\mathbf{M}} \subseteq \sigma'[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{M}}$ and $\sigma''[\![v]\!]^{\mathcal{P}\mathbf{M}} \subseteq \sigma''[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{M}}$, so $\sigma'[\![v]\!]^{\mathcal{P}\mathbf{M}} \cap \sigma''[\![v]\!]^{\mathcal{P}\mathbf{M}} \neq \varnothing$ implies $\sigma'[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{M}} \cap \sigma''[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{M}} \neq \varnothing$, hence v is a solution of $\sigma'[\![s]\!]^{[\![X]\!]} \sim \sigma''[\![s]\!]^{[\![X]\!]}$.

On the other hand, if v is a solution of $\{\ulcorner \xi \urcorner \sim \tau, \sigma'[\![s]\!]^{[\![X]\!]} \sim \sigma''[\![s]\!]^{[\![X]\!]}\}$ in an algebra $\mathbf{A}$ which is a structure of terms, then $\sigma'[\![v]\!]^{\mathcal{P}\mathbf{A}} \subseteq \sigma'[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{A}}$ and $\sigma''[\![v]\!]^{\mathcal{P}\mathbf{A}} \subseteq \sigma''[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{A}}$, hence $\sigma'[\![v]\!]^{\mathcal{P}\mathbf{A}} = \sigma'[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{A}}$ and $\sigma''[\![v]\!]^{\mathcal{P}\mathbf{A}} = \sigma''[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{A}}$, so

$$\sigma'[\![v]\!]^{\mathcal{P}\mathbf{A}} \cap \sigma''[\![v]\!]^{\mathcal{P}\mathbf{A}} = \sigma'[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{A}} \cap \sigma''[\![s]\!]^{[\![X]\!]}[\![v]\!]^{\mathcal{P}\mathbf{A}} \neq \varnothing,$$

hence v is a solution of $\sigma' \sim \sigma''$. ∎

F) **Definition.** A partial function $\mathfrak{f}$ is called *termal (termoidal) reductor*, if

(1) The argument of $\mathfrak{f}$ is a termal (termoidal) identity. The value of $\mathfrak{f}$ is a finite set of termal (termoidal) identities.

(2) If $\mathfrak{f}(\tau' \sim \tau'')$ is defined, then the system $\{\tau' \sim \tau''\}$ is termally equivalent to the system $\mathfrak{f}(\tau' \sim \tau'')$.

(3) If $\mathfrak{f}(\tau' \sim \tau'')$ is defined and some of its elements depends on ξ, then $\tau' \sim \tau''$ also depends on ξ.

(4) If the identity $\tau' \sim \tau''$ is not solving and $\mathfrak{f}(\tau' \sim \tau'')$ is not defined, then this identity is termally inconsistent.

(5) There exists no infinite sequence $\tau_1 \sim \sigma_1, \tau_2 \sim \sigma_2, \tau_3 \sim \sigma_3, \dots$, such that $\tau_{i+1} \sim \sigma_{i+1} \in \mathfrak{f}(\tau_i \sim \sigma_i)$ for any i.

G) **Definition.** A partial function $\mathfrak{f}$ is called *strong reductor*, if $\mathfrak{f}$ is termoidal reductor and whenever $\mathfrak{f}(\tau' \sim \tau'')$ is defined, the system $\{\tau' \sim \tau''\}$ is reducible to the system $\mathfrak{f}(\tau' \sim \tau'')$.[53]

H) Given a reductor $\mathfrak{f}$, it is possible to define two special transformations of a system over a `Sort`-indexed set X. We are going to call them *special solving transformations*.

First special solving transformation. If the system contains at least one identity for which $\mathfrak{f}$ is defined, then replace simultaneously all such identities $\tau' \sim \tau''$ in the system with the identities belonging to $\mathfrak{f}(\tau' \sim \tau'')$.

Second special solving transformation. If the system contains a solving identity $\ulcorner \xi \urcorner \sim \tau$ which, however, is not solving for the system, let s be the substitution

$$s\eta = \begin{cases} \tau, & \text{if } \eta = \xi, \\ \ulcorner \eta \urcorner, & \text{otherwise.} \end{cases}$$

We replace each identity $\tau' \sim \tau''$ of the system (except $\ulcorner \xi \urcorner \sim \tau$) with the identity $\tau'[s]^{[X]} \sim \tau''[s]^{[X]}$ (in the termal case) or $\tau'[\![s]\!]^{[\![X]\!]} \sim \tau''[\![s]\!]^{[\![X]\!]}$ (in the termoidal case).

I) **Proposition.** *If we apply a special solving transformation to a system, then this system is termally equivalent to the new one. In addition, if the reductor is a strong reductor, then the original system is reducible to the new one.*

<u>Proof.</u> If we apply the first special solving transformation to a system, then we are replacing some identities $\tau' \sim \tau''$ with all identities belonging to $\mathfrak{f}(\tau' \sim \tau'')$. According to the definition of a reductor (F2), the system $\{\tau' \sim \tau''\}$ is termally equivalent to $\mathfrak{f}(\tau' \sim \tau'')$, whence the original system is termally equivalent to the new system. In addition, if the reductor is a strong one, then according to (G) the system $\{\tau' \sim \tau''\}$ is reducible to the system $\mathfrak{f}(\tau' \sim \tau'')$, whence the original system is reducible to the new system.

[53]Compare this with (F2).

If we apply the second special solving transformation, then from (E) it follows that the original system is reducible to the new system (and in particular, termally equivalent). ∎

J) **Proposition.** *Given a reductor, if no special solving transformation can be applied to a system, then either the system is solved, or the system contains a termally inconsistent identity.*

Proof. If the system contains at least one identity which is not solving, then this identity is termally inconsistent, because otherwise we would be able to apply the first special solving transformation on this identity — see (F4) and (H).

Suppose all identities in the system are solving. Then all these identities must be solving for the system, because otherwise we would be able to apply the second special solving transformation. According to definition (A4), the system is solved. ∎

K) **Proposition.** *If we apply a special solving transformation to a system which is solved with respect to ξ, then the resulting system is solved with respect to ξ as well.*

Proof. Suppose that Θ is solved with respect to ξ and Θ contains an identity of the form $\ulcorner\xi\urcorner \sim \tau$. Then none of the other identities in Θ may depend on ξ.

According to (F3), if we apply the first special solving transformation to Θ, then we replace some identities $\tau' \sim \tau''$ with identities $\sigma' \sim \sigma''$ having the property that if $\sigma' \sim \sigma''$ depends on some η, then $\tau' \sim \tau''$ depends on η as well. Consequently, we replace $\tau' \sim \tau''$ with identities, of which none depends on ξ. Moreover, the identity $\ulcorner\xi\urcorner \sim \tau$ remains in the system after the transformation, so the system remains solved with respect to ξ.

If we apply the second special solving transformation to Θ, then Θ contains an identity of the form $\ulcorner\eta\urcorner \sim \sigma$ which is solving but is not solving for Θ and we apply to each identity in Θ (other than $\ulcorner\eta\urcorner \sim \sigma$) the substitution

$$s\zeta = \begin{cases} \sigma, & \text{if } \zeta = \eta, \\ \ulcorner\zeta\urcorner, & \text{otherwise.} \end{cases}$$

Since σ does not depend on ξ, if we apply s to a termal (termoidal) expression which does not depend on ξ we obtain a termal (termoidal) expression which does not depend on ξ as well. By the definition of solved system (A4), $\ulcorner\xi\urcorner \sim \tau$ is the only one identity of Θ depending on ξ, so $\xi \neq \eta$, hence if we apply s to the identity $\ulcorner\xi\urcorner \sim \tau$, $\ulcorner\xi\urcorner$ remains $\ulcorner\xi\urcorner$ and τ is replaced with termal (termoidal) expression which does not depend on ξ. On the other hand, if we apply s to some other identity of Θ (let that be $\tau' \sim \tau''$), since τ' and τ'' do not depend on ξ, the resulting identity will not depend on ξ either. According to the definition of solved system ($\mathsf{A4}$), the system remains solved with respect to ξ. ∎

L) **Proposition.** *Given a system Θ, if we apply to it the second special solving transformation about the identity $\ulcorner\xi\urcorner \sim \tau$, then the resulting system will be solved with respect to ξ.*

Proof. Let Θ be a system over the $\mathtt{Sort}$-indexed set X. Let s be the substitution

$$s\eta = \begin{cases} \tau, & \text{if } \eta = \xi, \\ \ulcorner\eta\urcorner, & \text{otherwise.} \end{cases}$$

To apply the second special solving transformation to Θ about $\ulcorner\xi\urcorner \sim \tau$ means to apply the substitution s to each identity $\tau' \sim \tau''$ of the system except to $\ulcorner\xi\urcorner \sim \tau$. Since τ does not depend on ξ (otherwise the identity $\ulcorner\xi\urcorner \sim \tau$ would not be solving and we would not be permitted to apply the second solving transformation), the substitution s maps to termal (termoidal) expressions which do not depend on ξ. Consequently, after we apply the second special solving transformation, the resulting system contains a solving identity $\ulcorner\xi\urcorner \sim \tau$, such that no other identity of the system depends on ξ. According to definition ($\mathsf{A4}$), the resulting system is solved with respect to ξ. ∎

M) **Proposition.** *Special solving transformations can not be applied to a finitary system infinitely many times.*

Proof. According to (L), if we apply the second special solving transformation about an identity $\ulcorner\xi\urcorner \sim \tau$, the resulting system is going to be solved with respect to ξ. On the other hand, according to (K), if a system is solved with respect to some ξ, then it remains solved to this ξ no matter what special special solving transformation we apply. These two facts together with the finitarity of the system imply that it is impossible to apply the second special solving transformation infinitely many times.

It remains to prove that if we do not apply the second special solving transformation, then it will be impossible to apply the first special solving transformation infinitely many times. Each time we apply the first special solving transformation we are replacing identities of the form $\tau' \sim \tau''$ with the elements of $\mathfrak{f}(\tau' \sim \tau'')$. According to the definition of reductor ($\mathsf{F1}$), we are replacing the identity $\tau' \sim \tau''$ with finitely many identities. In addition, according to the same definition ($\mathsf{F5}$), there exists no infinite sequence $\tau_1' \sim \tau_1'', \tau_2' \sim \tau_2'', \tau_3' \sim \tau_3'', \dots$, such that $\tau_{i+1}' \sim \tau_{i+1}'' \in \mathfrak{f}(\tau_i' \sim \tau_i'')$ for all i.

Consequently, the König's lemma implies that the first special solving transformation can not be applied infinitely many times. ■

N) **Definition.** (1) *Termal equaliser* for X is a function $\mathfrak{e}$, such that for any finitary termal system Θ over X, $\mathfrak{e}(\Theta)$ is a finite set of termal substitutions from X to $[X]$.

(2) *Termoidal equaliser* for X is a function $\mathfrak{e}$, such that for any finitary termoidal system Θ over X, $\mathfrak{e}(\Theta)$ is a finite set of termoidal substitutions from X to $[\![X]\!]$.

O) **Definition.** Given a reductor $\mathfrak{f}$, let $\mathfrak{e}_\mathfrak{f}$ be an equaliser, such that for any finitary system Θ, $\mathfrak{e}_\mathfrak{f}(\Theta)$ be a set of substitutions obtained in the following way.[54]

Let $\Theta_1 = \Theta$. Consider the following nondeterministic procedure: we apply repeatedly the first and the second special solving transformation (in arbitrary order) in order to produce systems $\Theta_2, \Theta_3, \Theta_4, \ldots$. On any step, if Θ_i contains an identity $\tau' \sim \tau''$ which is not solving and for which $\mathfrak{f}(\tau' \sim \tau'')$ is undefined, then stop — we do not produce Θ_{i+1}. If it becomes impossible to apply any special solving transformation on some step, then we stop as well.

According to (I), Θ_1 is termally equivalent to Θ_2, Θ_2 is termally equivalent to Θ_3, and so on, hence Θ_1 is termally equivalent to Θ_i for all i. In addition, if the reductor is a strong one, then Θ_1 is reducible to Θ_2, Θ_2 is reducible to Θ_3, and so on, hence Θ_1 is reducible to Θ_i for all i.

Suppose we reach Θ_i, such that Θ_i contains an identity $\tau' \sim \tau''$ which is not solving and for which $\mathfrak{f}(\tau' \sim \tau'')$ is undefined. In this case (F4) implies that $\tau' \sim \tau''$ is termally inconsistent, so Θ_i is termally inconsistent, hence Θ_1 is termally inconsistent. Let $\mathfrak{e}_\mathfrak{f}(\Theta) = \varnothing$, in this case.

Otherwise, that is if we never reach such Θ_i, then according to (M), after finitely many steps we will obtain a system Θ_n, such that no special solving transformation can be applied to Θ_n. From (J) it follows that the system Θ_n is solved, hence from (D) we obtain a substitution s, such that all solutions of Θ_n are instances of s and all instances of s in an algebra which is a structure of terms are solutions of Θ_n. But Θ_1 is termally equivalent to Θ_n, hence Θ_1 and Θ_n have same solutions in algebras which are structures of terms, so the solutions of Θ_1 in any algebra which is a system of terms are exactly the instances of s. In addition, if the reductor is a strong one, then Θ_1 is reducible to Θ_n, hence all solutions of Θ_1 are solutions of Θ_n, so all solutions of Θ_1 are instances of s. Let $\mathfrak{e}_\mathfrak{f}(\Theta)$ be some finite non-empty set of

[54] Notice that that this set is not determined uniquely. Therefore, in some cases we may need to use some weak form of the axiom of choice in order to prove that $\mathfrak{e}_\mathfrak{f}$ exists.

substitutions s obtained in this way (not necessarily all such substitutions, only one will suffice).

P) **Definition.** (1) An equaliser $\mathfrak{e}$ is *termally sound*, if for any system Θ, any instance of an element of $\mathfrak{e}(\Theta)$ in an algebra of terms is a solution of Θ in this algebra.

(2) An equaliser $\mathfrak{e}$ is *termally complete*, if for any system Θ, any solution of Θ in an algebra of terms is an instance of all elements of $\mathfrak{e}(\Theta)$.

(3) An equaliser $\mathfrak{e}$ is *near-complete*, if for any termally consistent system Θ, any solution of Θ (in any structure) is an instance of all elements of $\mathfrak{e}(\Theta)$.

Q) The reasoning in (O) show that for any reductor $\mathfrak{f}$, the equaliser $\mathfrak{e}_{\mathfrak{f}}$ is both termally sound and termally complete. In addition, if $\mathfrak{f}$ is a strong reductor, then $\mathfrak{e}_{\mathfrak{f}}$ is near-complete.

R) **Definition.** (1) Given a set Δ of termal expressions over the $\mathtt{Sort}$-indexed set X, the termal substitution $s : X \to [X]$ is *unifier* of Δ if for any $\tau', \tau'' \in \Delta$, $\tau'[s]^{[X]} = \tau''[s]^{[X]}$.

(2) Given a set Δ of termal expressions over the $\mathtt{Sort}$-indexed set X, the termal substitution $s : X \to [X]$ is *most general unifier* of Δ, if s is unifier of Δ and for any unifier s' of Δ, there exists a substitution $s'' : X \to [X]$, such that $s'\xi = (s\xi)[s'']^{[X]}$ for any $\xi \in X$ (or, equivalently, $s' = [s'']^{[X]} \circ s$).

S) It is a well known fact that if a finite set of terms has an unifier, then it has a most general unifier. The most general unifier is unique up to renaming of the variables/names. Let $\mathfrak{mgu}$ be equaliser, such that for any a termal system Θ of the form $\{\tau \sim \sigma_1, \dots, \tau \sim \sigma_n\}$, $\mathfrak{mgu}(\Theta)$ is one-element set containing some most general unifier of the set $\{\tau, \sigma_1, \dots, \sigma_n\}$, if such unifier exists, or the empty set, if such unifier does not exist. If s is some most general unifier of the set $\{\tau, \sigma_1, \dots, \sigma_n\}$, then s is variant of the element of $\mathfrak{mgu}(\Theta)$.

T) **Definition.** Given a termoidal substitution $s : X \to [\![X]\!]$, define a termal substitution $\overline{s} : X \to [X]$, such that $\overline{s}\xi = (s\xi)[\![\mathrm{nam}_X]\!]^{[X]}$ for any ξ. In other words, $\overline{s} = [\![\mathrm{nam}_X]\!]^{[X]} \circ s$.

U) **Proposition.** *Let $\mathfrak{e}$ be a termoidal equaliser for X, Θ be the system $\{\tau \sim \sigma_1, \dots, \tau \sim \sigma_n\}$ and Δ be the system $\{\tau[\![\mathrm{nam}_X]\!]^{[X]}, \sigma_1[\![\mathrm{nam}_X]\!]^{[X]}, \dots, \sigma_n[\![\mathrm{nam}_X]\!]^{[X]}\}$. Then:*

(1) *If $\mathfrak{e}$ is termally sound and $s \in \mathfrak{e}(\Theta)$, then $\overline{s}$ is unifier of Δ.*

(2) *If $\mathfrak{e}$ is both termally sound and termally complete and Δ has an unifier, then there exists some $s \in \mathfrak{e}(\Theta)$, such that $\overline{s}$ is most general unifier*

of Δ.

Proof. **(1)** According to the definition of "termally sound equaliser" (P1), any instance of s in a structure of terms is a solution of Θ. Notice that $\overline{s} = [\![\mathrm{nam}_X]\!]^{[X]} \circ s$ is an instance of s in $[X]$, so $\overline{s}$ has to be a solution of Θ in $[X]$, hence $\tau[\![\overline{s}]\!]^{[X]} = \sigma_i[\![\overline{s}]\!]^{[X]}$ for any $i \in \{1, \dots, n\}$. According to (16M), $\tau[\![\overline{s}]\!]^{[X]} = (\tau[\![\mathrm{nam}_X]\!]^{[X]})[\overline{s}]^{[X]}$ and $\sigma_i[\![\overline{s}]\!]^{[X]} = (\sigma_i[\![\mathrm{nam}_X]\!]^{[X]})[\overline{s}]^{[X]}$, so $(\tau[\![\mathrm{nam}_X]\!]^{[X]})[\overline{s}]^{[X]} = (\sigma_i[\![\mathrm{nam}_X]\!]^{[X]})[\overline{s}]^{[X]}$. Consequently, $\overline{s}$ is an unifier of Δ.

(2) Suppose that $s_1 : X \to [X]$ is most general unifier of Δ. Then $(\tau[\![\mathrm{nam}_X]\!]^{[X]})[s_1]^{[X]} = (\sigma_i[\![\mathrm{nam}_X]\!]^{[X]})[s_1]^{[X]}$ for any i. According to (16N), $(\tau[\![\mathrm{nam}_X]\!]^{[X]})[s_1]^{[X]} = \tau[\![s_1]\!]^{[X]}$ and $(\sigma_i[\![\mathrm{nam}_X]\!]^{[X]})[s_1]^{[X]} = \sigma_i[\![s_1]\!]^{[X]}$, so $\tau[\![s_1]\!]^{[X]} = \sigma_i[\![s_1]\!]^{[X]}$, hence s_1 is a solution of Θ in $[X]$. But $\mathfrak{e}$ is termally complete, hence there exists $s \in \mathfrak{e}(\Theta)$ and a Sort-indexed function $s_2 : X \to [X]$, such that $s_1 \ll [\![s_2]\!]^{\mathcal{P}[X]} \circ s$, which implies $s_1 = [\![s_2]\!]^{[X]} \circ s$. According to (16N), $[\![s_2]\!]^{[X]} = [s_2]^{[X]} \circ [\![\mathrm{nam}_X]\!]^{[X]}$, so $s_1 = [s_2]^{[X]} \circ [\![\mathrm{nam}_X]\!]^{[X]} \circ s = [s_2]^{[X]} \circ \overline{s}$.

Suppose that s_3 is some arbitrary unifier of Δ. Since s_1 is most general unifier, there exists a termal substitution s_4, such that $s_3 = [s_4]^{[X]} \circ s_1$, so $s_3 = [s_4]^{[X]} \circ [s_2]^{[X]} \circ \overline{s}$, hence, according to (11T), $s_3 = [s_4^{[X]} \circ s_2]^{[X]} \circ \overline{s}$. ∎

V) **Proposition.** *Let τ and σ be termal expressions over the* Sort-*indexed set X such that τ and σ have disjoint dependency. Let σ' be a variant of σ, such that σ' has disjoint dependency with τ. Let the* Sort-*indexed functions $f, g : X \to X$ be such that $\sigma' = \sigma[f]$, $\sigma = \sigma'[g]$ and $f\xi = \xi$ and $g\xi = \xi$ whenever the name $\ulcorner\xi\urcorner$ occurs in τ.*

If $s : X \to [X]$ is a most general unifier of τ and σ, then $s \circ g$ is a most general unifier of τ and σ'.

Proof. Since $f\xi = \xi$ and $g\xi = \xi$ whenever the name $\ulcorner\xi\urcorner$ occurs in τ, $\tau[f] = \tau$ and $\tau[g] = \tau$. Consequently, $\tau[s \circ g]^{[X]} = \tau[g][s]^{[X]} = \tau[s]^{[X]} = \sigma[s]^{[X]} = \sigma'[g][s]^{[X]} = \sigma'[s \circ g]^{[X]}$, so $s \circ g$ is an unifier of τ and σ'.

Suppose that s' is an unifier of τ and σ'. Then $\tau[s' \circ f]^{[X]} = \tau[f][s']^{[X]} = \tau[s']^{[X]} = \sigma'[s']^{[X]} = \sigma[f][s']^{[X]} = \sigma[s' \circ f]^{[X]}$, so $s' \circ f$ is an unifier of τ and σ. But s is a most general unifier of these termal expressions, so there exists a substitution $s'' : X \to [X]$, such that $s' \circ f = ([s'']^{[X]}) \circ s$. Without loss of generality we may assume that $f \circ g = \mathrm{id}_X$, so $s' = s' \circ f \circ g = ([s'']^{[X]}) \circ s \circ g$. Consequently, $s \circ g$ is a most general unifier of τ and σ'. ∎

W) **Definition.** We say that some property is true for *almost any* normal algebra if there exists a finite set of termally inconsistent systems, such that the property is true for any normal algebra $\mathbf{A}$, such that none of the systems belonging to this set has a solution in **A**.

X) **Lemma.** *If a property is true for almost any normal algebra and another property also is true for almost any normal algebra, then the conjunction of both properties is true for almost any normal algebra.*

Proof. Let Γ' be a set, such that the the first property is true for any algebra **A**, such that none of the systems belonging to Γ' has a solution in **A**. Let Γ'' be similarly defined for the second property. Then the conjunction of both properties will be true for any algebra **A**, such that none of the systems belonging to $\Gamma' \cup \Gamma''$ has a solution in **A**. Notice also that any system belonging to $\Gamma' \cup \Gamma''$ is termally inconsistent. ∎

Resolutive Deduction with Termoids

§19. ABSTRACT DEDUCTION

A) **Definition.** Given a set F, the relation $\Gamma \vdash \varphi$, where Γ and φ are respectively a subset and an element of F, is a *deductive relation* over F, if the following two conditions hold:

(1) If $\Gamma \subseteq F$ and $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$. (*reflexivity*)
(2) If $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for all $\psi \in \Gamma$, then $\Delta \vdash \varphi$. (*transitivity*)

Whenever $\vdash$ is a deductive relation, the usual notational convention will be used: we will write $\Gamma, \varphi \vdash \psi$ instead of $\Gamma \cup \{\varphi\} \vdash \psi$ and $\varphi, \psi \vdash \chi$ instead of $\{\varphi, \psi\} \vdash \chi$, etc.

B) **Proposition (*monotonicity*).** *If $\vdash$ is a deductive relation over F, then $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta \subseteq F$ imply $\Delta \vdash \varphi$.*

Proof. By reflexivity, $\Delta \vdash \psi$ for any $\psi \in \Gamma$. By transitivity, this implies $\Delta \vdash \varphi$. ∎

Deductive relations can be defined "by minimality". This is formalised by the following proposition.

C) **Proposition.** *Suppose Φ is a set and $\mathfrak{p}$ is a binary relation connecting a subset of Φ with an element of Φ. Then there exists a minimal deductive relation $\vdash$, such that $\mathfrak{p}(\Gamma, \varphi)$ implies $\Gamma \vdash \varphi$.*

Proof. Obviously an intersection of deductive relations is a deductive relation. Moreover, an intersection of relations extending $\mathfrak{p}$ is a relation extending $\mathfrak{p}$. Consequently, the intersection of all deductive relations over Φ extending $\mathfrak{p}$ is the minimal deductive relation extending $\mathfrak{p}$. ∎

As usually, to every "definition by minimality" there is a corresponding induction principle. For deductive relations, however, in most cases we can use the following simpler induction principle:

D) **Proposition (*simple inductive principle*).** *Given subsets Γ and Δ of Φ, let $\mathfrak{p}$ be a binary relation connecting a subset of Φ with an element of Φ and $\vdash$ be the minimal deductive relation extending $\mathfrak{p}$. Suppose that:*

1. *$\Gamma \subseteq \Delta$ and*
2. *if $\Theta \subseteq \Delta$ and $\mathfrak{p}(\Theta, \varphi)$, then $\varphi \in \Delta$ (for any Θ and φ).*

Then $\Gamma \vdash \varphi$ implies $\varphi \in \Delta$ for any φ.

Proof. Let $\mathfrak{q}(\Theta, \varphi)$ be the relation

$$\Theta \vdash \varphi \text{ and } (\Theta \text{ is not a subset of } \Delta \text{ or } \varphi \in \Delta)$$

We are going to prove that $\mathfrak{q}$ is a deductive relation extending $\mathfrak{p}$. This will imply that $\mathfrak{q}$ extends $\vdash$, since $\vdash$ is the minimal deductive relation extending $\mathfrak{p}$. Therefore, from $\Gamma \vdash \varphi$ it will follow $\mathfrak{q}(\Gamma, \varphi)$, so Γ is not a subset of Δ or $\varphi \in \Delta$. But Γ is a subset of Δ, so $\varphi \in \Delta$.

Condition (A1) is obvious: if $\varphi \in \Theta$, then on one hand, $\Theta \vdash \varphi$ (because $\vdash$ is deductive) and, on the other hand, Θ is not a subset of Δ or $\varphi \in \Theta \subseteq \Delta$.

In order to prove (A2), suppose that $\mathfrak{q}(\Theta, \varphi)$ and for any $\psi \in \Theta$, $\mathfrak{q}(\Xi, \psi)$. Since $\mathfrak{q}$ implies $\vdash$, $\Theta \vdash \varphi$ and for any $\psi \in \Theta$, $\Xi \vdash \psi$. But $\vdash$ is deductive, so $\Xi \vdash \varphi$. If Ξ is not a subset of Δ, then we obtain $\mathfrak{q}(\Xi, \varphi)$. Otherwise, let $\Xi \subseteq \Delta$. Since for any $\psi \in \Theta$, $\mathfrak{q}(\Xi, \psi)$, we obtain that for any $\psi \in \Theta$, $\psi \in \Delta$. Therefore, $\Theta \subseteq \Delta$, so from $\mathfrak{q}(\Theta, \varphi)$ we obtain $\varphi \in \Delta$, whence $\mathfrak{q}(\Xi, \varphi)$.

It remains to prove that $\mathfrak{q}$ extends $\mathfrak{p}$. Suppose that $\mathfrak{p}(\Theta, \varphi)$ for some Θ and φ. According to condition 2 of the Proposition, Θ is not a subset of Δ or $\varphi \in \Delta$. On the other hand $\mathfrak{p}$ implies $\vdash$. Consequently, $\mathfrak{q}(\Theta, \varphi)$ is true. ∎

E) We are going to call the condition $\Delta' \subseteq \Delta$ from (D) "induction hypothesis".

F) **Definition.** A deductive relation $\vdash$ is *finitary* if $\Gamma \vdash \varphi$ implies that there exists a finite subset Δ of Γ, such that $\Delta \vdash \varphi$.

G) **Proposition.** *Given a set Φ, let $\mathfrak{p}(\Gamma, \varphi)$ be a binary relation connecting a subset of Φ with an element of Φ, such that whenever $\mathfrak{p}(\Gamma, \varphi)$ is true, there exists a finite subset Δ of Γ, such that $\mathfrak{p}(\Delta, \varphi)$ is true. Then the minimal deductive relation extending $\mathfrak{p}$ is finitary.*

Proof. Let $\vdash$ be the minimal deductive relation extending $\mathfrak{p}$ and let $\Gamma \vdash' \delta$ be true if and only if $\Delta \vdash \delta$ for some finite subset Δ of Γ. We are going to prove that $\Gamma \vdash' \varphi$ is a deductive relation extending $\mathfrak{p}$. Since $\Gamma \vdash \varphi$ is the minimal deductive relation extending $\mathfrak{p}$, we obtain the required.

If $\mathfrak{p}(\Gamma, \varphi)$ is true, then $\mathfrak{p}(\Delta, \varphi)$ is true for some finite subset Δ of Γ, hence $\Delta \vdash \varphi$, so $\Gamma \vdash' \varphi$. Therefore, $\Gamma \vdash' \varphi$ extends $\mathfrak{p}(\Gamma, \varphi)$.

If $\varphi \in \Gamma$, then $\{\varphi\} \subseteq \Gamma$, so $\{\varphi\} \vdash \varphi$ implies $\Gamma \vdash' \varphi$. Therefore, $\Gamma \vdash' \varphi$ satisfies (A1).

Suppose that $\Gamma \vdash' \varphi$ and for any $\psi \in \Gamma$, $\Delta \vdash' \psi$. Then $\Gamma' \vdash \varphi$ for some finite subset Γ' of Γ and for any $\psi \in \Gamma$ there exists a finite subset Δ_ψ of Δ, such that $\Delta_\psi \vdash \psi$. Let $\Delta' = \bigcup_{\psi \in \Gamma'} \Delta_\psi$. The set Δ' is a finite union of finite sets, hence it is finite. Moreover, from (B) it follows $\Delta' \vdash \psi$ for any $\psi \in \Gamma'$. By transitivity, $\Delta' \vdash \varphi$, hence $\Delta \vdash' \varphi$. Therefore, $\Gamma \vdash' \varphi$ satisfies (A2). ∎

H) **Lemma.** *Any injective monotone function from a linearly ordered set to a partially ordered set is an order-embedding.*

Proof. Let $\mathfrak{f} : X \to Y$ be a monotone function from the linearly ordered set X to the partially ordered set Y. This means that $\eta \leq \xi$ implies $\mathfrak{f}\eta \leq \mathfrak{f}\xi$ for any $\eta, \xi \in X$. We have to prove that $\mathfrak{f}\eta \leq \mathfrak{f}\xi$ implies $\eta \leq \xi$. But if $\eta \leq \xi$ were not true, then $\xi \leq \eta$ because the order on X is linear. Then the monotonicity would imply $\mathfrak{f}\xi \leq \mathfrak{f}\eta$, so by antisymmetry we would obtain $\mathfrak{f}\eta = \mathfrak{f}\xi$ which would contradict the injectivity of $\mathfrak{f}$. ∎

I) **Lemma (*Bourbaki-Witt*).** *Let $\Theta = \langle \Theta, \preceq \rangle$ be a partially ordered set, such that every subset of Θ which is well-ordered by $\preceq$ has an upper bound.*[55] *Let $\mathfrak{f} : \Theta \to \Theta$ be an inflationary function, i.e. $\mathfrak{a} \preceq \mathfrak{f}\mathfrak{a}$ for any $\mathfrak{a} \in \Theta$. Then the function $\mathfrak{f}$ has a fixed point, i.e. there is $\mathfrak{a} \in \Theta$, such that $\mathfrak{f}\mathfrak{a} = \mathfrak{a}$.*

Proof. Let $\mathfrak{a} \prec \mathfrak{b}$ means "$\mathfrak{a} \preceq \mathfrak{b}$ and $\mathfrak{a} \neq \mathfrak{b}$".

Suppose that $\mathfrak{f}$ has no fixed points, i.e. $\mathfrak{a} \prec \mathfrak{f}\mathfrak{a}$ for all $\mathfrak{a} \in \Theta$. By transfinite recursion we are going to define an injective monotone map $\mathfrak{g}$ from the class of all ordinals to the set Θ, which is clearly a contradiction.

The set Θ is non-empty because the empty set is a well-ordered subset of Θ, so it has to have some upper bound. Define $\mathfrak{g}0$ to be an arbitrary element of Θ. Let $\mathfrak{g}(\xi + 1) = \mathfrak{f}(\mathfrak{g}\xi)$. And if ξ is a limit ordinal, then define $\mathfrak{g}\xi$ to be some upper bound of $\{\mathfrak{g}\eta : \eta < \xi\}$ if such upper bound exists[56] or

[55] In its usual formulation, the lemma of Bourbaki-Witt is weakened by the requirement that any chain (not necessarily a well-ordered one) has a least upper bound.

[56] We have to use the axiom of choice here. In its usual formulation, the Lemma of Bourbaki-Witt requires the existence of least upper bounds. Then we can define $\mathfrak{g}\xi$ to

$\mathfrak{g}\xi = \mathfrak{g}0$, otherwise.

In order to prove that $\eta < \xi$ implies $\mathfrak{g}\eta \prec \mathfrak{g}\xi$ we will use transfinite induction on ξ.

If ξ is a successor ordinal, then $\xi = \xi' + 1$ for some ξ', so $\mathfrak{g}\xi' \prec \mathfrak{f}(\mathfrak{g}\xi') = \mathfrak{g}(\xi' + 1) = \mathfrak{g}\xi$, hence if $\eta < \xi$, then $\eta \leq \xi'$, so by induction hypothesis $\mathfrak{g}\eta \preceq \mathfrak{g}\xi'$, hence $\mathfrak{g}\eta \prec \mathfrak{g}\xi$.

On the other hand, if ξ is a limit ordinal, then by induction hypothesis $\eta' < \eta''$ implies $\mathfrak{g}\eta' \prec \mathfrak{g}\eta''$ for any ordinals η' and η'', such that $\eta'' < \xi$. This and (H) imply that the set $\{\mathfrak{g}\eta : \eta < \xi\}$ is well-ordered by $\prec$, hence this set has an upper bound, so by definition, $\mathfrak{g}\xi$ is some upper bound of this set, hence $\mathfrak{g}\eta \preceq \mathfrak{g}\xi$ for any η, so $\mathfrak{g}\eta \prec \mathfrak{g}(\eta + 1)$ implies $\mathfrak{g}\eta \prec \mathfrak{g}\xi$ for any η.[57] ∎

The following theorem can be used to prove the completeness of deductive relations.

J) **Theorem.** *Suppose we are given a set, whose elements are called* sentences *and a set whose elements are called* conditions*. In addition, let $\prec$ be an irreflexive partial order on the conditions, $\mathfrak{a} \models \varphi$ be a partial binary relation*[58] *connecting a condition $\mathfrak{a}$ with a sentence φ and "$\mathfrak{a}$ forces Γ" be a binary relation connecting a condition $\mathfrak{a}$ with a set Γ of sentences.*

Suppose the following statements are true:

(1) *If $\mathfrak{a}$ forces a set Γ then $\mathfrak{a}$ forces all subsets of Γ.*

(2) *If $\mathfrak{a}$ forces $\{\varphi\}$ then $\mathfrak{a} \models \varphi$ is either true or undefined.*

(3) *If $\mathfrak{a}$ forces a set Γ and $\mathfrak{a} \models \varphi$ is undefined for at least one sentence φ, then there exists a condition $\mathfrak{b}$, such that $\mathfrak{a} \prec \mathfrak{b}$ and $\mathfrak{b}$ forces Γ.*

(4) *Any non-empty well-ordered by $\prec$ set of conditions Λ has an upper bound $\mathfrak{a}$, such that $\mathfrak{a}$ forces any set of sentences which is forced by all elements of Λ.*

Then, for any set of sentences Γ, which is forced by by at least one condition, there exists a condition $\mathfrak{b}$, such that $\mathfrak{b} \models \varphi$ for all $\varphi \in \Gamma$.

Proof. Take an arbitrary set of sentences Γ, which is forced by at least one condition. Define the following set of conditions:

$$\Theta = \{\mathfrak{a} : \mathfrak{a} \text{ forces } \Gamma\}$$

As a set of conditions, the set Θ is partially ordered by $\prec$. It is non-empty, because Γ is forced by at least one condition. Let Λ be an arbitrary subset

be the least upper bound of $\{\mathfrak{g}\eta : \eta < \xi\}$ and the Lemma can be proved without the axiom of choice.

[57] In fact we we have proved somewhat stronger proposition, namely that for any $\mathfrak{a} \in \Theta$ the function $\mathfrak{f}$ has a fixed point which is comparable to $\mathfrak{a}$, i.e. there is $\mathfrak{b} \in \Theta$, such that $\mathfrak{f}\mathfrak{b} = \mathfrak{b}$ and $\mathfrak{a} \preceq \mathfrak{b}$. In order to achieve this, simply define $\mathfrak{g}0 = \mathfrak{a}$.

[58] This means that $\mathfrak{a} \models \varphi$ is either true, or false, or undefined.

of Θ, which is well-ordered by $\prec$. If $\Lambda = \varnothing$, then any element of Θ is an upper bound of Λ. If $\Lambda \neq \varnothing$, then (4 implies Λ has an upper bound $\mathfrak{a}$, such that $\mathfrak{a}$ forces any set which is forced by all elements of Λ. Since $\Lambda \subseteq \Theta$, all elements of Λ force Γ, so $\mathfrak{a}$ forces Γ, hence $\mathfrak{a} \in \Theta$. Thus, we have proved that any subset of Θ which is well-ordered by $\prec$ has an upper bound belonging to Θ.

Now we are going to define an inflationary function $\mathfrak{f} : \Theta \to \Theta$. Define $\mathfrak{f}\mathfrak{a} = \mathfrak{a}$, if $\mathfrak{a} \models \varphi$ is defined for all sentences φ. Otherwise, $\mathfrak{a} \models \varphi$ is undefined for at least one sentence φ, so (3 and the axiom of choice permit us to define $\mathfrak{f}\mathfrak{a}$ in a way, such that $\mathfrak{a} \prec \mathfrak{f}\mathfrak{a}$.

From (I) it follows that the function $\mathfrak{f}$ has some fixed point $\mathfrak{b}$. According to the definition of $\mathfrak{f}$, this is possible only if $\mathfrak{b} \models \varphi$ is defined for all sentences φ.

Let φ be an arbitrary element of Γ. From $\mathfrak{b} \in \Theta$ it follows that $\mathfrak{b}$ forces Γ, so (1 implies that $\mathfrak{b}$ forces $\{\varphi\}$, hence (2 implies that $\mathfrak{b} \models \varphi$ is either true or undefined. But we have already proved that $\mathfrak{b} \models \varphi$ is always defined, so $\mathfrak{b} \models \varphi$ has to be true. Consequently, $\mathfrak{b} \models \varphi$ is defined and true for all $\varphi \in \Gamma$. ∎

§20. CLAUSES AND CLAUSOIDS

A) **Definition.** (1) *Literal* is an atomic formula or negation of an atomic formula. *Literaloid* is an atomic formuloid or negation of an atomic formuloid.

(2) The following notation will be used. If φ is an atomic formula or atomic formuloid, then $\overline{\varphi} = \neg\varphi$ and $\overline{\neg\varphi} = \varphi$. Obviously $\overline{\overline{\lambda}} = \lambda$ for any literal or literaloid λ.

B) **Definition.** (1) *Clauses* over a $\mathtt{Sort}$-indexed set X are defined inductively with the following rules: $\bot$ is a clause over X; any literal over X is a clause over X; if λ is a literal over X and δ is a clause over X and $\delta \neq \bot$, then $\lambda \vee \delta$ is a clause over X.

(2) *Clausoids* over X are defined analogously but with literaloids instead of literals.

C) **Definition.** (1) Given an algebra $\mathbf{A}$ and a clause δ over X, all clauses of the form $\delta[v]^{\mathbf{A}}$, where v is an arbitrary $\mathtt{Sort}$-indexed function from X to $|\mathbf{A}|$, are called *instances* of δ in $\mathbf{A}$.

(2) Given an algebra $\mathbf{A}$ and a clausoid δ over X, all clauses belonging to sets of the form $\delta[\![v]\!]^{\mathcal{P}\mathbf{A}}$, where v is an arbitrary $\mathtt{Sort}$-indexed function from X to $|\mathbf{A}|$, are called *instances* of δ in $\mathbf{A}$. Notice that the instances of

a clausoid are not clausoids but clauses.

D) **Example.** (1) Consider the clause $\mathtt{p}(\ulcorner\xi\urcorner,\ulcorner\eta\urcorner)$, where $\xi \in X$ and $\eta \in X$ have suitable sorts. Then the instances of this clause in an algebra $\mathbf{A}$ are all clauses $\mathtt{p}(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner)$, such that $\alpha \in |\mathbf{A}|$ and $\beta \in |\mathbf{A}|$ have suitable sorts.

(2) Consider the clausoid $\mathtt{p}(\ulcorner\ulcorner\xi\urcorner\urcorner,\ulcorner\ulcorner\eta\urcorner\urcorner)$, where $\xi \in X$ and $\eta \in X$ have suitable sorts. Then the instances of this clausoid in an algebra $\mathbf{A}$ are all clauses $\mathtt{p}(\ulcorner\alpha\urcorner,\ulcorner\beta\urcorner)$, such that $\alpha \in |\mathbf{A}|$ and $\beta \in |\mathbf{A}|$ have suitable sorts.

E) **Corollary.** *Let the algebra $\mathbf{A}$ be a structure of terms and δ be a clausoid over X. Then a clausoid is an instance of δ in $\mathbf{A}$ if and only if it is equal to $\delta[\![v]\!]^{\mathbf{A}}$ for some* `Sort`*-indexed function $v : X \to |\mathbf{A}|$.*

F) **Definition.** A clause or literal is *relational* if it is a relational formula.

G) **Proposition.** *All instances of a clause or clausoid are relational clauses.*

Proof. By definition (C), the instances of a clause or clausoid belong to the logical carrier of an algebra. By definition (12C2) all elements of the logical carrier of algebra are relational formulae. ∎

H) **Proposition.** *Given a logical structure $\mathbf{M}$, a clause or clausoid δ over X is universally valid in $\mathbf{M}$ if and only if all instances of δ in $\partial\mathbf{M}$ are true in $\mathbf{M}$.*

Proof. The proof differs depending on whether δ is a clause or a clausoid.

(clause) By definition (13A2), a clause δ over X is universally valid in $\mathbf{M}$ if and only if for any assignment function $v : X \to |\mathbf{M}|$ we have $\delta[v]^{\mathbf{M}} = 1$. But clauses are formulae and formulae may not contain names of logical sort, hence this is so if and only if for any assignment function $v : X \to |\partial\mathbf{M}|$ we have $\delta[v]^{\mathbf{M}} = 1$. Because of (13E) this is so if and only if for any assignment function $v : X \to |\partial\mathbf{M}|$ we have $(\delta[v]^{\partial\mathbf{M}})^{\mathbf{M}} = 1$. This is so if and only if all instances of δ in $\partial\mathbf{M}$ are true in $\mathbf{M}$.

(clausoid) By definition (14X) a clausoid δ over X is universally valid in $\mathbf{M}$ if and only if for any assignment function $v : X \to |\mathbf{M}|$ we have $\delta[\![v]\!]^{\mathcal{P}\mathbf{M}} = \{1\}$. Because of (14I5), this is so if and only if for any assignment function $v : X^\circ \to |\mathbf{M}|^\circ$ we have $\delta[\![v]\!]^{\mathcal{P}\mathbf{M}} = \{1\}$. Considering that $|\mathbf{M}|^\circ = |\partial\mathbf{M}|^\circ$, this is so if and only if for any assignment function $v : X^\circ \to |\partial\mathbf{M}|^\circ$ we have $\delta[\![v]\!]^{\mathcal{P}\mathbf{M}} = \{1\}$, if and only if for any assignment function $v : X \to |\partial\mathbf{M}|$ we have $\delta[\![v]\!]^{\mathcal{P}\mathbf{M}} = \{1\}$. Because of (14W2), this is so if and only if for any assignment function $v : X \to |\partial\mathbf{M}|$ we have $\{\psi^{\mathbf{M}} : \psi \in \delta[\![v]\!]^{\mathcal{P}(\partial\mathbf{M})}\} = \{1\}$, if and only if for any assignment function $v : X \to |\partial\mathbf{M}|$ all elements of $\delta[\![v]\!]^{\mathcal{P}(\partial\mathbf{M})}$ are true in $\mathbf{M}$, if and only if all instances of δ in $\partial\mathbf{M}$ are true in $\mathbf{M}$. ∎

I) **Corollary.** *If a set Γ of clauses (clausoids) is not universally satisfiable in an algebra $\mathbf{A}$, then the set of all instances in $\mathbf{A}$ of the elements of Γ is not satisfiable in $\mathbf{A}$.*

Proof. Suppose Γ is not universally satisfiable in $\mathbf{A}$ but the set Γ' of all instances in $\mathbf{A}$ of the elements of Γ is satisfiable in $\mathbf{A}$. Then there exists a logical structure $\mathbf{M}$, such that $\partial\mathbf{M} = \mathbf{A}$ and the elements of Γ' are true in $\mathbf{M}$. This, however, contradicts (H). ∎

J) **Definition.** (1) The *sequence* of a clause (clausoid) is defined inductively: $\langle\rangle$ is the sequence of $\bot$; if λ is a literal (literaloid), then $\langle\lambda\rangle$ is the sequence of λ; if $\langle\lambda_1, \dots, \lambda_n\rangle$ is the sequence of δ, then $\langle\lambda, \lambda_1, \dots, \lambda_n\rangle$ is the sequence of $\lambda \vee \delta$.

(2) A literal (literaloid) belongs to a clause (clausoid) if it belongs to its sequence.

K) **Corollary.** (1) *Any clause over X is a formula over X and any clausoid over X is a formuloid over X.*

(2) *Any clause or clausoid over X has unique sequence.*

(3) *The sequence of a clause over X is a sequence of literals over X. The sequence of a clausoid over X is a sequence of literaloids over X.*

(4) *For any finite sequence of literals (literaloids) over X there is unique clause (clausoid) over X whose sequence is the given sequence.*

Proof. By trivial induction. ∎

L) **Proposition.** (1) *If δ is a clause over X and $h : [X] \to [Y]$ is an arbitrary homomorphism, then $h\delta$ is a clause over Y. Moreover, if $\langle\lambda_1, \dots, \lambda_n\rangle$ is the sequence of δ, then $\langle h\lambda_1, \dots, h\lambda_n\rangle$ is the sequence of $h\delta$.*

(2) *If δ is a clausoid over X and $h : [\![X]\!] \to [\![Y]\!]$ is an arbitrary homomorphism, then $h\delta$ is a clausoid over Y. Moreover, if $\langle\lambda_1, \dots, \lambda_n\rangle$ is the sequence of δ, then $\langle h\lambda_1, \dots, h\lambda_n\rangle$ is the sequence of $h\delta$.*

(3) *If δ is a clausoid over X and $h : [\![X]\!] \to [Y]$ is an arbitrary homomorphism, then $h\delta$ is a clause over Y. Moreover, if $\langle\lambda_1, \dots, \lambda_n\rangle$ is the sequence of δ, then $\langle h\lambda_1, \dots, h\lambda_n\rangle$ is the sequence of $h\delta$.*

(4) *If δ is a clause over X and $h : [X] \to [\![Y]\!]$ is an arbitrary homomorphism, then $h\delta$ is a clausoid over Y. Moreover, if $\langle\lambda_1, \dots, \lambda_n\rangle$ is the sequence of δ, then $\langle h\lambda_1, \dots, h\lambda_n\rangle$ is the sequence of $h\delta$.*

(5) *Given an algebra* $\mathbf{A}$*, if* δ *is a clause over* X *and* $h : [X] \to \mathbf{A}$ *is an arbitrary homomorphism, then* $h\delta$ *is a relational clause over* $|\mathbf{A}|$*. Moreover, if* $\langle \lambda_1, \dots, \lambda_n \rangle$ *is the sequence of* δ*, then* $\langle h\lambda_1, \dots, h\lambda_n \rangle$ *is the sequence of* $h\delta$*.*

(6) *Given an algebra* $\mathbf{A}$*, if* δ *is a clausoid over* X *and* $h : [\![X]\!] \to \mathcal{P}\mathbf{A}$ *is an arbitrary quasimorphism, then* $h\delta$ *is a set of relational clauses over* $|\mathbf{A}|$*. Moreover, if* $\langle \lambda_1, \dots, \lambda_n \rangle$ *is the sequence of* δ*, then the sequences of the elements of* $h\delta$ *are all sequences* $\langle \mu_1, \dots, \mu_n \rangle$*, such that* $\mu_1 \in h\lambda_1, \dots, \mu_n \in h\lambda_n$*.*

(7) *Given an algebra* $\mathbf{A}$*, if* δ *is a clausoid over* X *and* $h : [\![X]\!] \to \mathbf{A}$ *is an arbitrary homomorphism, then* $h\delta$ *is a relational clause over* $|\mathbf{A}|$*. Moreover, if* $\langle \lambda_1, \dots, \lambda_n \rangle$ *is the sequence of* δ*, then* $\langle h\lambda_1, \dots, h\lambda_n \rangle$ *is the sequence of* $h\delta$*.*

Proof. **(1)** By simple induction on the length of the sequence. When the length is 0, the sequence is $\langle\rangle$, $\delta = \bot$ and the proposition is trivial. When the length is 1, then the sequence is $\langle \lambda \rangle$ for some literal λ and $\delta = \lambda$. If $\lambda = \mathtt{p}(\tau_1, \dots, \tau_n)$, then $h\delta = h(\mathtt{p}^{[X]}\langle \tau_1, \dots, \tau_n \rangle) = \mathtt{p}^{[Y]}\langle h\tau_1, \dots, h\tau_n \rangle = \mathtt{p}(h\tau_1, \dots, h\tau_n)$. On the other hand, if $\lambda = \neg\mathtt{p}(\tau_1, \dots, \tau_n)$, then $h\delta = \neg(h(\mathtt{p}^{[X]}\langle \tau_1, \dots, \tau_n \rangle)) = \neg(\mathtt{p}^{[Y]}\langle h\tau_1, \dots, h\tau_n \rangle) = \neg\mathtt{p}(h\tau_1, \dots, h\tau_n)$. Consequently, when δ is a literal, then $h\delta$ is a literal, hence $\langle h\lambda \rangle$ is the sequence of $h\delta$. Suppose that the proposition is true when the length is n and $\langle \lambda_1, \dots, \lambda_n \rangle$ is the sequence of δ. We have already proved that $h\lambda$ is literal for any literal λ, so $h(\lambda \vee \delta) = (h\lambda) \vee (h\delta)$ is a clause. Moreover, by induction hypothesis $\langle h\lambda_1, \dots, h\lambda_n \rangle$ is the sequence of $h\delta$, so the sequence of $h(\lambda \vee \delta) = (h\lambda) \vee (h\delta)$ is $\langle h\lambda, h\lambda_1, \dots, h\lambda_n \rangle$.

(2) is analogous to (1) but we use $[\![X]\!]$, $[\![Y]\!]$, $\mathtt{p}(\ulcorner\tau_1\urcorner, \dots, \ulcorner\tau_n\urcorner)$ and $\mathtt{p}(\ulcorner h\tau_1\urcorner, \dots, \ulcorner h\tau_n\urcorner)$ instead of $[X]$, $[Y]$, $\mathtt{p}(\tau_1, \dots, \tau_n)$ and $\mathtt{p}(h\tau_1, \dots, h\tau_n)$.

(3) is analogous to (1) but we use $[\![X]\!]$ and $\mathtt{p}(\ulcorner\tau_1\urcorner, \dots, \ulcorner\tau_n\urcorner)$ instead of $[X]$ and $\mathtt{p}(\tau_1, \dots, \tau_n)$.

(4) is analogous to (1) but we use $[\![Y]\!]$ and $\mathtt{p}(\ulcorner h\tau_1\urcorner, \dots, \ulcorner h\tau_n\urcorner)$ instead of $[Y]$ and $\mathtt{p}(h\tau_1, \dots, h\tau_n)$.

(5) is analogous to (1) but we use $\mathbf{A}$ and $\mathtt{p}(\ulcorner h\tau_1\urcorner, \dots, \ulcorner h\tau_n\urcorner)$ instead of $[Y]$ and $\mathtt{p}(h\tau_1, \dots, h\tau_n)$.

(6) By simple induction on the length of the sequence. When the length is 0, then the sequence is $\langle\rangle$, $\delta = \bot$, $h\delta = \{\bot\}$ and the proposition is trivial. When the length is 1, then the sequence is $\langle \lambda \rangle$ for some literaloid λ and $\delta = \lambda$. If $\lambda = \mathtt{p}(\tau_1, \dots, \tau_n)$, then $h\delta = h(\mathtt{p}^{[X]}\langle \tau_1, \dots, \tau_n \rangle) = \mathtt{p}^{\mathcal{P}\mathbf{A}}\langle h\tau_1, \dots, h\tau_n \rangle = \{\mathtt{p}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner)) : \alpha_1 \in h\tau_1, \dots, \alpha_n \in h\tau_n\}$. On the other hand, if $\lambda = \neg\mathtt{p}(\tau_1, \dots, \tau_n)$, then $h\delta = \neg(h(\mathtt{p}^{[X]}\langle \tau_1, \dots, \tau_n \rangle)) = \neg(\mathtt{p}^{\mathcal{P}\mathbf{A}}\langle h\tau_1, \dots, h\tau_n \rangle) = \{\neg\mathtt{p}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner) : \alpha_1 \in h\tau_1, \dots, \alpha_n \in h\tau_n\}$. Consequently, when δ is a literaloid, then $h\delta$ is a set of relational literaloids over $|\mathbf{A}|$, hence the sequences of the elements of $h\delta$ are all sequences $\langle \mu \rangle$, such that $\mu \in h\lambda$. Suppose that the proposition is true when the length is n and $\langle \lambda_1, \ldots, \lambda_n \rangle$ is the sequence of δ. We have already proved that $h\lambda$ is a set of literals for any literaloid λ, so $h(\lambda \vee \delta) = \{\mu \vee \varepsilon : \mu \in h\lambda \text{ and } \varepsilon \in h\delta\}$ is a set of clauses. Moreover, by induction hypothesis the sequences of the elements of $h\delta$ are all sequences $\langle \mu_1, \ldots, \mu_n \rangle$, such that $\mu_1 \in h\lambda_1, \ldots, \mu_n \in h\lambda_n$, so the sequence of the elements of $h(\lambda \vee \delta) = \{\mu \vee \varepsilon : \mu \in h\lambda \text{ and } \varepsilon \in h\delta\}$ are all sequences $\langle \mu, \mu_1, \ldots, \mu_n \rangle$, such that $\mu \in h\lambda, \mu_1 \in h\lambda_1, \ldots, \mu_n \in h\lambda_n$.

(7) is analogous to (1) but we use $[\![X]\!]$, $\mathbf{A}$, $\mathtt{p}(\ulcorner \tau_1 \urcorner, \ldots, \ulcorner \tau_n \urcorner)$ and $\mathtt{p}(\ulcorner h\tau_1 \urcorner, \ldots, \ulcorner h\tau_n \urcorner)$ instead of $[X]$, $[Y]$, $\mathtt{p}(\tau_1, \ldots, \tau_n)$ and $\mathtt{p}(h\tau_1, \ldots, h\tau_n)$. ∎

M) **Definition.** (1) Two clauses — δ over X and ε over Y — are *variants*, if there exist `Sort`-indexed functions $f : X \to Y$ and $g : Y \to X$, such that $\delta[f] = \varepsilon$ and $\varepsilon[g] = \delta$. The definition of *variant clausoids* is analogous, but we use $[\![f]\!]$ and $[\![g]\!]$ instead of $[f]$ and $[g]$.

(2) If $\langle \lambda_1, \ldots, \lambda_n \rangle$ is the sequence of δ and $\langle \mu_1, \ldots, \mu_n \rangle$ is the sequence of ε, then for any i, we say the literal (literaloid) λ_i is *corresponding* to μ_i.

N) Notice that if ε and δ are variants, then the sequences of ε and δ have the same length. Hence each literal or literaloid of ε has corresponding literal or literaloid in δ.

O) It is not difficult to see that "being variants" is equivalence relation. It is reflexive — we can use $h = g = \mathrm{id}_X$ to see that any clause (clausoid) over X is variant of itself. It is a symmetric relation — obviously if δ is variant of ε, then ε is variant of δ. And it is transitive as well — if δ is variant of ε and ε is variant of ζ and $f'\delta = \varepsilon$, $g'\varepsilon = \delta$, $f''\varepsilon = \zeta$ and $g''\zeta = \varepsilon$, then $(f'' \circ f')\varepsilon = \zeta$ and $(g' \circ g'')\zeta = \varepsilon$.

P) **Proposition.** *If the clausoids δ and ε over X are variants, then the clauses $\delta[\![\mathrm{nam}_X]\!]^{[X]}$ and $\varepsilon[\![\mathrm{nam}_X]\!]^{[X]}$ are variants as well.*

Proof. Let $\delta[\![f]\!] = \varepsilon$ and $\varepsilon[\![g]\!] = \delta$ for some `Sort`-indexed functions $f, g : X \to X$. According to (16D2), $(\delta[\![\mathrm{nam}_X]\!]^{[X]})[f] = \delta([f] \circ [\![\mathrm{nam}_X]\!]^{[X]}) = \delta([\![\mathrm{nam}_X]\!]^{[X]} \circ [\![f]\!]) = (\delta[\![f]\!])[\![\mathrm{nam}_X]\!]^{[X]} = \varepsilon[\![\mathrm{nam}_X]\!]^{[X]}$ and, analogously, $(\varepsilon[\![\mathrm{nam}_X]\!]^{[X]})[g] = \delta[\![\mathrm{nam}_X]\!]^{[X]}$. ∎

Q) **Proposition.** *If the clauses (clausoids) δ and ε are variants, then δ and ε have the same instances in any algebra.*

Proof. We will consider only the case when δ and ε are clausoids. The case when they are clauses is analogous.

Let $\varepsilon[\![g]\!] = \delta$. Any instance of δ belongs to $\delta[\![v]\!]^{\mathcal{P}\mathbf{A}}$, so it is an instance of ε because $\delta[\![v]\!]^{\mathcal{P}\mathbf{A}} = (\varepsilon[\![g]\!])[\![v]\!]^{\mathcal{P}\mathbf{A}} = \varepsilon[\![v \circ g]\!]^{\mathcal{P}\mathbf{A}}$. Analogously we can see that every instance of ε is an instance of δ. ∎

R) **Corollary.** *If a clause or clausoid is universally valid in a logical structure* $\mathbf{M}$*, all its variants are universally valid in* $\mathbf{M}$ *as well.*

Proof. Let δ be universally valid in $\mathbf{M}$ and δ' be a variant of δ. By (H), all instances of δ in $\partial\mathbf{M}$ are true in $\mathbf{M}$. By (Q), δ and δ' have same instances in $\partial\mathbf{M}$, hence all instances of δ' in $\partial\mathbf{M}$ are true in $\mathbf{M}$. By (H), δ' is universally valid in $\mathbf{M}$. ∎

S) **Proposition.** (1) *Let* $\mathbf{A}$ *be an algebra,* X *and* Y *be* `Sort`*-indexed sets,* $s : X \to [Y]$ *be a termal substitution and* δ *be a clause over* X*. Then all instances of* $\delta[s]^{[Y]}$ *in* $\mathbf{A}$ *are instances of* δ *in* $\mathbf{A}$*.*

(2) *Let* $\mathbf{A}$ *be an algebra which is a structure of terms.*[59] *Let* X *and* Y *be* `Sort`*-indexed sets,* $s : X \to [\![Y]\!]$ *be a termoidal substitution and* δ *be a clausoid over* X*. Then all instances of* $\delta[\![s]\!]^{[\![Y]\!]}$ *in* $\mathbf{A}$ *are instances of* δ *in* $\mathbf{A}$*.*

Proof. **(1)** Let δ' be an instance of $\delta[s]^{[Y]}$ in $\mathbf{A}$. Then $\delta' = \delta[s]^{[Y]}[u]^{\mathbf{A}}$ for some assignment function $u : Y \to |\mathbf{A}|$. From the lemma of the termal substitutions (11T) it follows that $\delta' = \delta[w]^{\mathbf{A}}$, where $w : X \to |\mathbf{A}|$ is the assignment function $w = ([u]^{\mathbf{A}}) \circ s$, hence δ' is an instance of δ in $\mathbf{A}$.

(2) Let δ' be an instance of $\delta[\![s]\!]^{[\![Y]\!]}$ in $\mathbf{A}$. Then $\delta' \in \delta[\![s]\!]^{[\![Y]\!]}[\![u]\!]^{\mathcal{P}\mathbf{A}}$ for some assignment function $u : Y \to |\mathbf{A}|$, hence $\delta' = \delta[\![s]\!]^{[\![Y]\!]}[\![u]\!]^{\mathbf{A}}$. From the lemma of the substitutions for terminators (14T2) it follows that $\delta' = \delta[\![w]\!]^{\mathbf{A}}$, where $w : X \to |\mathbf{A}|$ is the assignment function $w = ([\![u]\!]^{\mathbf{A}}) \circ s$, hence δ' is an instance of δ in $\mathbf{A}$. ∎

T) **Proposition.** *Let the* `Sort`*-indexed function* $f : X \to Y$ *be such that all its components* $f_\kappa : X_\kappa \to Y_\kappa$ *are injective. Then for any clause* δ *over* X*,* $\delta[f]$ *is a variant of* δ *and for any clausoid* δ *over* X*,* $\delta[\![f]\!]$ *is a variant of* δ*.*

Proof. We will prove the proposition for the case of clausoids. The other case is analogous.

For any injective function $f_\kappa : X_\kappa \to Y_\kappa$ there exists a function $g_\kappa : Y_\kappa \to X_\kappa$, such that $g_\kappa \circ f_\kappa = \mathrm{id}_{X_\kappa}$. Let $g : Y \to X$ be a `Sort`-indexed function composed of such functions g_κ. Then $g \circ f = \mathrm{id}_X$, so $(\delta[\![f]\!])[\![g]\!] = \delta[\![g \circ f]\!] = \delta$, hence δ and $\delta[\![f]\!]$ are variants. ∎

U) **Definition.** Two termal substitutions $s' : X \to [Y]$ and $s'' : X \to [Z]$ are *variants* if there exist `Sort`-indexed functions $f : Y \to Z$ and $g : Z \to Y$, such that $s' = [g] \circ s''$ and $s'' = [f] \circ s'$.

The definition of *variant termoidal substitutions* is analogous, but we use $[\![Y]\!]$, $[\![Z]\!]$, $[\![f]\!]$ and $[\![g]\!]$ instead of $[Y]$, $[Z]$, $[f]$ and $[g]$.

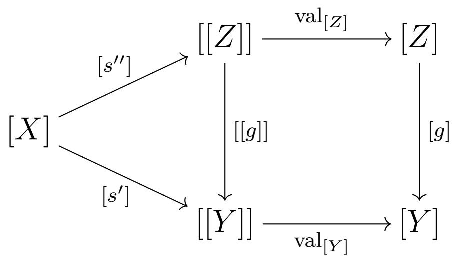
[59] See (14L) for the definition of structure of terms.

V) **Proposition.** (1) *If* $s' : X \to [Y]$ *and* $s'' : X \to [Z]$ *are variant termal substitutions, then for any clause* δ *over* X*,* $\delta[s']^{[Y]}$ *and* $\delta[s'']^{[Z]}$ *are variant clauses. Moreover, if the* `Sort`*-indexed functions* f *and* g *are such that* $s' = [g] \circ s''$ *and* $s'' = [f] \circ s'$*, then* $\delta[s']^{[Y]} = (\delta[s'']^{[Z]})[g]$ *and* $\delta[s'']^{[Z]} = (\delta[s']^{[Y]})[f]$*.*

(2) *If* $s' : X \to [\![Y]\!]$ *and* $s'' : X \to [\![Z]\!]$ *are variant termoidal substitutions, then for any clausoid* δ *over* X*,* $\delta[\![s']\!]^{[\![Y]\!]}$ *and* $\delta[\![s'']\!]^{[\![Z]\!]}$ *are variant clausoids. Moreover, if the* `Sort`*-indexed functions* f *and* g *are such that* $s' = [\![g]\!] \circ s''$ *and* $s'' = [\![f]\!] \circ s'$*, then* $\delta[\![s']\!]^{[\![Y]\!]} = (\delta[\![s'']\!]^{[\![Z]\!]})[\![g]\!]$ *and* $\delta[\![s'']\!]^{[\![Z]\!]} = (\delta[\![s']\!]^{[\![Y]\!]})[\![f]\!]$*.*

Proof. Only a proof for the termal case will be provided. The termoidal case is completely analogous.

Let f and g be such that $s' = [g] \circ s''$ and $s'' = [f] \circ s'$ and consider the following diagram:



The triangle is commutative because $s' = [g] \circ s''$ and the commutativity of the square follows from (11N) in the termal case and from (14I16) in the termoidal case. Consequently, for any clause δ over X, $\delta[s']^{[Y]} = (\delta[s'']^{[Z]})[g]$. Analogously, $\delta[s'']^{[Z]} = (\delta[s']^{[Y]})[f]$. ■

W) **Definition.** (1) A literal (literaloid) λ is *positive* if it does not contain a negation. It is *negative*, if it contains a negation.

(2) A clause (clausoid) is *positive*, if it contains no negative literals (literaloids). It is *non-positive*, if it is not positive.

(3) A clause (clausoid) is *negative*, if it contains no positive literals (literaloids). It is *non-negative*, if it is not negative.

X) This proposition tells us that we do not have to use clauses and clausoids over many different `Sort`-indexed sets. Let us fix one specific `Sort`-indexed set $\mathbb{X}$. It will be enough to impose on $\mathbb{X}$ only the following two properties:

- The components of $\mathbb{X}$ have sufficiently large cardinality, so that for any `Sort`-indexed set Y, such that we want to be able to use clauses or clausoids over Y, there exists a $\mathtt{Sort}$-indexed function $f : Y \to \mathbb{X}$ with injective components.

- Each component of $\mathbb{X}$ is an infinite set.

Given such a set $\mathbb{X}$, if δ is a clause (clausoid) over Y and the $\mathtt{Sort}$-indexed function $f : Y \to \mathbb{X}$ has injective components, then $\delta[f]$ (respectively, $\delta[\![f]\!]$) will be a variant of δ. Consequently, for any clause or clausoid we are able to find a variant of it over $\mathbb{X}$.

Y) For convenience, in few cases we will assume that the following additional properties are true:

- All elements of $\mathbb{X}$ are symbols different from the functional symbols, predicate symbols, logical symbols, brackets, comma and any other symbol we use to build termal expressions.
- We will also suppose that $\ulcorner \mathtt{x} \urcorner = \mathtt{x}$ for any $\mathtt{x} \in \mathbb{X}$.

This assumption is not necessary, it is here only for convenience. Because of it, we may write the clauses as $\mathtt{p}(\mathtt{x}, \mathtt{f}(\mathtt{y})) \vee \neg \mathtt{q}(\mathtt{x})$ instead of $\mathtt{p}(\ulcorner \mathtt{x} \urcorner, \mathtt{f}(\ulcorner \mathtt{y} \urcorner)) \vee \neg \mathtt{q}(\ulcorner \mathtt{x} \urcorner)$.

Z) **Remark.** Every clause δ over Y may contain only finitely many names of the elements of Y. This can be used in order to prove that there always exists a clause over $\mathbb{X}$ which is a variant of δ. This, however, is not true for the clausoids. This means that the statement in (X) "for any $\mathtt{Sort}$-indexed set Y, such that we want to be able to use clauses or clausoids over Y" is an informal one. Strictly speaking, we have to fix some cardinal and to restrict ourselves with $\mathtt{Sort}$-indexed sets whose components do not have greater cardinality. This limitation is not required in order to develop the theory of the resolution with clausoids; nevertheless, using only one $\mathtt{Sort}$-indexed set $\mathbb{X}$ for all clauses and clausoids, is going save as from some unwanted technicalities.

§21. SLD RESOLUTION

A) In this section we will assume that we are working with a terminator, where all termoidal expressions are finitary.[60]

B) **Definition.** (1) A *Horn clause (clausoid)* is a clause (clausoid) over $\mathbb{X}$ in whose sequence only the first literal (literaloid) may be positive and all other literals (literaloids) are negative.

[60] See (17A2) for the definition of "finitary".

(2) A *selection function* is a function $\mathfrak{sel}$, such that for any non-empty finite sequence of literals $\lambda_1, \ldots, \lambda_k$ over $\mathbb{X}$, $\mathfrak{sel}(\langle \lambda_1, \ldots, \lambda_k \rangle)$ is a natural number among $1, 2, \ldots, k$. We will assume that $\mathfrak{sel}(\langle \lambda_1, \ldots, \lambda_k \rangle) = \mathfrak{sel}(\langle \lambda_1[f], \ldots, \lambda_k[f] \rangle)$ for any bijective $\mathtt{Sort}$-indexed function f.

Any selection function $\mathfrak{sel}$ for literals over $\mathbb{X}$ can be extended standardly for literaloids over $\mathbb{X}$ in the following way: if $\lambda_1, \ldots, \lambda_k$ are literaloids over $\mathbb{X}$, then let $\mathfrak{sel}(\langle \lambda_1, \ldots, \lambda_k \rangle) = \mathfrak{sel}(\langle \lambda_1 [\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}, \ldots, \lambda_k [\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} \rangle)$.

(3) Given a selection function $\mathfrak{sel}$, an equaliser $\mathfrak{e}$, a negative Horn clause δ over $\mathbb{X}$ with sequence $\langle \lambda_1, \ldots, \lambda_k \rangle$ and a non-negative Horn-clause ε over $\mathbb{X}$ with sequence $\langle \mu_0, \mu_1, \ldots, \mu_m \rangle$, if $j = \mathfrak{sel}(\langle \lambda_1, \ldots, \lambda_k \rangle)$ and $s \in \mathfrak{e}(\{\overline{\lambda}_j \sim \mu_0\})$, then the clause whose sequence is

$$\langle \lambda_1[s]^{[\mathbb{X}]}, \ldots, \lambda_{j-1}[s]^{[\mathbb{X}]}, \mu_1[s]^{[\mathbb{X}]}, \ldots, \mu_m[s]^{[\mathbb{X}]}, \lambda_{j+1}[s]^{[\mathbb{X}]}, \lambda_k[s]^{[\mathbb{X}]} \rangle$$

is called $\mathfrak{e}$-$\mathfrak{sel}$-*SLD resolvent* of δ and ε.

The notion $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of clausoids is defined analogously, but instead of clauses, literals, $\lambda_i[s]^{[\mathbb{X}]}$ and $\mu_i[s]^{[\mathbb{X}]}$ we use clausoids, literaloids, $\lambda_i[\![s]\!]^{[\![\mathbb{X}]\!]}$ and $\mu_i[\![s]\!]^{[\![\mathbb{X}]\!]}$.

C) The machinery of Prolog uses SLD-resolution with the following trivial selection function: $\mathfrak{sel}(\zeta) = 1$ for every sequence ζ. In other words, Prolog always resolves the first literal of the clauses.

D) **Proposition.** *Let $\mathfrak{sel}$ be a selection function and $\mathfrak{e}$ be a termally sound and termally complete equaliser such that for any system Θ, the set $\mathfrak{e}(\Theta)$ contains at most one element. Then for no pair of clausoids δ and ε there exists more than one $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of δ and ε.*

Proof. Follows immediately from definition (B3). ■

Informally, the following Lemma says that if ζ' is a $\mathfrak{mgu}$-$\mathfrak{sel}$-SLD resolvent of the clauses δ' and ε' and δ' and ε' correspond to the clausoids δ and ε, then ζ' corresponds to some $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of δ and ε. The termal equaliser $\mathfrak{mgu}$ is defined in (18S).

E) **Lemma.** *Suppose $\mathfrak{sel}$ is a selection function, $\mathfrak{e}$ is a termally sound and termally complete equaliser, the clauses δ' and ε' have disjoint dependency and the clause ζ' is an $\mathfrak{mgu}$-$\mathfrak{sel}$-SLD resolvent of δ' and ε'. If the clausoids δ and ε are such that δ and ε have disjoint dependency, δ' is a variant of $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and ε' is a variant of $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$, then δ and ε have an $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent ζ, such that ζ' is a variant of $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$.*

Proof. Let $\langle \lambda_1, \ldots, \lambda_l \rangle$ be the sequence of δ, $\langle \lambda'_1, \ldots, \lambda'_l \rangle$ be the sequence of δ', $\langle \mu_0, \mu_1, \ldots, \mu_m \rangle$ be the sequence of ε and $\langle \mu'_0, \mu'_1, \ldots, \mu'_m \rangle$ be the sequence of ε'. Let $j = \mathfrak{sel}(\langle\lambda_1,\dots,\lambda_l\rangle) = \mathfrak{sel}(\langle\lambda'_1,\dots,\lambda'_l\rangle)$. Let s' be the substitution used to produce ζ' from δ' and ε'. Therefore, the sequence of ζ' is

$$\langle\lambda'_1[s']^{[\mathbb{X}]},\dots,\lambda'_{j-1}[s']^{[\mathbb{X}]},\mu'_1[s']^{[\mathbb{X}]},\dots,\mu'_m[s']^{[\mathbb{X}]},\lambda'_{j+1}[s']^{[\mathbb{X}]},\dots,\lambda'_l[s']^{[\mathbb{X}]}\rangle$$

Since δ' is a variant of $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and ε' is a variant of $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$, there exist `Sort`-indexed functions $f, g : \mathbb{X} \to \mathbb{X}$, such that $\delta' = \delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[f]$ and $\varepsilon' = \varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[g]$. Since δ and ε have disjoint dependency, according to (17I2), $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ also have disjoint dependency. Consequently, without loss of generality we may assume that f and g are bijective and $f = g$. So $\varepsilon' = \varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[f]$ and the sequence of ζ' is

$$\begin{aligned}\langle\lambda_1[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[s'\circ f]^{[\mathbb{X}]},\dots,\lambda_{j-1}[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[s'\circ f]^{[\mathbb{X}]},&\\ \mu_1[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[s'\circ f]^{[\mathbb{X}]},\dots,\mu_m[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[s'\circ f]^{[\mathbb{X}]},&\\ \lambda_{j+1}[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[s'\circ f]^{[\mathbb{X}]},\dots,\lambda_l[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[s'\circ f]^{[\mathbb{X}]}\rangle&\end{aligned}$$

Since s' is a most general unifier of $\overline{\lambda'_j}$ and μ'_0, $s'\circ f$ is a most general unifier of $\overline{\lambda}_j[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and $\mu_0[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$. From this and (18U2) we can conclude that there exists a substitution $s \in \mathfrak{e}(\{\overline{\lambda}_j \sim \mu_0\})$, such that $\overline{s} = [\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} \circ s$ is a most general unifier of $\overline{\lambda}_j[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and $\mu_0[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$. Since both $\overline{s}$ and $s'\circ f$ are most general unifiers of these literals, we can conclude that these substitutions are variants. Therefore ζ' is a variant of the clause whose sequence is

$$\begin{aligned}\langle\lambda_1[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]},\dots,\lambda_{j-1}[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]},&\\ \mu_1[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]},\dots,\mu_m[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]},&\\ \lambda_{j+1}[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]},\dots,\lambda_l[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]}\rangle&\end{aligned}$$

According to (16M) and (20L) this clause is equal to $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$, where ζ is the clausoid whose sequence is

$$\langle\lambda_1[\![s]\!]^{[\![\mathbb{X}]\!]},\dots,\lambda_{j-1}[\![s]\!]^{[\![\mathbb{X}]\!]},\mu_1[\![s]\!]^{[\![\mathbb{X}]\!]},\dots,\mu_m[\![s]\!]^{[\![\mathbb{X}]\!]},\lambda_{j+1}[\![s]\!]^{[\![\mathbb{X}]\!]},\dots,\lambda_l[\![s]\!]^{[\![\mathbb{X}]\!]}\rangle$$

Notice that ζ is an $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of δ and ε. ∎

F) **Lemma.** *Suppose the clauses δ' and ε' have disjoint dependency, $\mathfrak{sel}$ is a selection function and $\mathfrak{e}$ is a termally sound and termally complete equaliser, such that $\mathfrak{e}(\Theta)$ contains no more than one element for any system Θ. If the clausoids δ and ε are such that δ and ε have disjoint dependency, δ' is a variant of $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$, ε' is a variant of $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and*

δ and ε have an $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent ζ, then δ' and ε' have an $\mathfrak{mgu}$-$\mathfrak{sel}$-SLD resolvent ζ', such that ζ' is a variant of $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$.

Proof. Let $\langle\lambda_1,\ldots,\lambda_l\rangle$ be the sequence of δ, $\langle\lambda'_1,\ldots,\lambda'_l\rangle$ be the sequence of δ', $\langle\mu_0,\mu_1,\ldots,\mu_m\rangle$ be the sequence of ε and $\langle\mu'_0,\mu'_1,\ldots,\mu'_m\rangle$ be the sequence of ε'. Let $j=\mathfrak{sel}(\langle\lambda_1,\ldots,\lambda_l\rangle)=\mathfrak{sel}(\langle\lambda'_1,\ldots,\lambda'_l\rangle)$. Let s be the substitution used to produce ζ from δ and ε. Therefore, the sequence of ζ is

$$\langle\lambda_1[\![s]\!]^{[\![\mathbb{X}]\!]},\ldots,\lambda_{j-1}[\![s]\!]^{[\![\mathbb{X}]\!]},\mu_1[\![s]\!]^{[\![\mathbb{X}]\!]},\ldots,\mu_m[\![s]\!]^{[\![\mathbb{X}]\!]},\lambda_{j+1}[\![s]\!]^{[\![\mathbb{X}]\!]},\ldots,\lambda_l[\![s]\!]^{[\![\mathbb{X}]\!]}\rangle$$

so, according to (16M), the sequence of $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ is equal to

$$\begin{aligned}\langle\lambda_1[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]},\ldots,\lambda_{j-1}[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]},\\ \mu_1[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]},\ldots,\mu_m[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]},\\ \lambda_{j+1}[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]},\ldots,\lambda_l[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\overline{s}]^{[\mathbb{X}]}\rangle\end{aligned}$$

where $\overline{s}=[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}\circ s$.

Since δ' is a variant of $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and ε' is a variant of $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$, there exist `Sort`-indexed functions $f,g:\mathbb{X}\to\mathbb{X}$, such that $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}=\delta'[f]$ and $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}=\varepsilon'[g]$. Since δ and ε have disjoint dependency, according to (17l2), $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ also have disjoint dependency. Consequently, without loss of generality we may assume that f and g are bijective and $f=g$. So $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}=\varepsilon'[f]$ and the sequence of $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ is

$$\begin{aligned}\langle\lambda'_1[\overline{s}\circ f]^{[\mathbb{X}]},\ldots,\lambda'_{j-1}[\overline{s}\circ f]^{[\mathbb{X}]},\\ \mu'_1[\overline{s}\circ f]^{[\mathbb{X}]},\ldots,\mu'_m[\overline{s}\circ f]^{[\mathbb{X}]},\\ \lambda'_{j+1}[\overline{s}\circ f]^{[\mathbb{X}]},\ldots,\lambda'_l[\overline{s}\circ f]^{[\mathbb{X}]}\rangle\end{aligned}$$

Since $\mathfrak{e}(\Theta)$ contains no more than one element for any system Θ, from (18U) we can conclude that $\overline{s}$ is a most general unifier of $\overline{\lambda_j}[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and $\mu_0[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$. Therefore, $\overline{s}\circ f$ is a most general unifier of $\overline{\lambda'_j}$ and μ'_0. Consequently, there exists a substitution $s'\in\mathfrak{mgu}(\{\overline{\lambda'_j}\sim\mu'_0\})$, which is a most general unifier of $\overline{\lambda'_j}$ and μ'_0. Since both $\overline{s}\circ f$ and s' are most general unifiers of these literals, we can conclude that these substitutions are variants. Therefore $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ is a variant of the clause whose sequence is

$$\begin{aligned}\langle\lambda'_1[s']^{[\mathbb{X}]},\ldots,\lambda'_{j-1}[s']^{[\mathbb{X}]},\\ \mu'_1[s']^{[\mathbb{X}]},\ldots,\mu'_m[s']^{[\mathbb{X}]},\\ \lambda'_{j+1}[s']^{[\mathbb{X}]},\ldots,\lambda'_l[s']^{[\mathbb{X}]}\rangle\end{aligned}$$

According to definition (B3), the clause having this sequence is an $\mathfrak{mgu}$-$\mathfrak{sel}$-SLD resolvent of δ' and ε'. ■

G) **Lemma.** *We are applying to a finite sequence of natural numbers the following transformation: we remove an arbitrary element of the sequence and replace the removed natural number with arbitrarily many smaller natural numbers. It is impossible to apply transformations of this kind infinitely many times*

Proof. Suppose $\mathfrak{z}$ is a finite sequence of natural numbers, such that we can apply to $\mathfrak{z}$ transformations of mentioned kind infinitely many times. Let n be the greatest element of $\mathfrak{z}$.

Each time we apply a transformation removing a number equal to n, we a replacing it with smaller natural numbers. Since $\mathfrak{z}$ contains only finitely many elements equal to n and none of the transformations can create new numbers equal to n, we can apply only finitely many times transformations removing numbers equal to n. After the last such transformation, we will obtain a sequence $\mathfrak{z}_1$ with the following property: it is possible to apply infinitely many times transformations of the mentioned kind to $\mathfrak{z}_1$, such that none of them removes numbers equal to n.

Now, similar reasoning can show us that we can apply to $\mathfrak{z}_1$ only finitely many times transformations removing numbers equal to $n-1$. After the last such transformation we will obtain a sequence $\mathfrak{z}_2$ with the following property: it is possible to apply infinitely many times transformations of the mentioned kind to $\mathfrak{z}_2$, such that none of them removes numbers equal to n or $n-1$.

Analogously, from $\mathfrak{z}_2$ we can obtain a sequence $\mathfrak{z}_3$ with the following property: it is possible to apply infinitely many times transformations of the mentioned kind to $\mathfrak{z}_3$, such that none of them removes numbers equal to n, $n-1$ or $n-2$.

From $\mathfrak{z}_3$ we obtain $\mathfrak{z}_4$, then $\mathfrak{z}_5$ and so on. This is a contradiction because $\mathfrak{z}_n$ will have to have the following property: it is possible to apply infinitely many times transformations of the mentioned kind to $\mathfrak{z}_n$, such that none of them removes any element of $\mathfrak{z}_n$. ■

H) **Definition.** A set Γ of Horn clauses (clausoids) over $\mathbb{X}$ is called $\mathfrak{e}$-$\mathfrak{sel}$-*closed* if:

(1) δ and ε have disjoint dependency whenever δ is a negative element of Γ and ε is a non-negative element of Γ;

(2) Γ contains a variant of every $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of elements of Γ.

We shell see that if a set of clausoids is universally satisfiable in almost any normal algebra, then the set is universally satisfiable in some algebra with finite carriers. Therefore, from the following Theorem we can conclude that if the SLD-resolution with clausoids saturates and stops generating new clausoids, then the initial set of clausoids is satisfiable in an algebra with finite carriers.

l) **Theorem.** *Let $\mathfrak{e}$ be a near-complete equaliser, $\mathfrak{sel}$ be a selection function and Γ be a finite $\mathfrak{e}$-$\mathfrak{sel}$-complete set of Horn clausoids over $\mathbb{X}$, such that $\bot \notin \Gamma$. Then Γ is universally satisfiable in almost any normal algebra.*

Proof. Let Γ^- be the set of all negative elements of Γ and Γ^+ be the set of all non-negative elements of Γ. Since Γ is $\mathfrak{e}$-$\mathfrak{sel}$-closed set, all elements of Γ^- have disjoint dependency with all elements of Γ^+.

Since both Γ^- and Γ^+ are finite, there exist finitely many pairs $\langle \lambda, \mu \rangle$, such that λ is a literaloid of a clausoid belonging to Γ^-, μ is a literaloid of a clausoid belonging to Γ^+ and the system $\{\overline{\lambda} \sim \mu\}$ is termally inconsistent. Let Θ be the set all such termally inconsistent identities. The set Θ is finite.

Let $\mathbf{A}$ be an arbitrary normal algebra, such that none of the systems in Θ has a solution in $\mathbf{A}$. In order to prove the theorem, it will be enough to prove that the set $\Gamma = \Gamma^+ \cup \Gamma^-$ is universally satisfiable in $\mathbf{A}$.

Let Σ be a set of positive literals over $|\mathbf{A}|$ defined inductively by the following rule:

> If $\langle \lambda_0, \lambda_1, \dots, \lambda_n \rangle$ is the sequence of an element of Γ^+,[61] $v : \mathbb{X} \to |\mathbf{A}|$ is an arbitrary `Sort`-indexed function and each of the sets $\overline{\lambda}_1 [\![v]\!]^{\mathcal{P}\mathbf{A}}, \dots, \overline{\lambda}_n [\![v]\!]^{\mathcal{P}\mathbf{A}}$ contains at least one element of Σ, then all elements of $\lambda_0 [\![v]\!]^{\mathcal{P}\mathbf{A}}$ are elements of Σ.

We will define sets $\Sigma_0, \Sigma_1, \Sigma_2, \dots$, such that $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \dots$. Let $\Sigma_0 = \varnothing$ and Σ_{i+1} be the union of Σ_i and the set of all positive literals λ, such that there exists an element of Γ^+ whose sequence is $\langle \lambda_0, \lambda_1, \dots, \lambda_n \rangle$ and a `Sort`-indexed function $v : \mathbb{X} \to |\mathbf{A}|$, such that $\lambda \in \lambda_0 [\![v]\!]^{\mathcal{P}\mathbf{A}}$ and each of the sets $\overline{\lambda}_1 [\![v]\!]^{\mathcal{P}\mathbf{A}}, \dots, \overline{\lambda}_n [\![v]\!]^{\mathcal{P}\mathbf{A}}$ contains at least one element of Σ_i.

Since $\mathbf{A}$ is a normal structure, there exists a logical structure $\mathbf{K}$, such that $\mathbf{K}$ is a variant of $\mathbf{A}$ and the predicate symbols are interpreted in the following way: $\mathtt{p}^{\mathbf{K}} \langle \alpha_1, \dots, \alpha_n \rangle = 1$ if and only if $\mathtt{p}(\ulcorner \alpha_1 \urcorner, \dots, \ulcorner \alpha_n \urcorner) \in \Sigma$. Notice that a relational positive literal over $|\mathbf{K}|$ is true in $\mathbf{K}$ if and only if it is an element of Σ.

We are going to prove that the elements of $\Gamma = \Gamma^+ \cup \Gamma^-$ are universally valid in $\mathbf{K}$. This will imply that the set $\Gamma^+ \cup \Gamma^-$ is universally satisfiable in $\mathbf{A}$.

[61] Notice that λ_0 is a positive literaloid and $\lambda_1, \dots, \lambda_n$ are negative.

Suppose this is not so. There are two cases to consider – either some element of Γ^+ is not universally valid in $\mathbf{K}$, or some element of Γ^- is not universally valid in $\mathbf{K}$.

If an element of Γ^+ with sequence $\langle \lambda_0, \lambda_1, \ldots, \lambda_n \rangle$ is not universally valid in $\mathbf{K}$, then there exists an assignment function $v : \mathbb{X} \to |\mathbf{K}|$, such that $0 \in \lambda_0[\![v]\!]^{\mathcal{P}\mathbf{K}}$ and $1 \in \overline{\lambda}_i[\![v]\!]^{\mathcal{P}\mathbf{K}}$ for any $i \in \{1, \ldots, n\}$. Since the literaloids $\lambda_0, \overline{\lambda}_1, \ldots, \overline{\lambda}_n$ are positive, from (14W2) and the definition of the structure $\mathbf{K}$ it follows that the set $\lambda_0[\![v]\!]^{\mathcal{P}\mathbf{A}}$ is not a subset of Σ and each of the sets $\overline{\lambda}_1[\![v]\!]^{\mathcal{P}\mathbf{A}}, \ldots, \overline{\lambda}_n[\![v]\!]^{\mathcal{P}\mathbf{A}}$ contains at least one element of Σ. This, however, contradicts the definition of Σ.

Now, suppose there exists an element δ of Γ^- which is not universally valid in $\mathbf{K}$. Let $\langle \lambda_1, \ldots, \lambda_k \rangle$ be the sequence of δ. Since δ is not universally valid in $\mathbf{K}$, there exists an assignment function $v : \mathbb{X} \to |\mathbf{A}|$, such that $0 \in \delta[\![v]\!]^{\mathcal{P}\mathbf{K}}$. This is possible only if $1 \in \overline{\lambda}_i[\![v]\!]^{\mathcal{P}\mathbf{K}}$ for all $i \in \{1, \ldots, k\}$. Notice that the literaloids $\overline{\lambda}_i$ are positive.

If $\overline{\lambda}_i = \mathtt{p}(\ulcorner\tau_1\urcorner, \ldots, \ulcorner\tau_m\urcorner)$, then

$$\begin{aligned}
1 \in \overline{\lambda}_i[\![v]\!]^{\mathcal{P}\mathbf{K}} &\iff 1 \in \mathtt{p}(\ulcorner\tau_1\urcorner, \ldots, \ulcorner\tau_m\urcorner)[\![v]\!]^{\mathcal{P}\mathbf{K}} \\
&\iff 1 \in \mathtt{p}^{\mathcal{P}\mathbf{K}}\langle \tau_1[\![v]\!]^{\mathcal{P}\mathbf{K}}, \ldots, \tau_m[\![v]\!]^{\mathcal{P}\mathbf{K}} \rangle \\
&\iff 1 = \mathtt{p}^{\mathbf{K}}\langle \alpha_1, \ldots, \alpha_m \rangle \text{ for some } \alpha_i \in \tau_i[\![v]\!]^{\mathcal{P}\mathbf{K}} \\
&\iff \mathtt{p}(\ulcorner\alpha_1\urcorner, \ldots, \ulcorner\alpha_m\urcorner) \in \Sigma \text{ for some } \alpha_i \in \tau_i[\![v]\!]^{\mathcal{P}\mathbf{K}} \\
&\iff \mathtt{p}(\ulcorner\alpha_1\urcorner, \ldots, \ulcorner\alpha_m\urcorner) \in \Sigma \text{ for some } \alpha_i \in \tau_i[\![v]\!]^{\mathcal{P}\mathbf{A}} \\
&\iff \mathtt{p}(\ulcorner\tau_1\urcorner, \ldots, \ulcorner\tau_m\urcorner)[\![v]\!]^{\mathcal{P}\mathbf{A}} \cap \Sigma \neq \varnothing \\
&\iff \overline{\lambda}_i[\![v]\!]^{\mathcal{P}\mathbf{A}} \cap \Sigma \neq \varnothing
\end{aligned}$$

Therefore, each of the sets $\overline{\lambda}_i[\![v]\!]^{\mathcal{P}\mathbf{A}}$ contains at least one element of Σ.

Before we continue with the proof, let us give the following definition:

> **Definition.** Given a negative Horn clausoid δ with sequence $\langle \lambda_1, \ldots, \lambda_k \rangle$, we will say the sequence of natural numbers $\langle m_1, \ldots, m_k \rangle$ is a *numerical bound* of δ, if there exists a `Sort`-indexed function $v : \mathbb{X} \to |\mathbf{A}|$, such that $\overline{\lambda}_i[\![v]\!]^{\mathcal{P}\mathbf{A}} \cap \Sigma_{m_i} \neq \varnothing$ for any $i \in \{1, \ldots, k\}$. Notice that if a clausoid has a numerical bound, then it is not universally valid in $\mathbf{K}$.

Since each of the sets $\overline{\lambda}_i[\![v]\!]^{\mathcal{P}\mathbf{A}}$ contains at least one element of Σ, the clausoid δ has a numerical bound. Let $\langle m_1, \ldots, m_k \rangle$ be a numerical bound of δ. This means that $\overline{\lambda}_i[\![v]\!]^{\mathcal{P}\mathbf{A}} \cap \Sigma_{m_i} \neq \varnothing$ for any $i \in \{1, \ldots, k\}$.

Let $\alpha_1, \ldots, \alpha_k$ be such that $\alpha_i \in \overline{\lambda}_i[\![v]\!]^{\mathcal{P}\mathbf{A}} \cap \Sigma_{m_i}$ and let $j = \mathfrak{sel}(\langle \lambda_1, \ldots, \lambda_k \rangle)$. Since $\alpha_j \in \Sigma_{m_j}$, the definition of Σ implies that there exists an element ε of Γ^+ with sequence $\langle \mu_0, \mu_1, \ldots, \mu_l \rangle$ and a `Sort`-indexed function $v' : \mathbb{X} \to |\mathbf{A}|$, such that $\alpha_j \in \mu_0[\![v']\!]^{\mathcal{PA}}$ and each of the sets $\overline{\mu}_1[\![v']\!]^{\mathcal{PA}}, \ldots, \overline{\mu}_l[\![v']\!]^{\mathcal{PA}}$ contains at least one element of Σ_{m_j-1}. Let $\beta_1, \ldots, \beta_l$ be such that $\beta_i \in \overline{\mu}_i[\![v']\!]^{\mathcal{PA}} \cap \Sigma_{m_j-1}$ for any $i \in \{1, \ldots, l\}$.

Since ε has disjoint dependency with δ, according to (17H), there exists a `Sort`-indexed function $w : \mathbb{X} \to |\mathbf{A}|$, such that $\delta[\![v]\!] = \delta[\![w]\!]$ and $\varepsilon[\![v']\!] = \varepsilon[\![w]\!]$. Therefore, α_j belongs both to $\overline{\lambda}_j[\![v]\!]^{\mathcal{PA}} = \overline{\lambda}_j[\![w]\!]^{\mathcal{PA}}$ and to $\mu_0[\![v']\!]^{\mathcal{PA}} = \mu_0[\![w]\!]^{\mathcal{PA}}$, which means that w is a solution in $\mathbf{A}$ of the identity $\overline{\lambda}_j \sim \mu_0$. Now, the definition of the set Θ and the choice of $\mathbf{A}$ imply that this identity is termally consistent. But the equaliser $\mathfrak{e}$ is near-complete, so there exists some $s \in \mathfrak{e}(\{\overline{\lambda}_j \sim \mu_0\})$, such that w is an instance of s. According to the definition of "instance" (18C), there exists a `Sort`-indexed function w', such that $w \ll [\![w']\!]^{\mathcal{PA}} \circ s$. The lemma of the substitutions (14T1) now implies that for any termoidal expression ζ over $\mathbb{X}$,

$$\zeta[\![w]\!]^{\mathcal{PA}} \subseteq (\zeta[\![s]\!]^{[\![\mathbb{X}]\!]})[\![w']\!]^{\mathcal{PA}} \qquad (\sharp)$$

According to the definition of SLD resolvent, the clausoid with sequence $\langle \lambda_1[\![s]\!]^{[\![\mathbb{X}]\!]}, \ldots, \lambda_{j-1}[\![s]\!]^{[\![\mathbb{X}]\!]}, \mu_1[\![s]\!]^{[\![\mathbb{X}]\!]}, \ldots, \mu_l[\![s]\!]^{[\![\mathbb{X}]\!]}, \lambda_{j+1}[\![s]\!]^{[\![\mathbb{X}]\!]}, \ldots, \lambda_k[\![s]\!]^{[\![\mathbb{X}]\!]} \rangle$ is an $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of δ and ε. Considering that $\alpha_i \in \overline{\lambda}_i[\![v]\!] = \overline{\lambda}_i[\![w]\!]$ and $\beta_i \in \overline{\mu}_i[\![v']\!]^{\mathcal{PA}} = \overline{\mu}_i[\![w]\!]^{\mathcal{PA}}$ and taking into account ($\sharp$), we can conclude that the clause over $|\mathbf{A}|$ whose sequence is $\langle \overline{\alpha}_1, \ldots, \overline{\alpha}_{j-1}, \overline{\beta}_1, \ldots, \overline{\beta}_l, \overline{\alpha}_{j+1}, \overline{\alpha}_k \rangle$ is an instance in $\mathbf{A}$ of this $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent.

Since Γ^- ia an $\mathfrak{e}$-$\mathfrak{sel}$-closed set, Γ^- contains a variant δ' of this $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent. According to (20Q), variants have same instances, hence the clause with sequence $\langle \overline{\alpha}_1, \ldots, \overline{\alpha}_{j-1}, \overline{\beta}_1, \ldots, \overline{\beta}_l, \overline{\alpha}_{j+1}, \overline{\alpha}_k \rangle$ is an instance in $\mathbf{A}$ of δ'. Since $\alpha_i \in \Sigma_{m_i}$ and $\beta_i \in \Sigma_{m_j-1}$, the sequence

$$\langle m_1, \ldots, m_{j-1}, \underbrace{m_j - 1, \ldots, m_j - 1}_{l \text{ times}}, m_{j+1}, \ldots, m_k \rangle$$

is a numerical bound of the clausoid δ'. Notice that any clausoid with a numerical bound is not universally valid in $\mathbf{K}$.

So, we started with a clausoid $\delta \in \Gamma^-$ which is not universally valid in $\mathbf{K}$, we assumed δ has a numerical bound $\langle m_1, \ldots, m_k \rangle$ and we obtained a clausoid $\delta' \in \Gamma^-$, such that δ' is not universally valid in $\mathbf{K}$ and has a numerical bound

$$\langle m_1, \ldots, m_{j-1}, \underbrace{m_j - 1, \ldots, m_j - 1}_{l \text{ times}}, m_{j+1}, \ldots, m_k \rangle$$

Since the numerical bound of δ' can be obtained from the numerical bound of δ by means of a transformation of the kind described in (G), this is a contradiction. ∎

J) **Definition.** Given an equaliser $\mathfrak{e}$, a selection function $\mathfrak{sel}$ and a set Γ of non-negative Horn clauses over $\mathbb{X}$, an $\mathfrak{e}$-$\mathfrak{sel}$-*SLD search tree* for Γ is a finite or infinite rooted tree $\mathfrak{t}$ with the following properties:

(1) Any node of $\mathfrak{t}$ is labelled with a negative Horn clause.

(2) If a node labelled with δ has a child labelled with ε, then ε is a variant of an $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of δ and some variant δ' of an element of Γ such that δ and δ' have disjoint dependency.

(3) If a node is labelled with δ and ε is an $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of δ and some variant δ' of an element of Γ such that δ and δ' have disjoint dependency, then some child of this node is labelled with a variant of ε.

K) **Proposition.** *Let $\mathfrak{e}$ be an equaliser, such that $\mathfrak{e}(\Theta)$ contains no more than one element for any system Θ, $\mathfrak{sel}$ be a selection function, Γ' be a finite set of non-negative Horn clauses over $\mathbb{X}$, $\mathfrak{t}$ be a finite $\mathfrak{mgu}$-$\mathfrak{sel}$-SLD search tree for Γ' whose root is labelled with the clause ζ' and no node is labelled with $\bot$. If $\Gamma = \{\delta[\mathrm{Nam}_{\mathbb{X}}]^{[\![\mathbb{X}]\!]} : \delta \in \Gamma'\}$ and ζ' has disjoint dependency with all elements of Γ', then there exists a finite $\mathfrak{e}$-$\mathfrak{sel}$-closed set Δ of clausoids, such that $\Gamma \cup \{\zeta'[\mathrm{Nam}_{\mathbb{X}}]^{[\![\mathbb{X}]\!]}\} \subseteq \Delta$ and $\bot \notin \Delta$.*

<u>Proof.</u> Let $\zeta = \zeta'[\mathrm{Nam}_{\mathbb{X}}]^{[\![\mathbb{X}]\!]}$; then according to (16B1), $\zeta' = \zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and according to (17J), ζ has disjoint dependency with all elements of Γ. Moreover, since there are only finite number of names occurring in the clauses of Γ' and all components of $\mathbb{X}$ are infinite, there exists an injective `Sort`-indexed function $f : \mathbb{X} \to \mathbb{X}$, such that for any clausoid ε over $\mathbb{X}$, $\varepsilon[\![f]\!]$ has disjoint dependency with all elements of Γ. Notice that according to (20T), for any ε, ε and $\varepsilon[\![f]\!]$ are variants. In order to prove the proposition, we are going to build a finite $\mathfrak{e}$-$\mathfrak{sel}$-closed set Δ, such that $\Gamma \subseteq \Delta$, $\bot \notin \Delta$ and $\zeta \in \Delta$.

We are going to define set $\Sigma_0, \Sigma_1, \Sigma_2, \dots$. Let $\Delta_i = \Gamma \cup \Sigma_0 \cup \Sigma_1 \cup \dots \cup \Sigma_i$ and $\Delta = \Gamma \cup \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \dots$.

Let $\Sigma_0 = \{\zeta\}$. Since ζ is the only negative element of Σ_0, δ and ε have disjoint dependency whenever δ is a negative element of Δ_0 and ε is a non-negative element of Δ_0.

Let Σ_1 be the set of all clausoids of the form $\varepsilon[\![f]\!]$, where ε is an $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of an element of Σ_0 and an element of Γ. Due to the way f is defined, the set Δ_1 still has the property that δ and ε have disjoint dependency whenever δ is a negative element of Δ_1 and ε is a non-negative element of Δ_1.

Similarly, let Σ_2 be the set of all clausoids of the form $\varepsilon[\![f]\!]$, where ε is an $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of an element of Σ_1 and an element of Γ, Σ_3 be the set of all clausoids of the form $\varepsilon[\![f]\!]$, where ε is an $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of an element of Σ_2 and an element of Γ, and so on.

The set Δ still has the property that δ and ε have disjoint dependency whenever δ is a negative element of Δ and ε is a non-negative element of Δ. Therefore, Δ is an $\mathfrak{e}$-$\mathfrak{sel}$-closed set.

We will say that n is a numerical bound of the clause ε if the tree $\mathfrak{t}$ has a node labelled with a variant of ε, such that the maximal distance from this node to a leaf of $\mathfrak{t}$ is less than n. We will say that n is a numerical bound of the clausoid ε, if n is a numerical bound of $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$.

Since $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} = \zeta'$ labels the root of $\mathfrak{t}$, the clausoid ζ has a numerical bound. Let k be a numerical bound of ζ.

By induction on i, we are going to prove that $k-i$ is a numerical bound of all elements of Σ_i.

Since $\Sigma_0 = \{\zeta\}$, k is a numerical bound of the elements of Σ_0. Suppose that $k-i$ is a numerical bound of the elements of Σ_i and let $\eta \in \Sigma_{i+1}$. Then η has to be a variant of the $\mathfrak{e}$-$\mathfrak{sel}$-SLD resolvent of some clausoid $\delta \in \Sigma_i$ and a clausoid $\varepsilon \in \Gamma$. Since $\delta \in \Sigma_i$, δ has a numerical bound $k-i$, so there is a node in $\mathfrak{t}$ labelled with a variant δ' of $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$, such that the maximal distance from this node to a leaf of $\mathfrak{t}$ is less than $k-i$.

From the definition of Γ if follows that $\varepsilon = \varepsilon'[\mathrm{Nam}_{\mathbb{X}}]^{[\![\mathbb{X}]\!]}$ for some $\varepsilon' \in \Gamma'$ and, according to (16B1), $\varepsilon' = \varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$, so from this we can conclude that $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} \in \Gamma'$. According to (F), δ' and ε' have a $\mathfrak{mgu}$-$\mathfrak{sel}$-SLD resolvent η', such that η' is a variant of $\eta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$. Since $\mathfrak{t}$ is a $\mathfrak{mgu}$-$\mathfrak{sel}$-SLD search tree, the node labelled with δ' has a child labelled with a variant of η', hence also with a variant of $\eta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$. According to the definition of "numerical bound", $k-(i+1)$ is a numerical bound of η.

This completes the proof of the statement that $k-i$ is a numerical bound of all elements of Σ_i. From this we will be able to conclude that Δ is finite and that $\bot \notin \Delta$.

Indeed, since no clausoid can have a negative number as a numerical bound and $n-i$ is a numerical bound of the elements of Σ_i, the sets $\Sigma_{k+1}, \Sigma_{k+2}, \Sigma_{k+3}, \ldots$ are equal to $\varnothing$. On the other hand, from (D) it follows that the sets Σ_i are finite for any particular i. Therefore, Δ is finite.

Moreover, if ε is a clausoid having a numerical bound, then some node of $\mathfrak{t}$ has to be labelled with a variant of $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$, so $\varepsilon \neq \bot$. Therefore, $\bot \notin \Sigma_i$, so $\bot \notin \Delta$. ■

The following Corollary says that if Prolog fails to prove that a goal φ follows from a program Γ and during the process does not go into an infinite loop, then there exists a finite model of $\Gamma \cup \{\neg\varphi\}$. The author has published this result in [33].

L) **Corollary.** *Let Γ' be a finite set of non-negative Horn clauses over $\mathbb{X}$, $\mathfrak{sel}$ be a selection function and $\mathfrak{t}$ be a finite $\mathfrak{mgu}$-$\mathfrak{sel}$-SLD search tree for Γ', such that none of the leafs of $\mathfrak{t}$ is labelled with $\bot$. If the root of $\mathfrak{t}$ is labelled with ζ', then the set $\Gamma' \cup \{\zeta'\}$ is universally satisfiable in an algebra with finite carriers.*

Proof. Later in this work we will see that there exists termally sound, termally complete and near-complete equaliser $\mathfrak{e}$ for the so called epsilon-terminator, such that $\mathfrak{e}(\Theta)$ never contains more than one element. Let

$$\Gamma = \{\delta[\mathrm{Nam}_{\mathbb{X}}]^{[\![\mathbb{X}]\!]} : \delta \in \Gamma'\}$$

Without loss of generality we may assume that ζ' has disjoint dependency with all elements of Γ', so from (21K) we can conclude that there exists a finite $\mathfrak{e}$-$\mathfrak{sel}$-closed set Δ of epsilon-clausoids, such that $\Gamma \cup \{\zeta'[\mathrm{Nam}_{\mathbb{X}}]^{[\![\mathbb{X}]\!]}\} \subseteq \Delta$ and $\bot \notin \Delta$. According to (I), the set Δ is universally satisfiable in almost any algebra. If a finite set of clausoids is universally satisfiable in almost any algebra, then it is universally satisfiable in an algebra with finite carriers (we are going to prove this in 29M). It only remains to use (16J3) and (16B1) in order to conclude that the union of Γ and and the set of the clauses labelling nodes of $\mathfrak{t}$ is universally satisfiable in an algebra with finite carriers. ∎

M) **Remark.** Our definition of $\mathfrak{e}$-$\mathfrak{sel}$-SLD search tree differs from what Prolog actually does in the following ways:

- The clauses of a Prolog program are ordered and Prolog applies them in the specified order. No clause is applied more than once. On the other hand, in the definition of "$\mathfrak{e}$-$\mathfrak{sel}$-SLD search tree" no specific order is prescribed and nothing forbids a clause to be used more than once in order to produce many child nodes in the tree. This is not a problem because while not every $\mathfrak{e}$-$\mathfrak{sel}$-SLD search tree corresponds to what Prolog does, every search tree built by Prolog satisfies our definition.
- Almost all implementations of Prolog use unification without the so called "occurs check".[62] This is not a problem, since if a Prolog implementation without occurs check fails to prove a goal and stops after finite time, then every Prolog implementation with occurs check is going to fail and stop after finite time.

[62] The only exception I am aware of is Strawberry Prolog [9].

§22. PROPOSITIONAL POSITIVE HYPERRESOLUTION

A) Throughout this section $\mathbf{A}$ will be some fixed normal algebra.

B) **Definition.** (1) Given a clause δ with sequence $\langle \lambda_1, \ldots, \lambda_n \rangle$ and a literal λ occurring in δ, by $\delta \setminus \lambda$ we will denote the clause, whose sequence is obtained from $\langle \lambda_1, \ldots, \lambda_n \rangle$ by removing all occurrences of λ.

(2) Given a clause δ and a set Γ of literals occurring in δ, by $\delta \setminus \Gamma$ we will denote the clause, whose sequence is obtained from the sequence of δ by removing all occurrences of elements of Γ.

(3) A *clash sequence* is a sequence $\langle \delta, \varepsilon_1, \ldots, \varepsilon_n \rangle$, where $n \geq 1$, δ is a non-positive clause and $\varepsilon_1, \ldots, \varepsilon_n$ are positive clauses.

C) **Definition.** (1) Given a relational clause δ over $|\mathbf{A}|$ with sequence $\langle \lambda_1, \ldots, \lambda_n \rangle$ and a positive relational clause ε over $|\mathbf{A}|$, if λ_i is a negative literal, such that $\overline{\lambda_i}$ occurs in ε and there are no negative literals among $\lambda_1, \ldots, \lambda_{i-1}$, then the clause whose sequence is obtained from the sequence of δ by replacing λ_i with the sequence of $\varepsilon \setminus \overline{\lambda_i}$ is called *propositional positive resolvent* of δ and ε. The literal λ_i is called *resolved literal*.

(2) Let $\langle \delta, \varepsilon_1, \ldots, \varepsilon_n \rangle$ be a clash sequence of relational clauses over $|\mathbf{A}|$. Let $\delta_0, \ldots, \delta_n$ be such that $\delta_0 = \delta$ and δ_{i+1} be a propositional positive resolvent of δ_i and ε_{i+1} for any $i \in \{0, \ldots, n-1\}$. If δ_n is a positive clause, then δ_n is called *propositional positive hyperresolvent* defined by the clash sequence $\langle \delta, \varepsilon_1, \ldots, \varepsilon_n \rangle$.

D) **Corollary.** (1) *Let ε be a propositional positive hyperresolvent defined by the clash sequence $\langle \delta_0, \delta_1, \ldots, \delta_n \rangle$. Let $\lambda_1, \ldots, \lambda_k$ be the sequence of all negative literals of δ_0 in the same order as they occur in the sequence of δ_0. Then $k = n$, $\overline{\lambda_i}$ occurs in δ_i for all $i \in \{1, \ldots, n\}$ and the sequence of ε is obtained from the sequence of δ_0 by replacing each λ_i with the sequence of $\delta_i \setminus \overline{\lambda_i}$.*

(2) *If ε is a propositional positive hyperresolvent defined by the clash sequence $\langle \delta_0, \delta_1, \ldots, \delta_n \rangle$, then all positive literals of δ_0 occur in ε.*

Proof. **(1)** follows immediately from the previous definition.
(2) follows from (1). ∎

E) **Proposition.** (1) *The propositional positive resolvent of relational clauses δ and ε over $|\mathbf{A}|$ follows in $\mathbf{A}$ from $\{\delta, \varepsilon\}$.*

(2) *The propositional positive hyperresolvent of a clash sequence of relational clauses over $|\mathbf{A}|$ follows in $\mathbf{A}$ from the set of its elements.*

<u>Proof.</u> **(1)** Let $\langle \lambda_1, \dots, \lambda_n \rangle$ be the sequence of δ and the resolvent is obtained from the sequence of δ by replacing λ_i with the sequence of $\varepsilon \setminus \overline{\lambda}_i$. Suppose that $\mathbf{M}$ is a logical variant of $\mathbf{A}$ and both δ and ε are true in $\mathbf{M}$. If λ_i is false in $\mathbf{M}$, then δ contains some literal μ, such that $\mu \neq \lambda$ and μ is true in $\mathbf{M}$. But μ occurs in the resolvent too, hence the resolvent is true in $\mathbf{M}$. Otherwise, λ_i is true in $\mathbf{M}$, so $\overline{\lambda}_i$ is false in $\mathbf{M}$, hence ε contains at least one literal μ, such that $\mu \neq \overline{\lambda}_i$ and μ is true in $\mathbf{M}$. But μ occurs in the resolvent, hence the resolvent is true in $\mathbf{M}$.

(2) follows from (1). ∎

F) **Definition.** The relation $\Gamma \vdash \delta$, where Γ and δ are respectively a set of relational clauses over $|\mathbf{A}|$ and a relational clause over $|\mathbf{A}|$, is the minimal deductive relation, such that $\delta_1, \dots, \delta_n \vdash \varepsilon$ is true whenever ε is a hyperresolvent, defined by the clash sequence $\langle \delta_1, \dots, \delta_n \rangle$.

This definition is correct, because according to definition (C), all hyperresolvents of a clash sequence of relational clauses over $|\mathbf{A}|$, are always relational clauses over $|\mathbf{A}|$.

G) **Corollary.** *The deductive relation $\vdash$ is finitary.*

<u>Proof.</u> Follows immediately from (19G) where $\mathfrak{p}(\Gamma, \delta)$ is the relation "δ is a propositional positive hyperresolvent, defined by a clash sequence $\langle \delta_1, \dots, \delta_n \rangle$, such that $\Gamma = \{\delta_1, \dots, \delta_n\}$". ∎

H) **Lemma.** *Given a set Γ of relational clauses over $|\mathbf{A}|$, if $\Gamma \vdash \delta$ then δ follows from Γ in $\mathbf{A}$.*

<u>Proof.</u> We are going to apply the simple inductive principle (19D) on the relation $\vdash$. Let $\Delta = \{\varphi : \varphi \text{ follows from } \Gamma \text{ in } \mathbf{A}\}$. Obviously, $\Gamma \subseteq \Delta$. Moreover, if $\{\delta_1, \dots, \delta_n\} \subseteq \Delta$ and δ is an hyperresolvent, defined by the clash sequence $\langle \delta_1, \dots, \delta_n \rangle$, then $\delta \in \Delta$ because of (E2). Consequently, conditions 1 and 2 of (19D) are satisfied, hence $\Gamma \vdash \delta$ has to imply that δ follows from Γ in $\mathbf{A}$. ∎

I) **Definition.** (1) A *condition* is a set $\mathfrak{a}$ of relational literals over $|\mathbf{A}|$, such that $\mathfrak{a}$ is satisfiable in $\mathbf{A}$.

(2) The structure $\mathbf{M}$ *satisfies* the condition $\mathfrak{a}$, if $\mathbf{M}$ is a logical variant of $\mathbf{A}$ and all literals from $\mathfrak{a}$ are true in $\mathbf{M}$.

(3) The partial relation $\mathfrak{a} \models \varphi$ is true if φ is true in all satisfying $\mathfrak{a}$ structures. It is not true, if φ is false in all satisfying $\mathfrak{a}$ structures. Otherwise, $\mathfrak{a} \models \varphi$ is undefined.[63] We are going to use also the expressions "φ is true

[63] Since $\mathbf{A}$ is a normal structure, there exists logical structure which is a variant of $\mathbf{A}$,

in $\mathfrak{a}$", "φ is false in $\mathfrak{a}$" and "φ is undefined in $\mathfrak{a}$".

(4) A condition $\mathfrak{a}$ *forces* a set Γ of relational clauses, written $\mathfrak{a} \Vdash \Gamma$, if $\Gamma \cup \mathfrak{a} \vdash \bot$ is not true.

J) **Theorem.** *Let* $\mathbf{A}$ *be a normal algebra. Then a set* Γ *of relational clauses over* $|\mathbf{A}|$ *is not satisfiable in* $\mathbf{A}$ *if and only if* $\Gamma \vdash \bot$.

Proof. The "if" part follows from (H).

In order to prove the "only if" part, we are going to apply Theorem (J). The "sentences" in (19J) are the relational clauses over $|\mathbf{A}|$ and the relation $\prec$ is the subset inclusion of the conditions. Lemmas (K), (N), (S) and (T) imply (19J1), (19J2), (19J3) and (19J4), respectively.

Let Γ be a set of relational clauses over $|\mathbf{A}|$, such that $\Gamma \vdash \bot$ is not true. Let $\mathfrak{a} = \varnothing$. Since $\mathbf{A}$ is a normal structure, there exists a logical variant of $\mathbf{A}$, so $\mathfrak{a}$ is a condition. Moreover, obviously $\mathfrak{a}$ is a condition forcing Γ. From (19J) we obtain a condition $\mathfrak{b}$, such that $\mathfrak{b} \models \delta$ is true for all elements δ of Γ. Let $\mathbf{M}$ be an arbitrary logical variant of $\mathbf{A}$, such that all elements of $\mathfrak{b}$ are true in $\mathbf{M}$ (there is such structure, because $\mathfrak{b}$ is a condition). Then all elements of Γ are true in $\mathbf{M}$, hence Γ is satisfiable in $\mathbf{A}$. ■

K) **Lemma.** *If* $\mathfrak{a}$ *forces* Γ*, then* $\mathfrak{a}$ *forces all subsets of* Γ.

Proof. Let $\Delta \subseteq \Gamma$. If $\mathfrak{a} \Vdash \Gamma$, then $\Gamma \cup \mathfrak{a} \vdash \bot$ is not true, so $\Delta \cup \mathfrak{a} \vdash \bot$ is not true due to the monotonicity of $\vdash$ (19B), hence $\mathfrak{a} \Vdash \Delta$. ■

Often we are going to use the following Lemma without specific references. The reader is kindly asked to remember it.

L) **Lemma.** *Let* $\mathfrak{a}$ *be a condition and* λ *be a relational literal over* $|\mathbf{A}|$*. Then:*

(1) $\mathfrak{a} \models \lambda$ *is true if and only if* $\lambda \in \mathfrak{a}$.

(2) $\mathfrak{a} \models \lambda$ *is false if and only if* $\overline{\lambda} \in \mathfrak{a}$.

Proof. We are going to give a proof only to **(2)**. The other item of this lemma can be proved analogously.

The "if" part of the condition immediately follows from the definitions: if $\overline{\lambda} \in \mathfrak{a}$, then $\overline{\lambda}$ is true in all satisfying $\mathfrak{a}$ structures, so λ is false in all satisfying $\mathfrak{a}$ structures, hence $\mathfrak{a} \models \lambda$ is false.

In order to prove the "only if" part, suppose $\lambda = \mathtt{p}(\tau_1, \ldots, \tau_n)$ (the case $\lambda = \neg\mathtt{p}(\tau_1, \ldots, \tau_n)$ is analogous). Let $\langle\langle\kappa_1, \ldots, \kappa_n\rangle, \mathtt{Log}\rangle$ be the type of $\mathtt{p}$. The literal λ is relational, so for all $i \in \{1, \ldots, n\}$ there are $\alpha_i \in \mathbf{A}_{\kappa_i}$, such that $\tau_i = \ulcorner\alpha_i\urcorner$.

so it is impossible for $\mathfrak{a} \models \varphi$ to be both true and false.

Let $\mathbf{M}$ be an arbitrary satisfying $\mathfrak{a}$ structure. The literal λ is false in $\mathbf{M}$, so $\lambda^{\mathbf{M}} = \mathtt{p}^{\mathbf{M}}\langle\tau_1^{\mathbf{M}}, \dots, \tau_n^{\mathbf{M}}\rangle = \mathtt{p}^{\mathbf{M}}\langle\alpha_1, \dots, \alpha_n\rangle = 0$. Let $\mathbf{M}'$ be a structure with the same universe as $\mathbf{M}$, where the operation symbols are interpreted the same way as in $\mathbf{M}$ but with only one difference, namely $\mathtt{p}^{\mathbf{M}'}\langle\alpha_1, \dots, \alpha_n\rangle = 1$. Then $\mathtt{p}^{\mathbf{M}'}\langle\tau_1^{\mathbf{M}'}, \dots, \tau_n^{\mathbf{M}'}\rangle = \mathtt{p}^{\mathbf{M}'}\langle\alpha_1, \dots, \alpha_n\rangle = 1$, so the literal λ is true in $\mathbf{M}'$. But $\mathfrak{a} \models \lambda$ is false, so $\mathbf{M}'$ does not satisfy $\mathfrak{a}$, hence $\mathfrak{a}$ contains some literal μ which is false in $\mathbf{M}'$. On the other hand μ has to be true in $\mathbf{M}$ because $\mathbf{M}$ satisfies $\mathfrak{a}$.

If $\mathtt{q}(\tau_1', \dots, \tau_n')$ is an arbitrary relational atomic formula and $\langle\langle\kappa_1', \dots, \kappa_n'\rangle, \mathtt{Log}\rangle$ is the type of $\mathtt{q}$, then for all $i \in \{1, \dots, n\}$ there are $\beta_i \in \mathbf{A}_{\kappa_i'}$, such that $\tau_i' = \ulcorner\beta_i\urcorner$. Then $(\mathtt{q}(\tau_1', \dots, \tau_n'))^{\mathbf{M}} = \mathtt{q}^{\mathbf{M}}\langle\tau_1'^{\mathbf{M}}, \dots, \tau_n'^{\mathbf{M}}\rangle = \mathtt{q}^{\mathbf{M}}\langle\beta_1, \dots, \beta_n\rangle$ and $(\mathtt{q}(\tau_1', \dots, \tau_n'))^{\mathbf{M}'} = \mathtt{q}^{\mathbf{M}'}\langle\tau_1'^{\mathbf{M}'}, \dots, \tau_n'^{\mathbf{M}'}\rangle = \mathtt{q}^{\mathbf{M}'}\langle\beta_1, \dots, \beta_n\rangle$, so considering this and what is the only difference between $\mathbf{M}'$ and $\mathbf{M}$, we see that there can be only one relational literal, which is true in $\mathbf{M}$ and false in $\mathbf{M}'$, namely $\neg\mathtt{p}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner)$. Consequently, $\mu = \neg\mathtt{p}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner)$, so $\mu = \neg\mathtt{p}(\tau_1, \dots, \tau_n) = \overline{\lambda}$. But $\mu \in \mathfrak{a}$, so $\overline{\lambda} \in \mathfrak{a}$. ∎

M) **Lemma.** *Given a condition $\mathfrak{a}$ and a relational clause δ over $|\mathbf{A}|$, $\mathfrak{a} \models \delta$ is false if and only if $\mathfrak{a} \models \lambda$ is false for all literals λ in δ.*

Proof. ($\Longrightarrow$) Let λ be an arbitrary literal occurring in δ. Let $\mathbf{M}$ be an arbitrary satisfying $\mathfrak{a}$ structure. The clause δ is false in $\mathbf{M}$ because $\mathbf{M}$ satisfies $\mathfrak{a}$ and $\mathfrak{a} \models \delta$ is false. Therefore all occurring in δ literals have to be false in $\mathbf{M}$. In particular, λ is false in $\mathbf{M}$. Thus λ is false in all satisfying $\mathfrak{a}$ structures, hence $\mathfrak{a} \models \lambda$ is false.

($\Longleftarrow$) Let $\mathbf{M}$ be an arbitrary satisfying $\mathfrak{a}$ structure. All literals from δ are false in $\mathbf{M}$, hence δ is false in $\mathbf{M}$. Thus δ is false in all satisfying $\mathfrak{a}$ structures, hence $\mathfrak{a} \models \delta$ is false. ∎

N) **Lemma.** *If $\mathfrak{a} \Vdash \{\delta\}$, then $\mathfrak{a} \models \delta$ is true or undefined.*

Proof. Suppose $\mathfrak{a} \models \delta$ is false. Then δ is false in all satisfying $\mathfrak{a}$ structures, so by (M) all literals of δ are false in all satisfying $\mathfrak{a}$ structures, hence by (L) for all literals λ occurring in δ, the literals $\overline{\lambda}$ belong to $\mathfrak{a}$. From $\mathfrak{a} \Vdash \{\delta\}$ it follows that $\{\delta\} \cup \mathfrak{a} \vdash \bot$ is not true. Consequently, there exists a clause δ, such that:

- $\{\delta\} \cup \mathfrak{a} \vdash \bot$ is not true.
- For all literals λ in δ, $\overline{\lambda} \in \mathfrak{a}$.

From all clauses having these two properties, let ε be some with smallest possible number of literals.

If ε contains negative literals, let $\lambda_1, \ldots, \lambda_n$ be all negative literals in ε in the same order as they occur in the sequence of ε. Then by definition (B3), the sequence $\langle \varepsilon, \overline{\lambda}_1, \ldots, \overline{\lambda}_n \rangle$ is a clash sequence. Let ε' be the hyperresolvent defined by this clash sequence. The sequence of ε' is obtained from the sequence of ε by removing the occurrences of $\lambda_1, \ldots, \lambda_n$. The clause ε' contains smaller number of literals than ε. At the same time all literals of ε' are literals of ε too, hence $\overline{\lambda} \in \mathfrak{a}$ for all literals λ of ε'. On the other hand, $\{\varepsilon'\} \cup \mathfrak{a} \vdash \bot$ can not be true, since $\{\varepsilon\} \cup \mathfrak{a} \vdash \varepsilon'$. This is contradiction, because by choice, ε has the smallest possible number of literals among all clauses satisfying these properties.

If ε does not contain negative literals, we can reason analogously. In this case let λ be an arbitrary literal of ε. Then use the clash sequence $\langle \overline{\lambda}, \varepsilon \rangle$. ∎

O) **Lemma.** *Given a condition $\mathfrak{a}$ and a clause δ, if $\{\lambda, \overline{\lambda}\} \cap \mathfrak{a} \neq \varnothing$ for every occurring δ relational literal λ, then $\mathfrak{a} \models \delta$ is defined.*

Proof. From (L) it follows that $\mathfrak{a} \models \lambda$ is either true or false for every occurring in δ literal λ.

If δ contains a literal λ, such that $\mathfrak{a} \models \lambda$ is true, then λ is true in all satisfying $\mathfrak{a}$ structures, hence δ is true in all such structures, so in this case $\mathfrak{a} \models \delta$ is true.

Otherwise, δ does not contain such a literal, so all occurring in δ literals are false in all satisfying $\mathfrak{a}$ structures, hence δ is false in all such structures, so $\mathfrak{a} \models \delta$ is false. ∎

P) **Lemma.** *Given a relational literal λ over $|\mathbf{A}|$ and a condition $\mathfrak{a}$, $\mathfrak{a} \cup \{\lambda\}$ is a condition if and only if $\overline{\lambda} \notin \mathfrak{a}$.*

Proof. $\mathfrak{a} \cup \{\lambda\}$ is a condition if and only if it is a satisfiable in $\mathbf{A}$ set, if and only if λ is true in some satisfying $\mathfrak{a}$ structure, if and only if $\mathfrak{a} \models \lambda$ is not false, if and only if $\overline{\lambda} \notin \mathfrak{a}$ (due to L). ∎

Q) **Lemma.** *Given a set Γ of relational clauses over $|\mathbf{A}|$, a negative relational literal λ over $|\mathbf{A}|$ and a relational clause δ over $|\mathbf{A}|$, if $\delta \neq \lambda$ and $\Gamma \cup \{\lambda\} \vdash \delta$, then there exists a relational clause δ' over $|\mathbf{A}|$, such that $\Gamma \vdash \delta'$ and the sequence of δ can be obtained from the sequence of δ' by removing some occurrences of $\overline{\lambda}$.*

Proof. Let Δ be the set of all relational clauses ε over $|\mathbf{A}|$, such that there exists a relational clause ε' over $|\mathbf{A}|$, such that $\Gamma \vdash \varepsilon'$ and the sequence of ε can be obtained from the sequence of ε' by removing some occurrences of $\overline{\lambda}$. We are going to prove that $\Gamma \cup \{\lambda\} \vdash \varepsilon$ implies $\varepsilon \in \Delta \cup \{\lambda\}$ by the simple inductive principle (19D).

If $\varepsilon \in \Gamma \cup \{\lambda\}$, then obviously $\varepsilon \in \Delta \cup \{\lambda\}$ with $\varepsilon' = \varepsilon$.

Otherwise ε is a propositional positive hyperresolvent, defined by a clash sequence $\langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \rangle$, where each ε_i is an element of $\Delta \cup \{\lambda\}$. If $i \geq 1$, then $\varepsilon_i \in \Delta$, since $\varepsilon_1, \ldots, \varepsilon_n$ are positive clauses and λ is negative.

Regarding ε_0, we have two cases. If $\varepsilon_0 = \lambda$, then $n = 1$ and the sequence of ε is obtained from the sequence of ε_1 by removing all occurrences of $\overline{\lambda}$. By induction hypothesis there exists a clause ε_1', such that $\Gamma \vdash \varepsilon_1'$ and the sequence of ε_1 can be obtained from the sequence of ε_1' by removing some occurrences of $\overline{\lambda}$. Consequently we can use $\varepsilon' = \varepsilon_1'$ in order to see that $\varepsilon \in \Delta$.

If $\varepsilon_0 \neq \lambda$, then by induction hypothesis there exist clauses $\varepsilon_0', \varepsilon_1', \ldots, \varepsilon_n'$, such that for all i, $\Gamma \vdash \varepsilon_i'$ and the sequence of ε_i can be obtained from the sequence of ε_i' by removing some occurrences of $\overline{\lambda}$. Clearly, $\langle \varepsilon_0', \varepsilon_1', \ldots, \varepsilon_n' \rangle$ is a clash sequence and we can use it in order to derive a propositional positive hyperresolvent ε' using exactly the same resolved literals as with the sequence $\langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \rangle$ in order to derive ε. Then the sequence of ε can be obtained from the sequence of ε' by removing some occurrences of $\overline{\lambda}$, hence $\varepsilon \in \Delta$. ∎

R) **Lemma.** *Given a condition* $\mathfrak{a}$*, a relational literal* λ *over* $|\mathbf{A}|$ *and a set* Γ *of relational clauses over* $|\mathbf{A}|$*, if* $\{\lambda, \overline{\lambda}\} \cap \mathfrak{a} = \varnothing$ *and* $\mathfrak{a} \Vdash \Gamma$*, then either* $\mathfrak{a} \cup \{\lambda\} \Vdash \Gamma$*, or* $\mathfrak{a} \cup \{\overline{\lambda}\} \Vdash \Gamma$*.*

Proof. Since the condition of the Lemma is symmetric with regard to λ and $\overline{\lambda}$, without loss of generality we may assume that the literal λ is negative.

Suppose that both $\mathfrak{a} \cup \{\lambda\} \Vdash \Gamma$ and $\mathfrak{a} \cup \{\overline{\lambda}\} \Vdash \Gamma$ are false, that is $\mathfrak{a} \cup \Gamma \cup \{\lambda\} \vdash \bot$ and $\mathfrak{a} \cup \Gamma \cup \{\overline{\lambda}\} \vdash \bot$. From (Q) and $\mathfrak{a} \cup \Gamma \cup \{\lambda\} \vdash \bot$ it follows that $\mathfrak{a} \cup \Gamma \vdash \delta'$ for some clause δ', such that all elements of its sequence are equal to $\overline{\lambda}$.

Now, we are going to use the simple inductive principle (19D) in order to prove that for any clause ε, from $\mathfrak{a} \cup \Gamma \cup \{\overline{\lambda}\} \vdash \varepsilon$ it follows $\varepsilon = \overline{\lambda}$ or $\mathfrak{a} \cup \Gamma \vdash \varepsilon$. Then from $\mathfrak{a} \cup \Gamma \cup \{\overline{\lambda}\} \vdash \bot$ it will follow $\mathfrak{a} \cup \Gamma \vdash \bot$ and this will contradict $\mathfrak{a} \Vdash \Gamma$.

If $\varepsilon \in \mathfrak{a} \cup \Gamma \cup \{\overline{\lambda}\}$, then trivially $\varepsilon = \overline{\lambda}$ or $\mathfrak{a} \cup \Gamma \vdash \varepsilon$.

Otherwise, ε is a propositional positive hyperresolvent defined by a clash sequence $\langle \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \rangle$. By induction hypothesis, for all i either $\varepsilon_i = \overline{\lambda}$, or $\mathfrak{a} \cup \Gamma \vdash \varepsilon_i$. Define $\varepsilon_i' = \varepsilon_i$ if $\varepsilon_i \neq \overline{\lambda}$ and $\varepsilon_i' = \delta'$ if $\varepsilon_i = \overline{\lambda}$. Then $\mathfrak{a} \cup \Gamma \vdash \varepsilon_i'$ for all i. As $\overline{\lambda}$ is a positive literal, $\varepsilon_0 \neq \overline{\lambda}$, so $\varepsilon_0' = \varepsilon_0$. Clearly, ε is a propositional positive hyperresolvent defined by the clash sequence $\langle \varepsilon_0', \varepsilon_1', \ldots, \varepsilon_n' \rangle$. ∎

S) **Lemma.** *Given a condition* $\mathfrak{a}$ *and a set* Γ *of relational clauses* *over* $|\mathbf{A}|$*, if* $\mathfrak{a} \Vdash \Gamma$ *and* $\mathfrak{a} \models \delta$ *is undefined for at least one relational clause* δ *over* $|\mathbf{A}|$*, then there exists a condition* $\mathfrak{b}$*, such that* $\mathfrak{a} \subseteq \mathfrak{b}$*,* $\mathfrak{a} \neq \mathfrak{b}$ *and* $\mathfrak{b} \Vdash \Gamma$*.*

Proof. Since $\mathfrak{a} \models \delta$ is undefined, (O) implies that $\{\lambda, \overline{\lambda}\} \cap \mathfrak{a} = \varnothing$ for at least one literal λ over $|\mathbf{A}|$, hence (R) implies that either $\mathfrak{a} \cup \{\lambda\} \Vdash \Gamma$, or $\mathfrak{a} \cup \{\overline{\lambda}\} \Vdash \Gamma$. ∎

T) **Lemma.** *Given a non-empty set* Λ *of conditions, such that for any* $\mathfrak{a}', \mathfrak{a}'' \in \Lambda$ *either* $\mathfrak{a}' \subseteq \mathfrak{a}''$*, or* $\mathfrak{a}'' \subseteq \mathfrak{a}'$*, there exists a condition* $\mathfrak{b}$*, such that* $\mathfrak{a} \subseteq \mathfrak{b}$ *for all* $\mathfrak{b} \in \Lambda$ *and* $\mathfrak{b}$ *forces any set of relational clauses over* $|\mathbf{A}|$ *which is forced by all elements of* Λ*.*

Proof. Let $\mathfrak{b}$ be the union of all elements of Λ. From (13H) it follows that $\mathfrak{b}$ is a condition.

Suppose the set Γ is forced by all elements of Λ, but $\mathfrak{b} \Vdash \Gamma$ is false. Then $\mathfrak{b} \cup \Gamma \vdash \bot$, so from the finitarity of $\vdash$ we obtain a finite subset $\Delta \subseteq \mathfrak{b} \cup \Gamma$, such that $\Delta \vdash \bot$. Let $\mathfrak{b}' = \mathfrak{b} \cap \Delta$. Then $\mathfrak{b}'$ is a finite subset of $\mathfrak{b}$, such that $\Delta \subseteq \mathfrak{b}' \cup \Gamma$, hence $\mathfrak{b}' \cup \Gamma \vdash \bot$, so $\mathfrak{b}'$ does not force Γ. This is a contradiction, because all finite subsets of $\mathfrak{b}$ are subsets of some element of Λ, hence all finite subsets of $\mathfrak{b}$ force Γ. ∎

§23. POSITIVE HYPERRESOLUTION

A) **Definition.** (1) A clause δ *subsumes* the clause ε, if all literals of δ occur in ε.

(2) Given an algebra $\mathbf{A}$, a clause (clausoid) δ over $\mathbb{X}$ *subsumes* in $\mathbf{A}$ the clause (clausoid) ε if every instance of ε in $\mathbf{A}$ is subsumed by some instance of δ in $\mathbf{A}$.

(3) A *condensing function* is a function $\mathfrak{f}$ mapping clauses (clausoids) over $\mathbb{X}$ to clauses (clausoids) over $\mathbb{X}$, such that for any δ, $\mathfrak{f}(\delta)$ subsumes δ and the number of the negative literals in $\mathfrak{f}(\delta)$ is less than or equal to the number of the negative literals in δ.

(4) Given a logical structure $\mathbf{M}$, a condensing function $\mathfrak{f}$ is $\mathbf{M}$-*sound*, if for any universally valid in $\mathbf{M}$ clause (clausoid) δ over $\mathbb{X}$, $\mathfrak{f}(\delta)$ also is universally valid in $\mathbf{M}$.

Notice that every clause subsumes itself. In addition, if δ subsumes ε, then δ subsumes ε in every algebra.

B) **Proposition.** *Given a logical structure* $\mathbf{M}$ *and clauses (clausoids)* δ *and* ε *over* $\mathbb{X}$*, if* δ *is universally valid in* $\mathbf{M}$ *and* δ *subsumes* ε *in* $\partial\mathbf{M}$*, then* ε *is universally valid in* $\mathbf{M}$*.*

<u>Proof.</u> δ is universally valid in $\mathbf{M}$, hence (20H) implies that all instances in $\partial\mathbf{M}$ of δ are true in $\mathbf{M}$. But δ subsumes ε in $\partial\mathbf{M}$, hence every instance of ε in $\partial\mathbf{M}$ is subsumed by some instance of δ in $\partial\mathbf{M}$, so every instance of ε in $\partial\mathbf{M}$ is subsumed by some true in $\mathbf{M}$ clause or clausoid, hence every instance of ε in $\partial\mathbf{M}$ is true in $\mathbf{M}$, so by (20H) ε is universally valid in $\mathbf{M}$. ■

C) **Definition.** (1) Given a termal equaliser $\mathfrak{e}$, a clause δ over $\mathbb{X}$ with sequence $\langle\lambda_1, \ldots, \lambda_n\rangle$, a positive clause ε over $\mathbb{X}$ and a non-empty set Γ of literals occurring in ε, if λ_i is a negative literal, such that there are no negative literals among $\lambda_1, \ldots, \lambda_{i-1}$ and

$$s \in \mathfrak{e}(\{\lambda_i \sim \overline{\mu} : \mu \in \Gamma\}),$$

then the clause whose sequence is obtained from the sequence of $\delta[s]^{[\mathbb{X}]}$ by replacing the literal corresponding to λ_i with the sequence of $(\varepsilon \setminus \Gamma)[s]^{[\mathbb{X}]}$ is called *positive* $\mathfrak{e}$*-resolvent* of δ and ε. The literal λ_i is called *resolved literal*.

Given a termoidal equaliser $\mathfrak{e}$, the notions "positive $\mathfrak{e}$-resolvent of clausoids" and "resolved literaloid" are defined analogously, but instead of clauses, literals, $\delta[s]^{[\mathbb{X}]}$ and $(\varepsilon \setminus \Gamma)[s]^{[\mathbb{X}]}$, we use clausoids, literaloids, $\delta[\![s]\!]^{[\![\mathbb{X}]\!]}$ and $(\varepsilon \setminus \Gamma)[\![s]\!]^{[\![\mathbb{X}]\!]}$, respectively.

(2) Let $\mathfrak{e}$ be an equaliser and $\mathfrak{f}$ be a condensing function. Let $\langle\delta, \varepsilon_1, \ldots, \varepsilon_n\rangle$ be a clash sequence of clauses or clausoids over $\mathbb{X}$. Let $\delta_0, \ldots, \delta_n$ be such that $\delta_0 = \delta$ and δ_{i+1} is the result of the application of $\mathfrak{f}$ to some positive $\mathfrak{e}$-resolvent of a variant of δ_i and a variant of ε_{i+1} (both variants having disjoint dependency). If δ_n is a positive clause or clausoid, then δ_n is called *positive* $\mathfrak{ef}$*-hyperresolvent* defined by the clash sequence $\langle\delta, \varepsilon_1, \ldots, \varepsilon_n\rangle$.

D) Regarding definition (C1), notice that for any particular clauses (clausoids) δ and ε, there are finitely many possible sets Γ and finitely many systems $\{\lambda_i = \overline{\mu} : \mu \in \Gamma\}$, hence there are finitely many possible substitutions s. Therefore, for any particular clauses (clausoids) δ and ε, there are at most finitely many positive $\mathfrak{e}$-resolvents.

E) Also notice that if δ is a clause (clausoid) whose sequence contains n negative literals and ε is a positive clause (clausoid), then any positive $\mathfrak{e}$-resolvent of δ and ε contains $n-1$ negative literals. Because of this, if there exists a positive $\mathfrak{ef}$-hyperresolvent defined by the clash sequence $\langle\delta, \varepsilon_1, \ldots, \varepsilon_n\rangle$, then the sequence of δ contains exactly n negative literals (literaloids).

F) **Definition.** (1) A *reducing function* $\mathfrak{g}$ is a function mapping each set Γ of clauses (clausoids) over $\mathbb{X}$ to a set $\mathfrak{g}(\Gamma)$ of clauses (clausoids) over $\mathbb{X}$.

(2) Given an algebra $\mathbf{A}$, a reducing function $\mathfrak{g}$ is $\mathbf{A}$-sound, if for any set Γ of clauses (clausoids) over $\mathbb{X}$ each element of $\mathfrak{g}(\Gamma)$ is subsumed in $\mathbf{A}$ by some element of Γ.

(3) Given an algebra $\mathbf{A}$, a reducing function $\mathfrak{g}$ is $\mathbf{A}$-complete, if for any set Γ of clauses (clausoids) over $\mathbb{X}$ each element of Γ is subsumed in $\mathbf{A}$ by some element of $\mathfrak{g}(\Gamma)$.

G) **Definition.** (1) Given an equaliser $\mathfrak{e}$, a condensing function $\mathfrak{f}$, a reducing function $\mathfrak{g}$ and a set Γ of clauses (clausoids) over $\mathbb{X}$, let Γ' be the set of all positive $\mathfrak{ef}$-hyperresolvents defined by clash sequences whose elements belong to Γ. Then the set $\mathfrak{g}(\Gamma \cup \Gamma')$ will be denoted by $\mathfrak{res}(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$.

(2) For any natural number n, $\mathfrak{res}^n(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$ is the iterative application of the operator $\mathfrak{res}$. Namely, $\mathfrak{res}^0(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma) = \Gamma$ and $\mathfrak{res}^{n+1}(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma) = \mathfrak{res}(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \mathfrak{res}^n(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma))$.

(3) Let $\mathfrak{res}^*(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$ be the union of all $\mathfrak{res}^n(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$ for all n.

Notice that $\mathfrak{res}^n$ is not necessarily monotone on n. Because of the reducing function $\mathfrak{g}$, the situations when $\mathfrak{res}^n(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$ is not a subset of $\mathfrak{res}^{n+1}(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$ are common. Nevertheless, The following proposition compensates sufficiently for the non-monotonicity.

H) **Proposition.** *Given an equaliser $\mathfrak{e}$, a condensing function $\mathfrak{f}$, an algebra $\mathbf{A}$, an $\mathbf{A}$-complete reducing function $\mathfrak{g}$ and a set Γ of clauses (clausoids), if $\delta \in \mathfrak{res}^n(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$, then for any $m \geq n$ there exists $\varepsilon \in \mathfrak{res}^m(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$, such that δ is subsumed in $\mathbf{A}$ by ε.*

Proof. By induction on m. When $m = n$ the proposition becomes trivial (we can take $\varepsilon = \delta$).

Suppose that $\varepsilon \in \mathfrak{res}^m(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$ and δ is subsumed in $\mathbf{A}$ by ε. Let Δ be the set of all positive $\mathfrak{ef}$-hyperresolvents defined by clash sequences whose elements belong to $\mathfrak{res}^m(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$. Then $\mathfrak{res}^{m+1}(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma) = \mathfrak{g}(\mathfrak{res}^m(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma) \cup \Delta)$. Since $\mathfrak{g}$ is $\mathbf{A}$-complete and $\varepsilon \in \mathfrak{res}^m(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$, there exists some $\varepsilon' \in \mathfrak{res}^{m+1}(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$, such that ε is subsumed in $\mathbf{A}$ by ε'. Obviously, the relation "subsumed" is transitive, so δ is subsumed in $\mathbf{A}$ by ε'. ■

Soundness for Structures of Terms

I) The positive hyperresolution with termoids is not sound for arbitrary structures. It is possible for clausoids to be universally valid in a structure and yet — some their positive hyperresolvent not to be universally valid in the same structure. Nevertheless, we are going to prove that the positive hyperresolution is sound for structures of terms. Since the Herbrand structures are structures of terms and considering that a set of clausoids is universally satisfiable if and only if it is universally satisfiable in a Herbrand algebra (see 16K), we can conclude that the positive hyperresolution preserves the universal satisfiability. If a set Γ is universally satisfiable and we add to it positive hyperresolvents, the resulting set also is going to be universally satisfiable.

J) **Lemma.** *Given a logical structure* $\mathbf{M}$ *of terms,*[64] *a termally sound termal (termoidal) equaliser* $\mathfrak{e}$ *and universally valid in* $\mathbf{M}$ *clauses (clausoids)* δ_0 *and* δ_1 *over* $\mathbb{X}$*, if* ε *is a positive* $\mathfrak{e}$*-resolvent of some variants of* δ_0 *and* δ_1*, then* ε *is universally valid in* $\mathbf{M}$ *too.*

Proof. We will give the proof for the case of clausoids. The proof for the case of clauses is completely analogous.

Let ε' be an arbitrary instance of ε in $\partial\mathbf{M}$. We are going to prove that ε' is true in $\mathbf{M}$. From this and (20H) it will follow that ε is universally valid in $\mathbf{M}$.

Let δ_0' and δ_1' be variants of δ_0 and δ_1, such that ε is a positive $\mathfrak{e}$-resolvent of δ_0' and δ_1'. According to the definition of a positive $\mathfrak{e}$-resolvent (C1), there exists a non-empty set Γ of literals occurring in δ_1' and a literal λ', such that λ' is the first negative literal in the sequence of δ_0' and for some

$$s \in \mathfrak{e}(\{\lambda' = \overline{\mu} : \mu \in \Gamma\})$$

the sequence of ε can be obtained from the sequence of $\delta_0'[\![s]\!]^{[\![\mathbb{X}]\!]}$ by replacing the literal corresponding to λ' with the sequence of $(\delta_1' \setminus \Gamma)[\![s]\!]^{[\![\mathbb{X}]\!]}$. Since ε' is an instance in $\partial\mathbf{M}$ of ε, there exists an assignment function $v : \mathbb{X} \to |\partial\mathbf{M}|$, such that $\varepsilon' = \varepsilon[\![v]\!]^{\partial\mathbf{M}}$, see (20C1) and (20E). From (20L5) and (20L7) it follows that the sequence of ε' can be obtained from the sequence of $\delta_0'[\![s]\!]^{[\![\mathbb{X}]\!]}[\![v]\!]^{\partial\mathbf{M}}$ by replacing the literal corresponding to λ' with the sequence of $(\delta_1' \setminus \Gamma)[\![s]\!]^{[\![\mathbb{X}]\!]}[\![v]\!]^{\partial\mathbf{M}}$.

Let $w = ([\![v]\!]^{\partial\mathbf{M}}) \circ s$. From (11T) and (14T2) it follows that $\tau[\![s]\!]^{[\![\mathbb{X}]\!]}[\![v]\!]^{\partial\mathbf{M}} = \tau[\![w]\!]^{\partial\mathbf{M}}$ for any τ. Consequently, the sequence of ε' can be obtained from the sequence of $\delta_0'[\![w]\!]^{\partial\mathbf{M}}$ by replacing the literal corresponding to λ' with the sequence of $(\delta_1' \setminus \Gamma)[\![w]\!]^{\partial\mathbf{M}}$.

Since the equaliser $\mathfrak{e}$ is termally sound, w is a solution in $\mathbf{M}$ of all identities belonging to $\{\lambda' \sim \overline{\mu} : \mu \in \Gamma\}$, i.e. $\lambda'[\![w]\!]^{\partial\mathbf{M}} = \overline{\mu}[\![w]\!]^{\partial\mathbf{M}}$ for any $\mu \in \Gamma$. Consequently, the corresponding to λ' literal of $\delta_0'[\![w]\!]^{\partial\mathbf{M}}$ can not be true in $\mathbf{M}$ if at least one corresponding to a literaloid of Γ literal of $\delta_1[\![w]\!]^{\partial\mathbf{M}}$ is true in $\mathbf{M}$.

[64] In other words, $\mathbf{M}$ is a Herbrand structure.

On the other hand, $\delta'_0[\![w]\!]^{\partial\mathbf{M}}$ and $\delta'_1[\![w]\!]^{\partial\mathbf{M}}$ are instances in $\partial\mathbf{M}$ of δ'_0 and δ'_1, hence by (20Q) they are instances of δ_0 and δ_1 too. But δ_0 and δ_1 are universally valid in $\partial\mathbf{M}$, hence both $\delta'_0[\![w]\!]^{\partial\mathbf{M}}$ and $\delta'_1[\![w]\!]^{\partial\mathbf{M}}$ are true in $\partial\mathbf{M}$. This means that at least one literal of $\delta'_0[\![w]\!]^{\partial\mathbf{M}}$ is true in $\mathbf{M}$ and at least one literal of $\delta'_1[\![w]\!]^{\partial\mathbf{M}}$ is true in $\mathbf{M}$.

If at least one literal of $\delta'_0[\![w]\!]^{\partial\mathbf{M}}$ other than the one corresponding to λ' is true in $\mathbf{M}$, then ε' is true in $\mathbf{M}$ because all literals of $\delta'_0[\![w]\!]^{\partial\mathbf{M}}$ except the one corresponding to λ' occur in ε'. Otherwise, the literal corresponding to λ' is true, hence none of the literals in $\{\mu[\![w]\!]^{\partial\mathbf{M}} : w \in \Gamma\}$ can be true in $\mathbf{M}$. But $\delta'_1[\![w]\!]^{\partial\mathbf{M}}$ is true in $\mathbf{M}$, hence at least one literal of $\delta'_1[\![w]\!]^{\partial\mathbf{M}}$ which is not corresponding to a literaloid of Γ is true in $\mathbf{M}$. All literals of $\delta'_1[\![w]\!]^{\partial\mathbf{M}}$ which are not corresponding to a literaloid of Γ occur in ε', hence, again, in this case we can conclude that ε' is true in $\mathbf{M}$. ∎

K) **Corollary.** *Given a logical structure* $\mathbf{M}$ *of terms, a termally sound termal (termoidal) equaliser* $\mathfrak{e}$ *and an* $\mathbf{M}$*-sound condensing function* $\mathfrak{f}$*, if* Γ *is a set of universally valid in* $\mathbf{M}$ *clauses (clausoids) over* $\mathbb{X}$*, then all* $\mathfrak{ef}$*-hyperresolvents defined by clash sequences whose elements belong to* Γ *are universally valid in* $\mathbf{M}$.

Proof. Let δ be an $\mathfrak{ef}$-hyperresolvent defined by the clash sequence $\langle\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n\rangle$, where $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \in \Gamma$. By definition (C2), there exist clauses (clausoids) $\delta_0, \delta_1, \ldots, \delta_n$, such that $\delta_0 = \varepsilon_0$, $\delta_n = \delta$ and δ_{i+1} is the result of the application of $\mathfrak{f}$ to some positive $\mathfrak{e}$-resolvent of a variant of δ_i and a variant of ε_{i+1}. We are going to prove by induction on i that δ_i is universally valid in $\mathbf{M}$. This will complete the proof because $\delta = \delta_n$.

The case $i = 0$ is easy, as $\delta_0 = \varepsilon_0$ and we know that ε_0 is universally valid in $\mathbf{M}$.

Suppose that δ_i is universally valid in $\mathbf{M}$. Since δ_i and ε_{i+1} are universally valid, (20R) implies that all their variants are universally valid in $\mathbf{M}$ too, hence (J) implies that all $\mathfrak{e}$-resolvents of a variant of δ_i and a variant of ε_{i+1} are universally valid in $\mathbf{M}$. But $\mathfrak{f}$ is $\mathbf{M}$-sound and δ_{i+1} is the result of the application of $\mathfrak{f}$ to such variants, hence δ_{i+1} is universally valid in $\mathbf{M}$. ∎

The following Theorem states the soundness of the positive hyperresolution for logical structures of terms.

L) **Theorem.** *Given a logical structure* $\mathbf{M}$ *of terms, a termally sound equaliser* $\mathfrak{e}$*, an* $\mathbf{M}$*-sound condensing function* $\mathfrak{f}$ *and a* $\partial\mathbf{M}$*-sound reducing function* $\mathfrak{g}$*, if* Γ *is a set of universally valid in* $\mathbf{M}$ *clauses (clausoids) over* $\mathbb{X}$*, then* $\perp \notin \mathfrak{res}^*(\mathfrak{e}, \mathfrak{f}, \mathfrak{g}\,; \Gamma)$.

Proof. By induction on n we are going to prove that all elements of $\mathfrak{res}^n(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$ are universally valid in $\mathbf{M}$. Since $\perp$ is not true in $\mathbf{M}$, this completes the proof.

The case $n = 0$ is trivial, because $\mathfrak{res}^0(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma) = \Gamma$ and Γ is a set of universally valid in $\mathbf{M}$ clauses or clausoids.

Suppose that all elements of $\mathfrak{res}^n(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$ are universally valid in $\mathbf{M}$. Let Γ' be the set of all $\mathfrak{ef}$-hyperresolvents defined by clash sequences whose elements belong to $\mathfrak{res}^n(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$. The induction hypothesis and (K) imply that the elements of Γ' are universally valid in $\mathbf{M}$, hence all elements of $\mathfrak{res}^n(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma) \cup \Gamma'$ are universally valid in $\mathbf{M}$. Since $\mathfrak{g}$ is $\partial\mathbf{M}$-sound, each element of $\mathfrak{g}(\mathfrak{res}^n(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma) \cup \Gamma')$ is subsumed in $\partial\mathbf{M}$ by some element of $\mathfrak{res}^n(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma) \cup \Gamma'$, hence each element of $\mathfrak{g}(\mathfrak{res}^n(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma) \cup \Gamma')$ is subsumed in $\partial\mathbf{M}$ by some universally valid in $\mathbf{M}$ clause or clausoid. But by definition (G) $\mathfrak{res}^{n+1}(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma) = \mathfrak{g}(\mathfrak{res}^n(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma) \cup \Gamma')$, hence each element of $\mathfrak{res}^{n+1}(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$ is subsumed in $\partial\mathbf{M}$ by some universally valid in $\mathbf{M}$ clause or clausoid. Now (B) implies that the elements of $\mathfrak{res}^{n+1}(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$ are universally valid in $\mathbf{M}$. ∎

Completeness for Almost Any Normal Algebra

M) **Lemma.** *Let the* `Sort`*-indexed functions $d', d'' : \mathbb{X} \to \mathbb{X}$ be such that both d' and d'' have injective components and for any sort κ the images of d'_κ and d''_κ have empty intersection. Then:*

(1) *For any* `Sort`*-indexed functions $f', f'' : \mathbb{X} \to Y$ there exists a* `Sort`*-indexed function $f : \mathbb{X} \to Y$, such that $f' = f \circ d'$ and $f'' = f \circ d''$.*

(2) *For any termal (termoidal) expression τ over $\mathbb{X}$, $\tau[d']$ and $\tau[d'']$ (resp. $\tau[\![d']\!]$ and $\tau[\![d'']\!]$) have disjoint dependency.*

Proof. **(1)** Take an arbitrary sort κ and $\xi \in \mathbb{X}_\kappa$. If there exists $\eta \in \mathbb{X}$, such that $d'_\kappa\eta = \xi$, then let $f_\kappa\xi = f'\eta$. If there exists $\eta \in \mathbb{X}$, such that $d''_\kappa\eta = \xi$, then let $f_\kappa\xi = f''\eta$. Otherwise let $f_\kappa\xi$ be arbitrary. According to this definition, $f_\kappa(d'_\kappa\eta) = f'\eta$ and $f_\kappa(d''_\kappa\eta) = f''\eta$ for any $\eta \in \mathbb{X}_\kappa$. Consequently, $f' = f \circ d'$ and $f'' = f \circ d''$.

(2) The lemma is obvious in the case of termal expressions. Let τ be a termoidal expression. Let $\mathbb{X}'$ and $\mathbb{X}''$ be `Sort`-indexed subsets of $\mathbb{X}$, such that the image of d' is a subset of $\mathbb{X}'$, the image of d'' is a subset of $\mathbb{X}''$ and $\mathbb{X}'$ and $\mathbb{X}''$ have no common elements. According to (14I4), $\tau[\![d']\!]$ is a termoidal expression over $\mathbb{X}'$ and $\tau[\![d'']\!]$ is a termoidal expression over $\mathbb{X}''$. Suppose that there exist some $\xi \in \mathbb{X}$, such that both $\tau[\![d']\!]$ and $\tau[\![d'']\!]$ depend on ξ. According to definition (17C), there exist functions $g', g'' : \mathbb{X} \to \mathbb{X}$, such that g' and g'' are identity over all elements of $\mathbb{X}$ except ξ and $\tau[\![d']\!][\![g']\!] \neq \tau[\![d']\!]$ and $\tau[\![d'']\!][\![g'']\!] \neq \tau[\![d'']\!]$. Since

$\tau[\![d']\!][\![g']\!] = \tau[\![g' \circ d']\!] = \tau[\![(g' \restriction \mathbb{X}') \circ d']\!] = \tau[\![d']\!][\![g' \restriction \mathbb{X}']\!]$, the function $g' \restriction \mathbb{X}'$ is not identity, hence ξ belongs to $\mathbb{X}'$. Analogously, ξ belongs to $\mathbb{X}''$. This is a contradiction. ∎

N) **Lemma (*the lifting lemma for resolvents*).** *Given a near-complete equaliser $\mathfrak{e}$ and clauses (clausoids) δ_0 and δ_1 over $\mathbb{X}$, there exist variants δ_0'' and δ_1'' of δ_0 and δ_1, such that δ_0'' and δ_1'' have disjoint dependency and the following is true for almost any normal algebra $\mathbf{A}$:*

If the relational clauses δ_0' and δ_1' over $|\mathbf{A}|$ are respective instances in $\mathbf{A}$ of δ_0 and δ_1 and ε' is a propositional positive resolvent of δ_0' and δ_1', then there exists a clause (clausoid) ε over $\mathbb{X}$, such that ε' is an instance in $\mathbf{A}$ of ε and ε is a positive $\mathfrak{e}$-resolvent of δ_0'' and δ_1''.

Proof. We will give the proof for the case of clausoids. The differences for the case of clauses will be provided in footnotes.

Let the `Sort`-indexed functions $d_0, d_1 : \mathbb{X} \to \mathbb{X}$ be with injective components and for any sort κ the images of the components $(d_0)_\kappa$ and $(d_1)_\kappa$ have empty intersection. From (20T) it follows that $\delta_0[\![d_0]\!]$ is a variant of δ_0 and $\delta_1[\![d_1]\!]$ is a variant of δ_1. Let $\delta_0'' = \delta_0[\![d_0]\!]$ and $\delta_1'' = \delta_1[\![d_1]\!]$. According to (M2), δ_0'' and δ_1'' have disjoint dependency.

We are going to use the functions d_0 and d_1 in order to define a finite set Γ of termally inconsistent systems, such that the property stated in this Lemma is true for any algebra $\mathbf{A}$, such that none of the systems of Γ has a solution in $\mathbf{A}$. Let Γ be the set of all termally inconsistent systems whose identities have the form $\lambda[\![d_0]\!] \sim \overline{\mu[\![d_1]\!]}$ where λ is a negative literal of δ_0 and μ is a positive literal of δ_1.

Let $\mathbf{A}$ be an arbitrary algebra, sush that none of the systems in Γ has a solution in $\mathbf{A}$.

Let $\langle \lambda_1', \ldots, \lambda_n' \rangle$, $\langle \lambda_1, \ldots, \lambda_n \rangle$, $\langle \mu_1', \ldots, \mu_m' \rangle$ and $\langle \mu_1, \ldots, \mu_m \rangle$ be the sequences, respectively, of δ_0', δ_0, δ_1' and δ_1. Let λ_j' be a negative literal, such that the literals $\lambda_1', \ldots, \lambda_{j-1}'$ are positive. Let $\mu_{k_1}', \ldots, \mu_{k_t}'$ be all literals of δ_1', such that $\lambda_j' = \overline{\mu_{k_i}'}$, $i \in \{k_1, \ldots, k_t\}$. Then by the definition of propositional positive resolvent, the sequence of ε' can be obtained from the sequence of δ_0' by replacing λ_j' with the sequence of $\delta_1' \setminus \{\mu_{k_1}', \ldots, \mu_{k_t}'\}$.

Let the `Sort`-indexed functions $v_0 : \mathbb{X} \to |\mathbf{A}|$ and $v_1 : \mathbb{X} \to |\mathbf{A}|$ be such that $\delta_0' \in \delta_0[\![v_0]\!]^{\mathcal{P}\mathbf{A}}$ and $\delta_1' \in \delta_1[\![v_1]\!]^{\mathcal{P}\mathbf{A}}$.[65] Because of (20L), $\lambda_i' \in \lambda_i[\![v_0]\!]^{\mathcal{P}\mathbf{A}}$ and $\mu_i' \in \mu_i[\![v_1]\!]^{\mathcal{P}\mathbf{A}}$ for any i.[66]

According to (M1), there exists a `Sort`-indexed function $v : \mathbb{X} \to |\mathbf{A}|$, such that $v_0 = v \circ d_0$ and $v_1 = v \circ d_1$.

[65] Such that $\delta_0' = \delta_0[v_0]^{\mathbf{A}}$ and $\delta_1' = \delta_1[v_1]^{\mathbf{A}}$.
[66] $\lambda_i' = \lambda_i[v_0]^{\mathbf{A}}$ and $\mu_i' = \mu_i[v_1]^{\mathbf{A}}$

Notice that $\lambda'_j \in \lambda_j[\![v_0]\!]^{\mathcal{P}\mathbf{A}} = \lambda_j[\![v \circ d_0]\!]^{\mathcal{P}\mathbf{A}} = (\lambda_j[\![d_0]\!])[\![v]\!]^{\mathcal{P}\mathbf{A}}$ and at the same time for any $r \in \{k_1, \ldots, k_t\}$ we have $\lambda'_j = \overline{\mu'_r} \in \overline{\mu_r[\![v_1]\!]^{\mathcal{P}\mathbf{A}}} = \overline{\mu_r[\![v \circ d_1]\!]^{\mathcal{P}\mathbf{A}}} = \overline{(\mu_r[\![d_1]\!])[\![v]\!]^{\mathcal{P}\mathbf{A}}} = (\overline{\mu_r[\![d_1]\!]})[\![v]\!]^{\mathcal{P}\mathbf{A}}$.[67] Consequently, v is a solution of Θ in $\mathbf{A}$, where Θ is the system of all identities of the form

$$\lambda_j[\![d_0]\!] = \overline{\mu_r[\![d_1]\!]},$$

where $r \in \{k_1, \ldots, k_t\}$.

Since Θ has a solution in $\mathbf{A}$, Θ can not be an element of Γ. Therefore, Θ is termally consistent.

Since the equaliser $\mathfrak{e}$ is near-complete, there exists $s \in \mathfrak{e}(\Theta)$ and a `Sort`-indexed function $w : \mathbb{X} \to |\mathbf{A}|$, such that $v \ll ([\![w]\!]^{\mathcal{P}\mathbf{A}}) \circ s$.[68]

Let ε be the clausoid whose sequence is obtained from the sequence of $(\delta_0[\![d_0]\!])[\![s]\!]^{[\![\mathbb{X}]\!]}$ by replacing $(\lambda_j[\![d_0]\!])[\![s]\!]^{[\![\mathbb{X}]\!]}$ with the sequence of $(\delta_1[\![d_1]\!] \setminus \{\mu_{k_1}[\![d_1]\!], \ldots, \mu_{k_t}[\![d_1]\!]\})[\![s]\!]^{[\![\mathbb{X}]\!]}$. By the definition of positive $\mathfrak{e}$-resolvent, ε is a positive $\mathfrak{e}$-resolvent of $\delta_0[\![d_0]\!]$ and $\delta_1[\![d_1]\!]$, hence it is a positive $\mathfrak{e}$-resolvent of variants of δ_0 and δ_1.

It remains to show that ε' is an instance of ε. Because of the way ε' and ε are defined and (20L), it is enough to show that δ'_0 is an instance of $(\delta_0[\![d_0]\!])[\![s]\!]^{[\![\mathbb{X}]\!]}$ and δ'_1 is an instance of $(\delta_1[\![d_1]\!])[\![s]\!]^{[\![\mathbb{X}]\!]}$.

From $v \ll ([\![w]\!]^{\mathcal{P}\mathbf{A}}) \circ s$ and (14T1) it follows that for any termoidal expression τ over $\mathbb{X}$ we have $\tau[\![v]\!]^{\mathcal{P}\mathbf{A}} \subseteq (\tau[\![s]\!]^{[\![\mathbb{X}]\!]})[\![w]\!]^{\mathcal{P}\mathbf{A}}$.[69] In particular, when $\tau = \delta_0[\![d_0]\!]$ we obtain[70] that

$$\begin{aligned}
\delta'_0 &\in \delta_0[\![v_0]\!]^{\mathcal{P}\mathbf{A}} && \text{by the definition of } v_0\\
&= \delta_0[\![v \circ d_0]\!]^{\mathcal{P}\mathbf{A}} && \text{because } v_0 = v \circ d_0\\
&= (\delta_0[\![d_0]\!])[\![v]\!]^{\mathcal{P}\mathbf{A}} && \text{from (14I7)}\\
&\subseteq ((\delta_0[\![d_0]\!])[\![s]\!]^{[\![\mathbb{X}]\!]})[\![w]\!]^{\mathcal{P}\mathbf{A}} && \text{when } \tau = \delta_0[\![d_0]\!]
\end{aligned}$$

[67] Notice that $\lambda'_j = \lambda_j[v_0]^{\mathbf{A}} = \lambda_j[v \circ d_0]^{\mathbf{A}} = (\lambda_j[d_0])[v]^{\mathbf{A}}$ and at the same time for any $r \in \{k_1, \ldots, k_t\}$ we have $\lambda'_j = \overline{\mu'_r} = \overline{\mu_r[v_1]^{\mathbf{A}}} = \overline{\mu_r[v \circ d_1]^{\mathbf{A}}} = \overline{(\mu_r[d_1])[v]^{\mathbf{A}}} = (\overline{\mu_r[d_1]})[v]^{\mathbf{A}}$.

[68] Such that $v = ([w]^{\mathbf{A}}) \circ s$.

[69] From $v = ([w]^{\mathbf{A}}) \circ s$ and (11T) it follows that $\tau[v]^{\mathbf{A}} = (\tau[s]^{[\mathbb{X}]})[w]^{\mathbf{A}}$.

[70] We obtain that

$$\begin{aligned}
\delta'_0 &= \delta_0[v_0]^{\mathbf{A}} && \text{by the definition of } v_0\\
&= \delta_0[v \circ d_0]^{\mathbf{A}} && \text{because } v_0 = v \circ d_0\\
&= (\delta_0[d_0])[v]^{\mathbf{A}} && \text{from (11H)}\\
&= ((\delta_0[d_0])[s]^{[\mathbb{X}]})[w]^{\mathbf{A}} && \text{when } \tau = \delta_0[d_0]
\end{aligned}$$

Hence δ_0' is an instance of $(\delta_0[\![d_0]\!])[\![s]\!]^{[\![\mathbb{X}]\!]}$. Analogously it can be shown that δ_1' is an instance of $(\delta_1[\![d_1]\!])[\![s]\!]^{[\![\mathbb{X}]\!]}$. ∎

O) **Lemma** ***(the lifting lemma for*** $\mathfrak{ef}$***-hyperresolvents).*** *Given a near-complete equaliser* $\mathfrak{e}$*, a condensing function* $\mathfrak{f}$ *and clausoids* $\delta_0, \delta_1, \ldots, \delta_n$ *over* $\mathbb{X}$*, the following is true for almost any normal algebra* $\mathbf{A}$*:*

If the relational clauses $\delta_0', \delta_1', \ldots, \delta_n'$ *over* $|\mathbf{A}|$ *are subsumed by some instances in* $\mathbf{A}$ *of* $\delta_0, \delta_1, \ldots, \delta_n$*, respectively, and* ε' *is a propositional positive hyperresolvent defined by the clash sequence* $\langle\delta_0', \delta_1', \ldots, \delta_n'\rangle$*, then either* ε' *is subsumed by an instance of an element of* $\{\delta_0, \delta_1, \ldots, \delta_n\}$ *or* ε' *is subsumed by an instance of some positive* $\mathfrak{ef}$*-hyperresolvent defined by a clash sequence whose elements are among* $\delta_0, \delta_1, \ldots, \delta_n$*.*

Proof. First, we are going to define by recursion finite sets $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ Let $\Gamma_0 = \{\delta_0, \delta_1, \ldots, \delta_n\}$. For any i, there exist finitely many pairs $\zeta', \zeta'' \in \Gamma_i$, so we can use Lemmas (N) and (18X) in order to find finitely many variants of the clausoids of these pairs, such that for almost any algebra $\mathbf{A}$ any propositional positive resolvent of instances in $\mathbf{A}$ of elements of Γ_i is an instance in $\mathbf{A}$ of a positive $\mathfrak{e}$-resolvent of some of these finitely many variants. In addition to this, according to (D), there are finitely many such positive $\mathfrak{e}$-resolvents. Let Γ_{i+1} be the set containing all elements of Γ_i and all results of the application of $\mathfrak{f}$ to such $\mathfrak{e}$-resolvents.

Let t be the number of the negative literaloids in δ_0.

For almost any algebra $\mathbf{A}$, any propositional positive resolvent of instances in $\mathbf{A}$ of elements of Γ_t is an instance in $\mathbf{A}$ of a positive $\mathfrak{e}$-resolvent of some variants of elements of Γ_t. Let $\mathbf{A}$ be one such algebra.

Let $\lambda_1, \ldots, \lambda_n$ be all negative literals of δ_0' in that order. Let $\delta_0'', \delta_1'', \ldots, \delta_n''$ be relational clauses over $|\mathbf{A}|$, such that δ_i' is subsumed by δ_i'' for any i and δ_i'' is an instance of δ_i for any i.

If $\overline{\lambda}_i$ does not occur in δ_i'' for some i, then δ_i'' subsumes $\delta_i' \setminus \overline{\lambda}_i$, hence it subsumes ε' too, so in this case ε' is subsumed by an instance of an element of $\{\delta_0, \delta_1, \ldots, \delta_n\}$.

Analogously, if none of $\lambda_1, \ldots, \lambda_n$ occurs in δ_0'', then δ_0'' subsumes ε', so in this case ε' is subsumed by an instance of an element of $\{\delta_0, \delta_1, \ldots, \delta_n\}$ too.

Let Δ be the set of all relational clauses over $|\mathbf{A}|$, such that all their literals occur either in δ_0' or in $\delta_i' \setminus \overline{\lambda}_i$ for some i. Since each of the literals $\lambda_1, \ldots, \lambda_n$ is negative, all positive literals occurring in an element of Δ occur in ε' too, hence if an element of Δ is a positive clause, then it subsumes ε'. Consequently, in order to prove the Lemma, it will be enough to show that some instance of a positive $\mathfrak{ef}$-hyperresolvent defined by a clash sequence whose elements are among $\delta_0, \delta_1, \ldots, \delta_n$ is an element of Δ.

Notice that any clause subsuming an element of Δ belongs to Δ itself.

We are going to find a natural number m, such that $m \leq t$, and to define clausoids $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m$ over $\mathbb{X}$, such that for any i:

1. ε_i belongs to Γ_i and an instance of ε_i belongs to Δ;
2. ε_{i+1} is the result of the application of $\mathfrak{f}$ to some positive $\mathfrak{c}$-resolvent of a variant of ε_i and a variant of an element of $\{\delta_0, \delta_1, \ldots, \delta_n\}$;

Let $\varepsilon_0 = \delta_0$. Then $\varepsilon_0 \in \Gamma_0$ and δ''_0 is an instance of ε_0 and belongs to Δ.

If ε_i does not contain any negative literals for some i, then we are done — let $m = i$. Notice that $m \neq 0$ as we have already considered the case when δ''_0 contains zero negative literals.[71] According to 2. and definition (C2), ε_m is a positive $\mathfrak{cf}$-hyperresolvent defined by a clash sequence whose elements are among $\delta_0, \delta_1, \ldots, \delta_n$. At the same time, according to 1., an instance of ε_m belongs to Δ.

Otherwise, continue. Let ε''_i be an instance of ε_i, such that $\varepsilon''_i \in \Delta$. The definition of Δ implies that all negative literals of ε''_i are among $\lambda_1, \ldots, \lambda_n$, hence the first negative literal occurring in the sequence of ε''_i is equal to λ_{k_i} for some k_i. Let ζ''_{i+1} be the clause whose sequence is obtained from the sequence of ε''_i by replacing λ_{k_i} with the sequence of $\delta''_{k_1} \setminus \overline{\lambda}_{k_i}$.

By definition (22C1), ζ''_{i+1} is a propositional positive resolvent of ε''_i and δ''_{k_i}. But ε''_i and δ''_{k_i} are instances of ε_i and δ_{k_i}, respectively, and both ε_i and δ_{k_i} belong to Γ_i. Therefore, there exists some positive $\mathfrak{c}$-resolvent ζ_{i+1} of variants of ε_i and δ_{k+1}, such that $\mathfrak{f}(\zeta_{i+1}) \in \Gamma_{i+1}$ and ζ''_{i+1} is an instance of ζ_{i+1}. Let $\varepsilon_{i+1} = \mathfrak{f}(\zeta_{i+1})$; then $\varepsilon_{i+1} \in \Gamma_{i+1}$.

According to the definition of condensing function, ε_{i+1} subsumes ζ_{i+1} in $\mathbf{A}$. But ζ''_{i+1} is an instance in $\mathbf{A}$ of ζ_{i+1}, hence ζ''_{i+1} is subsumed by an instance of ε_{i+1}. On the other hand, from $\varepsilon''_i \in \Delta$ and the definition of ζ''_{i+1} it follows that $\zeta''_{i+1} \in \Delta$. Consequently, an instance of ε_{i+1} subsumes an element of Δ, hence an instance of ε_{i+1} belongs to Δ.

It only remains to see that this process is finite and that $m \leq t$. This is so, because according to the definition of condensing function (A3), the application of $\mathfrak{f}$ does not increase the number of the negative literaloids. Consequently, ε_{i+1} contains less or equal negative literals than ζ_{i+1}. In addition, ζ_{i+1} contains exactly one negative literaloid less than ε_i, hence the clausoid ε_{i+1} contains strictly less negative literaloids than ε_i. Therefore, for any i, the number of the negative literaloids in ε_i is smaller than or equal to $t - i$. ∎

[71] If δ''_0 contains zero negative literals, then none of $\lambda_1, \ldots, \lambda_n$ occurs in δ''_0.

P) **Definition.** A set Γ of clauses (clausoids) is *essentially finite* if there exists a finite subset Δ of Γ, such that for any $\delta \in \Gamma$ some variant of δ belongs to Δ.

Q) **Theorem.** *Given a near-complete equaliser* $\mathfrak{e}$*, a condensing function* $\mathfrak{f}$*, a reducing function* $\mathfrak{g}$ *which is* **A***-complete for almost any normal algebra* **A***, and a set* Γ *of clausoids, if* $\mathfrak{res}^*(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$ *is essentially finite and* $\bot \notin \mathfrak{res}^*(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$*, then* Γ *is universally satisfiable in almost any normal algebra.*

Proof. The essential finiteness of $\mathfrak{res}^*(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$ and (23E) imply that there exists a finite set of clash sequences whose clausoids belong to $\mathfrak{res}^*(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$ and who are able to derive variants of all hyperresolvents belonging to $\mathfrak{res}^*(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$. Therefore, for almost any normal algebra we are permitted to apply (O) for clash sequences whose elements belong to $\mathfrak{res}^*(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$.

Let **A** be one such algebra. Suppose that Γ is not universally satisfiable in **A**. According to (20I), the set Γ' of all instances in **A** of the elements of Γ is not satisfiable in **A**, so (22J) implies that $\Gamma' \vdash \bot$, where the relation $\vdash$ is defined in (22F).

By using the simple inductive principle (19D), we can prove that whenever $\Gamma' \vdash \delta'$, there exists a natural number n, such that for any $m \geq n$ there exists a clausoid $\delta \in \mathfrak{res}^m(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$, such that δ' is subsumed by an instance in **A** of δ. This will give us a contradiction because $\bot$ can be subsumed only by $\bot$ and $\bot$ can be an instance only of $\bot$, but $\bot \notin \mathfrak{res}^*(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$.

If $\delta' \in \Gamma'$, then δ' is an instance in **A** of some $\varepsilon \in \Gamma$. Since $\Gamma = \mathfrak{res}^0(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$, according to (23H), for any $m \geq 0$, there exists $\delta \in \mathfrak{res}^m(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$, such that ε is subsumed in **A** by δ. Therefore, δ' is subsumed by an instance of δ.

Now, suppose that δ' is a propositional positive hyperresolvent defined by the clash sequence $\langle \delta'_0, \delta'_1, \ldots, \delta'_k \rangle$. By induction hypothesis, there exist natural numbers $n_0, n_1, \ldots, n_k$, such that whenever $m \geq n_i$, there exists a clausoid $\delta \in \mathfrak{res}^m(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$, such that δ'_i is subsumed by an instance of δ. Let $n = \max\{n_0, n_1, \ldots, n_k\}$. Then for any $m \geq n$ there exist clausoids $\delta_0, \delta_1, \ldots, \delta_k$, such that for any i, $\delta_i \in \mathfrak{res}^m(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$ and δ'_i is subsumed by an instance of δ_i. Since $\delta_0, \delta_1, \ldots, \delta_k$ belong to $\mathfrak{res}^m(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$, we may use (O) and conclude that δ' is subsumed by an instance of an element of $\mathfrak{res}^{m+1}(\mathfrak{e},\mathfrak{f},\mathfrak{g}\,;\Gamma)$. ■

Gamma-, Delta- and Epsilon-terminators

§24. THE GAMMA-TERMINATOR

A) For any functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1,\ldots,\kappa_n\rangle,\lambda\rangle$, let $\mathtt{f}_1^{-1},\ldots,\mathtt{f}_n^{-1}$ be some new symbols, different from any operation symbol, parentheses, comma, or any other formal symbol we use. For any sort κ, let $\triangle_\kappa$ be a new symbol different from all mentioned above symbols.

B) **Definition.** Let X be an arbitrary $\mathtt{Sort}$-indexed set. We define the *gamma-semitermoids* over X inductively:

(1) If $\mathtt{y}\in X_\kappa$, then $\mathrm{nam}_{X,\kappa}(\mathtt{y})$ is a gamma-semitermoid of sort κ over X for any algebraic sort κ.

(2) If there exists at least one term of sort κ over X, then $\triangle_\kappa$ is a gamma-semitermoid of sort κ over X.

(3) If $\mathtt{f}$ is a functional symbol of type $\langle\langle\kappa_1,\ldots,\kappa_n\rangle,\lambda\rangle$ and $\tau_1,\ldots,\tau_n$ are gamma-semitermoids over X of sorts $\kappa_1,\ldots,\kappa_n$, respectively, then the string $\mathtt{f}(\tau_1,\ldots,\tau_n)$ is a gamma-semitermoid of sort λ over X.

(4) If $\mathtt{f}$ is a functional symbol of type $\langle\langle\kappa_1,\ldots,\kappa_n\rangle,\lambda\rangle$ and τ is a gamma-semitermoid over X of sort λ, then the string $\mathtt{f}_i^{-1}(\tau)$ is a gamma-semitermoid of sort κ_i over X for any $i\in\{1,\ldots,n\}$.

C) Intuitively, gamma-semitermoids are terms in an extended language — one where for any functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1,\ldots,\kappa_n\rangle,\lambda\rangle$, the symbol $\mathtt{f}_i^{-1}$ is like a functional symbol of type $\langle\langle\lambda\rangle,\kappa_i\rangle$ and the symbol $\triangle_\kappa$ is like a functional symbol of type $\langle\langle\rangle,\kappa\rangle$.

For example, given a sort κ, if the type of $\mathtt{f}$ is $\langle\langle\kappa,\kappa\rangle,\kappa\rangle$ and the type of $\mathtt{c}$ is $\langle\langle\rangle,\kappa\rangle$, then $\mathtt{f}(\mathtt{c},\mathtt{c})$, $\mathtt{f}(\mathtt{c},\triangle_\kappa)$ and $\mathtt{f}_1^{-1}(\mathtt{f}(\mathtt{c},\triangle_\kappa))$ are gamma-semitermoids of sort κ.

D) It can be seen by induction, that if no symbol in a gamma-semitermoid of sort κ over X is of the form $\mathtt{f}_i^{-1}$ or $\triangle_\kappa$, then this gamma-semitermoid is a term of sort κ over X.

Also by induction, it can be seen that each term of sort κ over X is a gamma-semitermoid of sort κ over X.

E) **Definition.** We define the *associated gamma-semitermoid* of a gamma-semitermoid inductively.

(1) $\mathrm{nam}_{X,\kappa}(\mathtt{y})$ is the associated gamma-semitermoid of $\mathrm{nam}_{X,\kappa}(\mathtt{y})$.

(2) $\triangle_\kappa$ is the associated gamma-semitermoid of $\triangle_\kappa$.

(3) If $\sigma_1, \ldots, \sigma_n$ are associated gamma-semitermoids of $\tau_1, \ldots, \tau_n$, respectively, then $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ is an associated gamma-semitermoid of $\mathtt{f}(\tau_1, \ldots, \tau_n)$.

(4) If $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ is an associated gamma-semitermoid of τ, then σ_i is an associated gamma-semitermoid of $\mathtt{f}_i^{-1}(\tau)$.

When the associated gamma-semitermoid of a gamma-semitermoid is a term, we are going to say that it is an *associated term* of this gamma-semitermoid.

The following corollary follows immediately from the above definition.

F) **Corollary.** (1) *Each gamma-semitermoid has at most one associated gamma-semitermoid.*

(2) *No symbol of the form $\mathtt{f}_i^{-1}$ occurs in the associated gamma-semitermoid.*

(3) *$\mathtt{f}(\tau_1, \ldots, \tau_n)$ does not have an associated gamma-semitermoid if at least one of the gamma-semitermoids $\tau_1, \ldots, \tau_n$ does not have an associated gamma-semitermoid.*

(4) *If τ does not have an associated gamma-semitermoid or its associated gamma-semitermoid is not of the form $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$, then $\mathtt{f}_i^{-1}(\tau)$ does not have an associated gamma-semitermoid.*

G) **Proposition.** (1) *The associated gamma-semitermoid of a gamma-semitermoid over X is a gamma-semitermoid over X.*

(2) *If no symbol of the form $\mathtt{f}_i^{-1}$ occurs in the gamma-semitermoid τ, then τ is associated gamma-semitermoid of itself.*

(3) *If no symbol of the form $\triangle_\kappa$ occurs in an associated gamma-semitermoid, then this gamma-semitermoid is an associated term.*

(4) *If τ is a gamma-semitermoid over X of sort κ and τ has an associated term, then its associated term is a term over X of sort κ.*

Proof. **(1)** follows immediately from the definitions by induction on the gamma-semitermoid over X.

(2) can be proved by induction on τ.

(3) and **(4)** follow from (D) and (1). ∎

H) **Definition.** Given a structure $\mathbf{M}$ we define the set of the values of a gamma-semitermoid over $|\mathbf{M}|$ in $\mathbf{M}$ inductively on the gamma-semitermoid:

(1) The set of the values of $\mathrm{nam}^{\gamma}_{|\mathbf{M}|}\,\mu$ is $\{\mu\}$.

(2) The set of the values of $\triangle_\kappa$ is the whole carrier $\mathbf{M}_\kappa$.

(3) The set of the values of the gamma-semitermoid $\mathtt{f}(\tau_1,\ldots,\tau_n)$ is $\{\mathtt{f}^{\mathbf{M}}\langle\mu_1,\ldots,\mu_n\rangle : \mu_1 \in A_1,\ldots,\mu_n \in A_n\}$, where the $A_1,\ldots,A_n$ are the sets of the values of $\tau_1,\ldots,\tau_n$ in $\mathbf{M}$, respectively.

(4) The set of the values of the gamma-semitermoid $\mathtt{f}_i^{-1}(\tau)$ is $\{\mu_i : \text{there exist } \mu_1,\ldots,\mu_{i-1},\mu_{i+1},\ldots,\mu_n, \text{ such that } \mathtt{f}^{\mathbf{M}}\langle\mu_1,\ldots,\mu_n\rangle \in A\}$, where A is the set of the values of τ in $\mathbf{M}$.

I) **Corollary.** (1) *Given a structure* $\mathbf{M}$ *and a gamma-semitermoid* τ *over* $|\mathbf{M}|$ *of sort* κ, *all elements of the set of the values of* τ *in* $\mathbf{M}$ *belong to* $\mathbf{M}_\kappa$.

(2) *If a gamma-semitermoid* τ *over* $|\mathbf{M}|$ *is a term, then the the set of the values of* τ *in* $\mathbf{M}$ *contains exactly one element, namely the value in* $\mathbf{M}$ *of* τ *as a term.*

(3) *If a gamma-semitermoid contains no symbol of the form* $\mathtt{f}_i^{-1}$, *then the set of its values in any normal structure is non-empty.*

Proof. **(1)** follows immediately from the above definition.

(2) is true because the above definition and the definition of value of a term (11J) are basically identical with respect to terms.

(3) can be proved by induction on the gamma-semitermoid. The set of the values of a gamma-semitermoid of the form $\mathrm{nam}^{\gamma}_{|\mathbf{M}|}\,\mu$ is non-empty by definition. The set of the values of a gamma-semitermoid of the form $\triangle_\kappa$ is non-empty, since the existence of a term over $|\mathbf{M}|$ of sort κ implies that the carrier $\mathbf{M}_\kappa$ is a non-empty set. The set of the values of a gamma-semitermoid of the form $\mathtt{f}(\tau_1,\ldots,\tau_n)$ is non-empty also by definition, as long as the sets of the values of $\tau_1,\ldots,\tau_n$ are non-empty (what they are by induction hypothesis). ∎

J) **Proposition.** *Given a structure* $\mathbf{M}$, *if* τ *is a gamma-semitermoid over* $|\mathbf{M}|$ *and* σ *is its associated gamma-semitermoid, then all elements of the set of the values of* σ *in* $\mathbf{M}$ *belong to the set of the values of* τ *in* $\mathbf{M}$.

Proof. According to (G1), σ is a gamma-semitermoid over $|\mathbf{M}|$ of the same sort as the sort of τ, hence the elements of the set of the values of σ

are of the proper sort.

We are going to use induction on the gamma-semitermoid τ.

If $\tau = \ulcorner\mu\urcorner$ or $\tau = \triangle_\kappa$, then, by the definitions, τ is its own associated gamma-semitermoid.

Let $\tau = \mathtt{f}(\tau_1, \ldots, \tau_n)$ and $\sigma_1, \ldots, \sigma_n$ be the gamma-semitermoids associated to $\tau_1, \ldots, \tau_n$, respectively (if there were no such gamma-semitermoids, then τ would have no associated gamma-semitermoid). Let $A_1, \ldots, A_n$ be the sets of the values of $\sigma_1, \ldots, \sigma_n$ in $\mathbf{M}$, respectively. By induction hypothesis, the elements of $A_1, \ldots, A_n$ belong the the sets of the values of $\tau_1, \ldots, \tau_n$, respectively. Since $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ is the gamma-semitermoid associated to τ and the elements of the set of its values in $\mathbf{M}$ have the form $\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle$ for some $\mu_1 \in A_1, \ldots, \mu_n \in A_n$, from definition (H3) it follows that these elements belong to the set of the values of τ in $\mathbf{M}$.

Let $\tau = \mathtt{f}_i^{-1}(\tau')$. Since τ has an associated gamma-semitermoid, from the definition of associated gamma-semitermoid (E) it follows that τ' has an associated gamma-semitermoid of the form $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ and σ_i is the associated gamma-semitermoid of τ. Since no symbol of the form $\mathtt{f}_i^{-1}$ occurs in an associated gamma-semitermoid, (I3) implies that the sets of the values of $\sigma_1, \ldots, \sigma_n$ in $\mathbf{M}$ are non-empty. Let $\mu_1, \ldots, \mu_n$ are some elements of these sets. Let μ be an arbitrary element of the set of the values of σ_i. Then $\mathtt{f}^{\mathbf{A}}\langle\mu_1, \ldots, \mu_{i-1}, \mu, \mu_{i+1}, \ldots, \mu_n\rangle$ belongs to the set of the values of $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ in $\mathbf{M}$ and, by induction hypothesis, it also belongs to the set of the values of τ' in $\mathbf{M}$. By definition (H4), μ belongs to the set of the values of τ in $\mathbf{M}$. As μ was an arbitrarily chosen element of the set of the values of σ_i, this set must be a subset of the set of the values of τ in $\mathbf{M}$. ∎

K) **Lemma.** *Given an arbitrary* `Sort`*-indexed set X, the set of the values in $[X]$ of a gamma-semitermoid over $|[X]|$ is equal to the set of the values in $[X]$ of its associated gamma-semitermoid.*

Proof. By induction on the gamma-semitermoid we are going to prove that the set of the values of any gamma-semitermoid is a subset of the set of the values of its associated gamma-semitermoid. This is enough to prove the lemma, as the opposite set inclusion follows from (J).

The associated gamma-semitermoid of a name or of a gamma-semitermoid of the form $\triangle_\kappa$ is the gamma-semitermoid itself, so there is nothing to prove.

If the gamma-semitermoid is $\mathtt{f}(\tau_1, \ldots, \tau_n)$, then its associated gamma-semitermoid is $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$, where $\sigma_1, \ldots, \sigma_n$ are the respective associated gamma-semitermoids of $\tau_1, \ldots, \tau_n$. By induction hypothesis, the sets of the values of $\tau_1, \ldots, \tau_n$ are equal to the respective sets of the values of $\sigma_1, \ldots, \sigma_n$, hence by the definition (H3), the sets of the values of $\mathtt{f}(\tau_1, \ldots, \tau_n)$ and its associated gamma-semitermoid are equal.

Let the gamma-semitermoid be $\mathtt{f}_i^{-1}(\tau)$. Choose an arbitrary element of the set of the values of this gamma-semitermoid in $[X]$. Then the set of the values of τ in $[X]$ contains an element of the form $\mathtt{f}^{[X]}\langle\rho_1, \ldots, \rho_n\rangle$, such that ρ_i is equal to the chosen element. Since $\mathtt{f}_i^{-1}(\tau)$ has an associated gamma-semitermoid, definition (E) implies that the associated gamma-semitermoid of τ has the form $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$. By induction hypothesis, $\mathtt{f}^{[X]}\langle\rho_1, \ldots, \rho_n\rangle$ belongs to the set of the values of $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ in $[X]$. Considering definitions (H3) and (11E1), we obtain that ρ_i belongs to the set of the values of σ_i in $[X]$ for any $i \in \{1, \ldots, n\}$. Since σ_i is the associated gamma-semitermoid of $\mathtt{f}_i^{-1}(\tau)$, this completes the proof. ∎

L) We can not build a terminator based on gamma-semitermoids for the following reason. Let $\mathtt{f}$ be a binary functional symbol of suitable type and $\mathtt{c}$ be a nullary functional symbol (a constant symbol). Then $\mathtt{f}_1^{-1}(\mathtt{c})$ is a gamma-semitermoid. Suppose the structure $\mathbf{M}$ is such that the value of the function $\mathtt{f}^{\mathbf{M}}$ is never equal to $\mathtt{c}^{\mathbf{M}}$. It is not difficult to see that the set of the values of $\mathtt{f}_1^{-1}(\mathtt{c})$ is the empty set. According to (14I12), however, the value of any termoid in any structure is a non-empty set. This is why we are going to define gamma-termoids as a special kind of termoids.

Intuitively, τ is a gamma-termoid, if τ is a gamma-semitermoid and symbols like $\mathtt{f}_i^{-1}$ are applied only to expressions who a values of $\mathtt{f}$.

M) **Definition.** τ is a *gamma-termoid* over X if τ is a gamma-semitermoid over X and it has an associated term (i.e. it has an associated gamma-semitermoid which is a term).

N) **Example.** Assuming $\mathtt{f}$, $\mathtt{g}$ and $\mathtt{c}$ are functional symbols of suitable types and $\triangle_\kappa$ is admitted by (B2):

(1) Each term over X is a gamma-termoid over X.

(2) $\mathtt{f}_2^{-1}(\mathtt{g}_1^{-1}(\mathtt{g}(\mathtt{f}(\mathtt{c}, \mathtt{a}), \mathtt{c})))$ is a gamma-termoid and $\mathtt{f}_2^{-1}(\mathtt{g}_2^{-1}(\mathtt{g}(\mathtt{f}(\mathtt{c}, \mathtt{c}), \mathtt{c})))$ is not. The first of these gamma-semitermoids has an associated term $\mathtt{a}$ while the second one has no associated gamma-semitermoid.

(3) $\mathtt{f}_1^{-1}(\mathtt{f}(\mathtt{c}, \triangle_\kappa))$ is an example of a gamma-termoid containing the symbol $\triangle_\kappa$. Its associated term is $\mathtt{c}$. On the other hand, $\triangle_\kappa$ is the associated gamma-semitermoid of $\mathtt{f}_2^{-1}(\mathtt{f}(\mathtt{c}, \triangle_\kappa))$, so this gamma-semitermoid is not a gamma-termoid.

O) **Proposition.** *If τ is both a gamma-termoid over X and gamma-semitermoid over Y, then τ is a gamma-termoid over Y.*

Proof. In (E) we have defined the notion “associated gamma-semitermoid of a gamma-semitermoid", not "associated gamma-semitermoid of a gamma-semitermoid over X". Consequently, the associated gamma-semitermoid of τ is always the same regardless of whether we consider τ a gamma-semitermoid over X, or over Y. If it is a term, then it is a term. ■

P) **Lemma.** *The set of the values of any gamma-termoid over $|[X]|$ contains exactly one element, namely the value in $[X]$ of its associated term.*

Proof. Follows immediately from (K) and (I2). ■

Q) **Definition.** For any $\mathtt{Sort}$-indexed set X, let $[\![X]\!]_\gamma$ be the algebra, such that:

(1) The algebraic carrier of sort κ of $[\![X]\!]_\gamma$ is the set of all gamma-termoids over X of sort κ.

(2) For any functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1,\dots,\kappa_n\rangle,\lambda\rangle$ and gamma-termoids $\tau_1,\dots,\tau_n$ of sorts $\kappa_1,\dots,\kappa_n$, respectively, let

$$\mathtt{f}^{[\![X]\!]_\gamma}(\tau_1,\dots,\tau_n)=\mathtt{f}(\tau_1,\dots,\tau_n)$$

where on the right side of the equality sign stays a formal expression.

This definition is correct because:

First, the elements of the algebraic carriers of $[\![X]\!]_\gamma$ are exactly the gamma-termoids, hence $\mathtt{f}(\tau_1,\dots,\tau_n)$ belongs to the carrier of sort λ of $[\![X]\!]_\gamma$.

Second, any algebra is uniquely determined by its algebraic carriers and the interpretation of the functional symbols (see 12Q1).

R) **Definition.** Given $\mathtt{Sort}$-indexed sets X and Y and a $\mathtt{Sort}$-indexed function $f : X \to Y$, let $[\![f]\!]_\gamma : [\![X]\!]_\gamma \to [\![Y]\!]_\gamma$ be the homomorphism, who, when applied to a gamma-termoid τ, replaces all occurrences of names $\mathrm{nam}_{X,\lambda}(\mathtt{z})$ in τ with $\mathrm{nam}_{Y,\lambda}(f_\lambda \mathtt{z})$ (i.e. $[\![f]\!]_\gamma$ replaces all occurrences of $\ulcorner\mathtt{z}\urcorner$ with $\ulcorner f\mathtt{z}\urcorner$).

We are going to use postfix notation for this homomorphism. Thus $\tau[\![f]\!]_\gamma$ means to apply $[\![f]\!]_\gamma$ to τ. As an extension of the notation, we are going to write $\tau[\![f]\!]_\gamma$ even when τ is not a gamma-termoid, but only a gamma-semitermoid — let $\tau[\![f]\!]_\gamma$ be the expression which is obtained from τ by replacing all occurrences of names $\mathrm{nam}_{X,\lambda}(\mathtt{z})$ in τ with $\mathrm{nam}_{Y,\lambda}(f_\lambda \mathtt{z})$ (i.e. in $\tau[\![f]\!]_\gamma$ all occurrences of $\ulcorner\mathtt{z}\urcorner$ in τ are replaced with $\ulcorner f\mathtt{z}\urcorner$).

The following proposition shows that the above definition is correct:

S) **Corollary.** *Let $f : X \to Y$ be a $\mathtt{Sort}$-indexed function. Then:*

(1) *If τ is a gamma-semitermoid over X, then $\tau[\![f]\!]_\gamma$ is a gamma-semitermoid over Y.*

(2) *If* σ *is the associated gamma-semitermoid of the gamma-semitermoid* τ *over* X*, then* $\sigma[\![f]\!]_\gamma$ *is the associated gamma-semitermoid of* $\tau[\![f]\!]_\gamma$.

(3) *If* τ *is a gamma-termoid over* X*, then* $\tau[\![f]\!]_\gamma$ *is a gamma-termoid over* Y.

(4) *There exists unique homomorphism from* $[\![X]\!]_\gamma$ *to* $[\![Y]\!]_\gamma$*, such that the result of its application to any gamma-termoid* τ *is equal to* $\tau[\![f]\!]_\gamma$.

Proof. **(1)** By induction on τ.

If $\ulcorner\xi\urcorner$ is a gamma-semitermoid over X, then $\xi \in X$, hence $f\xi \in Y$, so $\ulcorner f\xi\urcorner$ is a gamma-semitermoid over Y.

If $\triangle_\kappa$ is a gamma-semitermoid over X, then there exists at least one term ρ over X of sort κ (see definition B2), hence $\rho[f]$ is a term over Y of sort κ, so $\triangle_\kappa$ also is a gamma-semitermoid over Y, hence $\triangle_\kappa[\![f]\!]_\gamma$ is a gamma-semitermoid, since $\triangle_\kappa[\![f]\!]_\gamma = \triangle_\kappa$.

If $\tau = \mathtt{f}(\tau_1, \ldots, \tau_n)$, then $\tau[\![f]\!]_\gamma = \mathtt{f}(\tau_1[\![f]\!]_\gamma, \ldots, \tau_n[\![f]\!]_\gamma)$ which is a gamma-semitermoid, since by induction hypothesis $\tau_i[\![f]\!]_\gamma$ is a gamma-semitermoid for any $i \in \{1, \ldots, n\}$.

If $\tau = \mathtt{f}_i^{-1}(\tau')$, then $\tau[\![f]\!]_\gamma = \mathtt{f}_i^{-1}(\tau'[\![f]\!]_\gamma)$ which is a gamma-semitermoid, since by induction hypothesis $\tau'[\![f]\!]_\gamma$ is a gamma-semitermoid.

(2) Again by induction on τ.

If $\tau = \ulcorner\xi\urcorner$, then $\xi \in X$ and $\sigma = \tau = \ulcorner\xi\urcorner$, so $\tau[\![f]\!]_\gamma = \sigma[\![f]\!]_\gamma = \ulcorner f\xi\urcorner$, hence $\sigma[\![f]\!]_\gamma$ is the associated gamma-semitermoid of $\tau[\![f]\!]_\gamma$.

If $\tau = \triangle_\kappa$, then $\sigma = \triangle_\kappa$, hence $\tau[\![f]\!]_\gamma = \sigma[\![f]\!]_\gamma = \triangle_\kappa$, so $\sigma[\![f]\!]_\gamma$ is the associated gamma-semitermoid of $\tau[\![f]\!]_\gamma$.

If $\tau = \mathtt{f}(\tau_1, \ldots, \tau_n)$, then $\sigma = \mathtt{f}(\sigma_1, \ldots, \sigma_n)$ where $\sigma_1, \ldots, \sigma_n$ are the respective associated gamma-semitermoids of $\tau_1, \ldots, \tau_n$. By induction hypothesis $\sigma_i[\![f]\!]_\gamma$ is the associated gamma-semitermoid of $\tau_i[\![f]\!]_\gamma$ for any $i \in \{1, \ldots, n\}$, hence $\sigma[\![f]\!]_\gamma = \mathtt{f}(\sigma_1[\![f]\!]_\gamma, \ldots, \sigma_n[\![f]\!]_\gamma)$ is the associated gamma-semitermoid of $\tau[\![f]\!]_\gamma = \mathtt{f}(\tau_1[\![f]\!]_\gamma, \ldots, \tau_n[\![f]\!]_\gamma)$.

If $\tau = \mathtt{f}_i^{-1}(\tau')$, then the associated gamma-semitermoid of τ' has the form $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$, where $\sigma_i = \sigma$. By induction hypothesis, the associated gamma-semitermoid of $\tau'[\![f]\!]_\gamma$ is $\mathtt{f}(\sigma_1, \ldots, \sigma_n)[\![f]\!]_\gamma = \mathtt{f}(\sigma_1[\![f]\!]_\gamma, \ldots, \sigma_n[\![f]\!]_\gamma)$, hence the associated gamma-semitermoid of $\tau[\![f]\!]_\gamma = \mathtt{f}_i^{-1}(\tau'[\![f]\!]_\gamma)$ is $\sigma_i[\![f]\!]_\gamma$.

(3) From (1) it follows that $\tau[\![f]\!]_\gamma$ is a gamma-semitermoid over Y, so it remains to show that the associated gamma-semitermoid of $\tau[\![f]\!]_\gamma$ is a term.

Let σ be the associated term of τ. Then (2) implies that $\sigma[\![f]\!]_\gamma$ is the associated gamma-semitermoid of $\tau[\![f]\!]_\gamma$. But σ is a term, so according to the definition of $[\![.]\!]_\gamma$, $\sigma[\![f]\!]_\gamma = \sigma[f]$, hence $\sigma[\![f]\!]_\gamma$ is a term.

(4) follows from (3) and (12Q2). We only have to notice that for any functional symbol $\mathtt{f}$ and gamma-termoids $\tau_1, \ldots, \tau_n$ of suitable sorts, $(\mathtt{f}^{[\![X]\!]_\gamma}\langle\tau_1, \ldots, \tau_n\rangle)[\![f]\!]_\gamma = (\mathtt{f}(\tau_1, \ldots, \tau_n))[\![f]\!]_\gamma = \mathtt{f}(\tau_1[\![f]\!]_\gamma, \ldots, \tau_n[\![f]\!]_\gamma) = \mathtt{f}^{[\![Y]\!]_\gamma}\langle\tau_1[\![f]\!]_\gamma, \ldots, \tau_n[\![f]\!]_\gamma\rangle$. ∎

T) **Lemma.** *Given a gamma-termoid τ over $|\mathbf{M}|$ and a homomorphism $h : \mathbf{M} \to \mathbf{K}$, if μ belongs to the set of the values of τ in $\mathbf{M}$, then $h\mu$ belongs to the set of the values of $\tau[\![h]\!]_\gamma$ in $\mathbf{K}$.*

Proof. By induction on the gamma-semitermoid τ we are going to prove that if σ is obtained from τ by replacing each occurrence of a name $\ulcorner\xi\urcorner$ with $\ulcorner f\xi\urcorner$ and μ belongs to the set of the values of τ in $\mathbf{M}$, then $h\mu$ belongs to the set of the values of σ in $\mathbf{M}$.

If τ is a name, then $\tau = \ulcorner\mu\urcorner$ and the set of the values of τ in $\mathbf{M}$ is $\{\mu\}$, so $\sigma = \ulcorner h\mu\urcorner$, hence the set of the values of σ in $\mathbf{K}$ is $\{h\mu\}$.

If $\tau = \triangle_\kappa$, then $\sigma = \triangle_\kappa$. The set of the values of σ in $\mathbf{K}$ is the whole carrier $\mathbf{K}_\kappa$, hence $h\mu$ belongs to this set.

Let $\tau = \mathtt{f}(\tau_1, \ldots, \tau_n)$. Let $A_1, \ldots, A_n$ be the respective sets of the values of $\tau_1, \ldots, \tau_n$ in $\mathbf{M}$. If μ belongs to the set of the values of τ in $\mathbf{M}$, then $\mu = \mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle$ for some $\mu_1 \in A_1, \ldots, \mu_n \in A_n$. Since h is a homomorphism, $h\mu = h(\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle) = \mathtt{f}^{\mathbf{K}}\langle h\mu_1, \ldots, h\mu_n\rangle$. By induction hypothesis, $h\mu_i$ belongs to the set of the values of $\tau_i[\![h]\!]_\gamma$ in $\mathbf{K}$ for any $i \in \{1, \ldots, n\}$, hence $h\mu$ belongs to the set of the values of $\tau[\![h]\!]_\gamma = \mathtt{f}(\tau_1[\![h]\!]_\gamma, \ldots, \tau_n[\![h]\!]_\gamma)$ in $\mathbf{K}$.

Let $\tau = \mathtt{f}_i^{-1}(\tau')$. Let A be the set of the values of τ' in $\mathbf{M}$. If μ belongs to the set of the values of τ in $\mathbf{M}$, then there exist $\mu_1, \ldots, \mu_n$, such that $\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle \in A$ and $\mu = \mu_i$. By induction hypothesis, $h(\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle)$ belongs to the set of the values of $\tau'[\![h]\!]_\gamma$ in $\mathbf{K}$, hence $\mathtt{f}^{\mathbf{K}}\langle h\mu_1, \ldots, h\mu_n\rangle = h(\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle)$ belongs to the set of the values of $\tau'[\![h]\!]_\gamma$ in $\mathbf{K}$, so by definition (H4), $h\mu_i = h\mu$ belongs to the set of the values of $\tau[\![h]\!]_\gamma = (\mathtt{f}_i^{-1}(\tau'))[\![h]\!]_\gamma$ in $\mathbf{K}$. ∎

U) **Definition.** Given a structure $\mathbf{M}$, let $\mathrm{Val}^\gamma_{\mathbf{M}} : [\![|\mathbf{M}|]\!]_\gamma \to \mathcal{P}\mathbf{M}$ be the only homomorphism, such that for any gamma-termoid τ over $|\mathbf{M}|$, $\mathrm{Val}^\gamma_{\mathbf{M}}\,\tau$ is equal to the set of the values of τ in $\mathbf{M}$.

As an extension of the notation, we are going to write $\mathrm{Val}^\gamma_{\mathbf{M}}\,\tau$ even when τ is only a gamma-semitermoid over $|\mathbf{M}|$, not necessarily a gamma-termoid. In this case too, let $\mathrm{Val}^\gamma_{\mathbf{M}}\,\tau$ be equal to the set of the values of τ in $\mathbf{M}$.

According to (12Q2), in order to prove that $\mathrm{Val}^\gamma_{\mathbf{M}}$ is indeed a homomorphism, we only have to notice that for any functional symbol $\mathtt{f}$ of type

$\langle\langle\kappa_1,\dots,\kappa_n\rangle,\lambda\rangle$ and any termoids $\tau_1\dots,\tau_n$ of sorts $\kappa_1,\dots,\kappa_n$,

$$\begin{aligned}
&\mathrm{Val}^{\gamma}_{\mathbf{M}}(\mathtt{f}^{[\![|\mathbf{M}|]\!]_\gamma}\langle\tau_1,\dots,\tau_n\rangle) = \\
&\quad = \mathrm{Val}^{\gamma}_{\mathbf{M}}(\mathtt{f}(\tau_1,\dots,\tau_n)) && \text{from (Q2)}\\
&\quad = \{\mathtt{f}^{\mathbf{M}}\langle\mu_1,\dots,\mu_n\rangle : \mu_1\in\mathrm{Val}^{\gamma}_{\mathbf{M}}\,\tau_1,\dots,\mu_n\in\mathrm{Val}^{\gamma}_{\mathbf{M}}\,\tau_n\} && \text{from (H3)}\\
&\quad = \mathtt{f}^{\mathcal{P}\mathbf{M}}\langle\mathrm{Val}^{\gamma}_{\mathbf{M}}\,\tau_1,\dots,\mathrm{Val}^{\gamma}_{\mathbf{M}}\,\tau_n\rangle
\end{aligned}$$

V) **Definition.** Given a `Sort`-indexed set X, let $\mathrm{Vals}^{\gamma}_X : [\![|[\![X]\!]_\gamma|]\!]_\gamma \to [\![X]\!]_\gamma$ be the only homomorphism, such that for any gamma-termoid τ over $|[\![X]\!]_\gamma|$, $\mathrm{Vals}^{\gamma}_X\,\tau$ is the result of the replacement in τ of all names $\mathrm{nam}_{|[\![X]\!]_\gamma|}\,\sigma$ with σ.

As an extension of the notation, we are going to write $\mathrm{Vals}^{\gamma}_X\,\tau$ even when τ is only a gamma-semitermoid over $|[\![X]\!]_\gamma|$, not necessarily a gamma-termoid. In this case too, let $\mathrm{Val}^{\gamma}_X\,\tau$ be the result of the replacement in τ of all names $\mathrm{nam}_{|[\![X]\!]_\gamma|}\,\sigma$ with σ.

W) **Example.** Assuming `f` and `c` are functional symbols of suitable types, and ξ is an element of the `Sort`-indexed set X,

$$\mathtt{f}(\ulcorner\mathtt{f}(\mathtt{c},\ulcorner\xi\urcorner)\urcorner,\mathtt{f}(\ulcorner\mathtt{c}\urcorner,\ulcorner\ulcorner\xi\urcorner\urcorner))$$

is a term and gamma-termoid over $|[\![X]\!]|$. To apply Vals_X to this gamma-termoid means to remove upper level of the symbols $\ulcorner\,.\,\urcorner$:

$$\mathtt{f}(\mathtt{f}(\mathtt{c},\ulcorner\xi\urcorner),\mathtt{f}(\mathtt{c},\ulcorner\xi\urcorner))$$

The result is a term and gamma-termoid over X.

The following proposition shows that the definition of Vals_X is correct:

X) **Proposition.** (1) *If τ is a gamma-semitermoid over $|[\![X]\!]_\gamma|$, then $\mathrm{Vals}^{\gamma}_X\,\tau$ is a gamma-semitermoid over X.*

(2) *For any gamma-semitermoid τ over $|[\![X]\!]_\gamma|$ let $\mathfrak{g}\tau$ be the result of the replacement in τ of all names $\ulcorner\sigma\urcorner$ with the associated term of σ.*[72] *Then $\mathfrak{g}\tau$ is a gamma-semitermoid over X.*

(3) *If σ is the associated gamma-semitermoid of the gamma-semitermoid τ over $|[\![X]\!]_\gamma|$, then $\mathfrak{g}\sigma$ is the associated gamma-semitermoid of* $\mathrm{Vals}^{\gamma}_X\,\tau$.

(4) *If τ is a gamma-termoid over $|[\![X]\!]_\gamma|$, then* $\mathrm{Vals}^{\gamma}_X\,\tau$ *is a gamma-termoid over X.*

(5) *There exists unique homomorphism from $[\![|[\![X]\!]_\gamma|]\!]_\gamma$ to $[\![X]\!]_\gamma$, such that the result of its application to any gamma-termoid τ is equal to* $\mathrm{Vals}^{\gamma}_X\,\tau$.

[72] Notice that $\ulcorner\sigma\urcorner$ is a name of an element of $|[\![X]\!]_\gamma|$, hence σ is a gamma-termoid over X, so σ has an associated term.

<u>Proof.</u> **(1)** By induction on τ.

If $\tau = \ulcorner\sigma\urcorner$, then according to (B), $\sigma \in |[\![X]\!]_\gamma|$, hence σ is a gamma-termoid over X, but $\mathrm{Vals}^\gamma_X\,\tau = \sigma$, so $\mathrm{Vals}^\gamma_X\,\tau$ is a gamma-termoid over X.

If $\tau = \triangle_\kappa$, then $\mathrm{Vals}^\gamma_X\,\tau = \triangle_\kappa$.

If $\tau = \mathtt{f}(\tau_1,\ldots,\tau_n)$, then $\mathrm{Vals}^\gamma_X\,\tau = \mathtt{f}(\mathrm{Vals}^\gamma_X\,\tau_1,\ldots,\mathrm{Vals}^\gamma_X\,\tau_n)$ which is a gamma-semitermoid over X, since by induction hypothesis $\mathrm{Vals}^\gamma_X\,\tau_i$ is a gamma-semitermoid over X for any $i \in \{1,\ldots,n\}$.

If $\tau = \mathtt{f}_i^{-1}(\tau')$, then $\mathrm{Vals}^\gamma_X\,\tau = \mathtt{f}_i^{-1}(\mathrm{Vals}^\gamma_X\,\tau')$ which is a gamma-semitermoid over X, since by induction hypothesis $\mathrm{Vals}^\gamma_X\,\tau'$ is a gamma-semitermoid over X.

(2) By induction on τ.

If $\tau = \ulcorner\sigma\urcorner$, then $\mathfrak{g}\tau$ is the associated term of σ, hence $\mathfrak{g}\tau$ is a gamma-semitermoid over X.

If $\tau = \triangle_\kappa$, then $\mathfrak{g}\tau = \tau = \triangle_\kappa$. Since $\tau = \triangle_\kappa$ is a gamma-semitermoid over $|[\![X]\!]_\gamma|$ of sort κ, there exists a term σ over $|[\![X]\!]_\gamma|$ of sort κ. But the value of the term σ in $[\![X]\!]_\gamma$ is an element of the algebraic carrier of $[\![X]\!]_\gamma$ of sort κ, hence this value is a gamma-termoid of sort κ over X, so the associated term of this gamma-termoid is a term of sort κ over X, hence there exists a term of sort κ over X, so $\mathfrak{g}\tau = \triangle_\kappa$ is a gamma-termoid over X.

If $\tau = \mathtt{f}(\tau_1,\ldots,\tau_n)$, then $\mathfrak{g}\tau = \mathtt{f}(\mathfrak{g}\tau_1,\ldots,\mathfrak{g}\tau_n)$ which is a gamma-semitermoid over X, since by induction hypothesis $\mathfrak{g}\tau_i$ is a gamma-semitermoid over X for any $i \in \{1,\ldots,n\}$.

If $\tau = \mathtt{f}_i^{-1}(\tau')$, then $\mathfrak{g}\tau = \mathtt{f}_i^{-1}(\mathfrak{g}\tau')$ which is a gamma-semitermoid over X, since by induction hypothesis $\mathfrak{g}\tau'$ is a gamma-semitermoid over X.

(3) By induction on τ.

If $\tau = \ulcorner\rho\urcorner$, then $\sigma = \tau = \ulcorner\rho\urcorner$, so $\mathrm{Vals}^\gamma_X\,\tau = \rho$ and $\mathfrak{g}\sigma = \pi$, where π is the associated term of ρ.

If $\tau = \triangle_\kappa$, then $\sigma = \tau = \triangle_\kappa$ and also $\mathrm{Vals}^\gamma_X\,\tau = \triangle_\kappa$ and $\mathfrak{g}\sigma = \triangle_\kappa$.

If $\tau = \mathtt{f}(\tau_1,\ldots,\tau_n)$, then $\sigma = \mathtt{f}(\sigma_1,\ldots,\sigma_n)$, where $\sigma_1\ldots,\sigma_n$ are the respective associated gamma-semitermoids of $\tau_1,\ldots,\tau_n$. By induction hypothesis $\mathfrak{g}\sigma_i$ is the associated gamma-semitermoid of $\mathrm{Vals}^\gamma_X\,\tau_i$ for any $i \in \{1,\ldots,n\}$, so $\mathfrak{g}\sigma = \mathtt{f}(\mathfrak{g}\sigma_1,\ldots,\mathfrak{g}\sigma_n)$ is the associated gamma-semitermoid of $\mathrm{Vals}^\gamma_X\,\tau = \mathtt{f}(\mathrm{Vals}^\gamma_X\,\tau_1,\ldots,\mathrm{Vals}^\gamma_X\,\tau_n)$.

If $\tau = \mathtt{f}_i^{-1}(\tau')$, then the associated gamma-semitermoid of τ' has the form $\mathtt{f}(\sigma_1,\ldots,\sigma_n)$ and $\sigma = \sigma_i$. By induction hypothesis $\mathfrak{g}(\mathtt{f}(\sigma_1,\ldots,\sigma_n)) = \mathtt{f}(\mathfrak{g}\sigma_1,\ldots,\mathfrak{g}\sigma_n)$ is the associated gamma-semitermoid of $\mathrm{Vals}^\gamma_X\,\tau'$, so the associated gamma-semitermoid of $\mathrm{Vals}^\gamma_X\,\tau = \mathtt{f}_i^{-1}(\mathrm{Vals}^\gamma_X\,\tau')$ is $\mathfrak{g}\sigma_i$, i.e. $\mathfrak{g}\sigma$.

(4) From (1) it follows that $\mathrm{Vals}_X\,\tau$ is a gamma-semitermoid over X. Let σ be the associated term of τ; then from (3) it follows that $\mathfrak{g}\sigma$ is the associated gamma-semitermoid of $\mathrm{Vals}_X\,\tau$. But $\mathfrak{g}\sigma$ also is a term (obviously, when $\mathfrak{g}$ is applied to a term, the result also is a term), so $\mathrm{Vals}_X \tau$ is a gamma-termoid.

(5) follows from (4) and (12Q2). We only have to notice that for any functional symbol $\mathtt{f}$ of type $\langle\langle \kappa_1, \ldots, \kappa_n\rangle, \lambda\rangle$ and any gamma-termoids $\tau_1 \ldots, \tau_n$ over $|[\![X]\!]_\gamma|$ of sorts $\kappa_1, \ldots, \kappa_n$, we have $\mathrm{Vals}^\gamma_X(\mathtt{f}^{[\![|[\![X]\!]_\gamma|]\!]_\gamma}\langle\tau_1, \ldots, \tau_n\rangle) = \mathtt{f}^{[\![X]\!]_\gamma}\langle \mathrm{Vals}^\gamma_X \tau_1, \ldots, \mathrm{Vals}^\gamma_X \tau_n\rangle$. This is so because anything Vals^γ_X replaces is a name and $\mathtt{f}$ is not a name, $\mathrm{Vals}^\gamma_X(\mathtt{f}^{[\![|[\![X]\!]_\gamma|]\!]_\gamma}\langle\tau_1, \ldots, \tau_n\rangle) = \mathrm{Vals}^\gamma_X(\mathtt{f}(\tau_1, \ldots, \tau_n)) = \mathtt{f}(\mathrm{Vals}^\gamma_X \tau_1, \ldots, \mathrm{Vals}^\gamma_X \tau_n) = \mathtt{f}^{[\![X]\!]_\gamma}\langle \mathrm{Vals}^\gamma_X \tau_1, \ldots, \mathrm{Vals}^\gamma_X \tau_n\rangle$. ■

Y) **Definition.** Given a $\mathtt{Sort}$-indexed set X, let $\mathrm{Nam}^\gamma_X : X^\circ \to |[\![X]\!]_\gamma|$ be the $\mathtt{Sort}$-indexed function, such that for any $\mathtt{x} \in X^\circ$

$$\mathrm{Nam}^\gamma_X(\mathtt{x}) = \mathrm{nam}_X(\mathtt{x})$$

Z) **Definition.** The quadruple $\langle [\![.]\!]_\gamma, \mathrm{Val}^\gamma, \mathrm{Vals}^\gamma, \mathrm{Nam}^\gamma\rangle$ is the *gamma-terminator*.

In order to avoid ambiguities, the termoidal expressions and the formuloids defined by this terminator will be called "gamma-termoidal expression" and "gamma-formuloid". Because of (Q), the notion of termoid corresponding to this terminator (see definition 14J) is identical with the notion of gamma-termoid, as defined in (M).

We have to prove that the above definition is correct.

Proof. We are going to prove the requirements of definition (14I) one by one.

(1) $[\![X]\!]_\gamma$ is algebra by definition (Q).

(2) If $f : X \to Y$ is an arbitrary $\mathtt{Sort}$-indexed function, according to definition (R), $[\![f]\!]_\gamma$ is a homomorphism from $[\![X]\!]_\gamma$ to $[\![Y]\!]_\gamma$.

(3) If τ is a gamma-semitermoid over X and at the same time a gamma-semitermoid over Y, then τ is a gamma-semitermoid over $X \cap Y$ as well (see definition B).

(4) Definition (R) implies that if the values of $f' : X' \to Y'$ and $f'' : X'' \to Y''$ are equal over the objects whose names occur in the gamma-termoid τ, then $\tau[\![f']\!]_\gamma = \tau[\![f'']\!]_\gamma$. Consequently, the homomorphisms $[\![f']\!]_\gamma$ and $[\![f'']\!]_\gamma \restriction [\![X']\!]_\gamma$ are identical over the algebraic carriers, hence according to (12H2) they are identical. On the other hand, definition (R) trivially implies that $\tau[\![f'']\!]_\gamma = \tau([\![f'']\!]_\gamma \restriction [\![X']\!]_\gamma)$ for any $\tau \in |[\![X']\!]|$.

(5) The definition of gamma-semitermoids over X (B) does not refer to the elements of $X_{\mathtt{Log}}$, so $[\![X]\!]_\gamma = [\![X^\circ]\!]_\gamma$. Gamma-termoids contain no names of logical sort, hence immediately from definition (R) it follows that $[\![f]\!]_\gamma = [\![f^\circ]\!]_\gamma$.

(6) and **(7)** follow from the definitions.

(8) follows from definition (Y).

(9) Given a `Sort`-indexed function $f : X \to Y$, for any $\xi \in X^\circ$ we have $(\llbracket f \rrbracket_\gamma \circ \mathrm{Nam}^\gamma_X)\xi = (\mathrm{Nam}^\gamma_X\, \xi)\llbracket f \rrbracket_\gamma = \ulcorner \xi \urcorner \llbracket f \rrbracket_\gamma = \ulcorner f\xi \urcorner = \ulcorner f^\circ \xi \urcorner = \mathrm{Nam}^\gamma_X\,(f^\circ \xi) = (\mathrm{Nam}^\gamma_X \circ f^\circ)\xi$.

$$\begin{array}{ccc} X^\circ & \xrightarrow{\mathrm{Nam}^\gamma_X} & |\llbracket X \rrbracket_\gamma| \\ {\scriptstyle f^\circ}\downarrow & & \downarrow{\scriptstyle \llbracket f \rrbracket_\gamma} \\ Y^\circ & \xrightarrow{\mathrm{Nam}^\gamma_Y} & |\llbracket Y \rrbracket_\gamma| \end{array}$$

(10) According to definition (U), Val^γ is a homomorphism, hence it also is a quasimorphism.

(11) A simple induction can be used in order to prove that if the structures $\mathbf{M}$ and $\mathbf{K}$ are variants, then for any gamma-termoid τ its set of values in $\mathbf{M}$ is equal to its set of values in $\mathbf{K}$. Since $\mathbf{M}$ and $\partial\mathbf{M}$ are variants, $\mathrm{Val}^\gamma_{\mathbf{M}}\,\tau = \mathrm{Val}^\gamma_{\partial\mathbf{M}}\,\tau$ for any gamma-termoid τ.

(12) Any gamma-termoid has an associated term (see E), so the value of the associated term belongs to the set of the values of the gamma-termoid (see J), hence the algebraic components of $\mathrm{Val}^\gamma_{\mathbf{M}}$ map to non-empty sets, so by (14E) all components of $\mathrm{Val}^\gamma_{\mathbf{M}}$ map to non-empty sets.

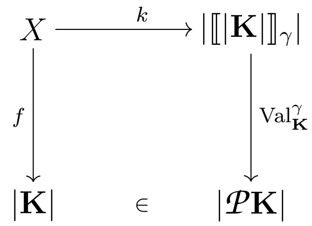
In order to prove that $\mathrm{Val}^\gamma_{[X]}$ maps to one-element sets, notice that (P) implies that the algebraic components of this homomorphism map to one-element sets, hence, due to (14F), all components of this homomorphism map to one-element sets.

(13) Let $h : \mathbf{M} \to \mathbf{K}$ be a homomorphism. From (T) it follows that for any τ belonging to an algebraic carrier of $\llbracket |\mathbf{M}| \rrbracket$, $(h^{\mathcal{P}} \circ \mathrm{Val}^\gamma_{\mathbf{M}})\tau \subseteq (\mathrm{Val}^\gamma_{\mathbf{K}} \circ \llbracket h \rrbracket_\gamma)\tau$, so from (14G) we obtain that the same also is true for τ belonging to the logical carrier of $\llbracket |\mathbf{M}| \rrbracket$.

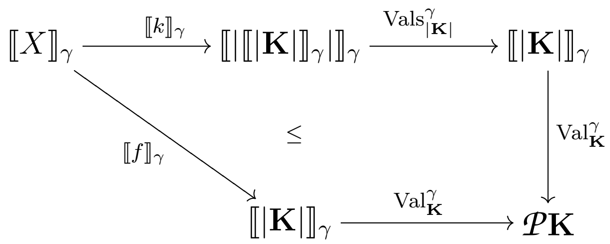
$$\begin{array}{ccc} \llbracket |\mathbf{M}| \rrbracket_\gamma & \xrightarrow{\mathrm{Val}^\gamma_{\mathbf{M}}} & \mathcal{P}\mathbf{M} \\ {\scriptstyle \llbracket h \rrbracket_\gamma}\downarrow & \geq & \downarrow{\scriptstyle h^{\mathcal{P}}} \\ \llbracket |\mathbf{K}| \rrbracket_\gamma & \xrightarrow{\mathrm{Val}^\gamma_{\mathbf{K}}} & \mathcal{P}\mathbf{K} \end{array}$$

(14) According to definition (V), Vals^γ is a homomorphism.

(15) Given a `Sort`-indexed set X, a structure $\mathbf{K}$, a `Sort`-indexed function $k : X \to |\llbracket |\mathbf{K}| \rrbracket_\gamma|$ and a `Sort`-indexed function $f : X \to |\mathbf{K}|$, suppose that $f \ll \mathrm{Val}^\gamma_{\mathbf{K}} \circ k$.



Due to (14G), in order to prove that $(\mathrm{Val}^\gamma_{\mathbf{K}} \circ [\![f]\!]_\gamma)\tau \subseteq (\mathrm{Val}^\gamma_{\mathbf{K}} \circ \mathrm{Vals}^\gamma_{|\mathbf{K}|} \circ [\![k]\!]_\gamma)\tau$ for any $\tau \in |[\![X]\!]_\gamma|$, it is enough to consider only the algebraic carriers, i.e. it is enough to consider only the case when τ is a gamma-termoid over X.



We are going to prove by induction on τ that this is true not only when τ is a gamma-termoid, but when it also is a gamma-semitermoid over X.

If τ is a name, $\tau = \ulcorner\xi\urcorner$ for some $\xi \in X$. Then $(\mathrm{Val}^\gamma_{\mathbf{K}} \circ [\![f]\!]_\gamma)\tau = \mathrm{Val}^\gamma_{\mathbf{K}}(\ulcorner\xi\urcorner[\![f]\!]_\gamma) = \mathrm{Val}^\gamma_{\mathbf{K}}(\ulcorner f\xi\urcorner) = \{f\xi\}$. Since $f \ll \mathrm{Val}^\gamma_{\mathbf{K}} \circ k$, this set is a subset of $(\mathrm{Val}^\gamma_{\mathbf{K}} \circ k)\xi = \mathrm{Val}^\gamma_{\mathbf{K}}(k\xi) = \mathrm{Val}^\gamma_{\mathbf{K}}(\mathrm{Vals}^\gamma_{|\mathbf{K}|}\ulcorner k\xi\urcorner) = \mathrm{Val}^\gamma_{\mathbf{K}}(\mathrm{Vals}^\gamma_{|\mathbf{K}|}(\ulcorner\xi\urcorner[\![k]\!]_\gamma)) = (\mathrm{Val}^\gamma_{\mathbf{K}} \circ \mathrm{Vals}^\gamma_{|\mathbf{K}|} \circ [\![k]\!]_\gamma)\tau$

If $\tau = \triangle_\kappa$, then $(\mathrm{Val}^\gamma_{\mathbf{K}} \circ [\![f]\!]_\gamma)\tau = \mathrm{Val}^\gamma_{\mathbf{K}}(\triangle_\kappa[\![f]\!]_\gamma) = \mathrm{Val}^\gamma_{\mathbf{K}}\triangle_\kappa$. On the other hand $(\mathrm{Val}^\gamma_{\mathbf{K}} \circ \mathrm{Vals}^\gamma_{|\mathbf{K}|} \circ [\![k]\!]_\gamma)\tau = \mathrm{Val}^\gamma_{\mathbf{K}}(\mathrm{Vals}^\gamma_{|\mathbf{K}|}(\triangle_\kappa[\![k]\!]_\gamma)) = \mathrm{Val}^\gamma_{\mathbf{K}}(\mathrm{Vals}^\gamma_{|\mathbf{K}|}\triangle_\kappa) = \mathrm{Val}^\gamma_{\mathbf{K}}\triangle_\kappa$.

If $\tau = \mathtt{f}(\tau_1, \ldots, \tau_n)$ for some functional symbol $\mathtt{f}$ and gamma-semitermoids $\tau_1, \ldots, \tau_n$ of suitable sorts, then

$$\begin{aligned}(\mathrm{Val}^\gamma_{\mathbf{K}} \circ [\![f]\!]_\gamma)\tau &= \mathrm{Val}^\gamma_{\mathbf{K}}(\mathtt{f}(\tau_1, \ldots, \tau_n)[\![f]\!]_\gamma)\\ &= \mathrm{Val}^\gamma_{\mathbf{K}}(\mathtt{f}(\tau_1[\![f]\!]_\gamma, \ldots, \tau_n[\![f]\!]_\gamma))\\ &= \mathrm{Val}^\gamma_{\mathbf{K}}(\mathtt{f}^{[\![|\mathbf{K}|]\!]_\gamma}\langle\tau_1[\![f]\!]_\gamma, \ldots, \tau_n[\![f]\!]_\gamma\rangle)\\ &= \mathtt{f}^{\mathcal{P}\mathbf{K}}\langle\mathrm{Val}^\gamma_{\mathbf{K}}(\tau_1[\![f]\!]_\gamma), \ldots, \mathrm{Val}^\gamma_{\mathbf{K}}(\tau_n[\![f]\!]_\gamma)\rangle\\ &= \mathtt{f}^{\mathcal{P}\mathbf{K}}\langle(\mathrm{Val}^\gamma_{\mathbf{K}} \circ [\![f]\!]_\gamma)\tau_1, \ldots, (\mathrm{Val}^\gamma_{\mathbf{K}} \circ [\![f]\!]_\gamma)\tau_n\rangle\end{aligned}$$

By induction hypothesis, the last set is a subset of

$$\begin{aligned}
&\mathtt{f}^{\mathcal{P}\mathbf{K}}\langle(\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}\circ[\![k]\!]_{\gamma})\tau_1,\ldots,(\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}\circ[\![k]\!]_{\gamma})\tau_n\rangle = \\
&\quad=\mathtt{f}^{\mathcal{P}\mathbf{K}}\langle(\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|})(\tau_1[\![k]\!]_{\gamma}),\ldots,(\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|})(\tau_n[\![k]\!]_{\gamma})\rangle \\
&\quad=(\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|})(\mathtt{f}^{[\![|[\![|\mathbf{K}|]\!]_{\gamma}|]\!]_{\gamma}}\langle\tau_1[\![k]\!]_{\gamma},\ldots,\tau_n[\![k]\!]_{\gamma}\rangle) \\
&\quad=(\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|})(\mathtt{f}(\tau_1[\![k]\!]_{\gamma},\ldots,\tau_n[\![k]\!]_{\gamma})) \\
&\quad=(\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|})(\mathtt{f}(\tau_1,\ldots,\tau_n)[\![k]\!]_{\gamma}) \\
&\quad=(\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}\circ[\![k]\!]_{\gamma})\tau
\end{aligned}$$

Lastly, let $\tau = \mathtt{f}_i^{-1}(\tau')$, where $\mathtt{f}$ is a functional symbol of type $\langle\langle\kappa_1,\ldots,\kappa_n\rangle,\lambda\rangle$. Let $\mathfrak{g}$ be the function, such that for any subset Π of $\mathbf{K}_\lambda$, $\mathfrak{g}\Pi$ is the set of all $\mu\in\mathbf{K}_{\kappa_i}$, such that there exist $\mu_1\in\mathbf{K}_{\kappa_1},\ldots,\mu_n\in\mathbf{K}_{\kappa_n}$, such that $\mathtt{f}^{\mathbf{K}}\langle\mu_1,\ldots,\mu_n\rangle\in\Pi$ and $\mu=\mu_i$. Then

$$\begin{aligned}
(\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ[\![f]\!]_{\gamma})\tau &= \mathrm{Val}^{\gamma}_{\mathbf{K}}(\mathtt{f}_i^{-1}(\tau')[\![f]\!]_{\gamma}) \\
&= \mathrm{Val}^{\gamma}_{\mathbf{K}}(\mathtt{f}_i^{-1}(\tau'[\![f]\!]_{\gamma})) \\
&= \mathfrak{g}(\mathrm{Val}^{\gamma}_{\mathbf{K}}(\tau'[\![f]\!]_{\gamma})) \\
&= \mathfrak{g}((\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ[\![f]\!]_{\gamma})\tau')
\end{aligned}$$

By induction hypothesis, $(\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ[\![f]\!]_{\gamma})\tau' \subseteq (\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}\circ[\![k]\!]_{\gamma})\tau$, hence the above set is a subset of

$$\begin{aligned}
\mathfrak{g}((\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}\circ[\![k]\!]_{\gamma})\tau') &= \mathfrak{g}(\mathrm{Val}^{\gamma}_{\mathbf{K}}(\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}(\tau'[\![k]\!]_{\gamma}))) \\
&= \mathrm{Val}^{\gamma}_{\mathbf{K}}(\mathtt{f}_i^{-1}(\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}(\tau'[\![k]\!]_{\gamma}))) \\
&= \mathrm{Val}^{\gamma}_{\mathbf{K}}(\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}(\mathtt{f}_i^{-1}(\tau'[\![k]\!]_{\gamma}))) \\
&= \mathrm{Val}^{\gamma}_{\mathbf{K}}(\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}(\mathtt{f}_i^{-1}(\tau')[\![k]\!]_{\gamma})) \\
&= (\mathrm{Val}^{\gamma}_{\mathbf{K}}\circ\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}\circ[\![k]\!]_{\gamma})\tau
\end{aligned}$$

(16) and **(17)** follow immediately from the definitions and (12H2).

$$\begin{array}{ccc}
[\![|[\![X]\!]_{\gamma}|]\!]_{\gamma} & \xrightarrow{\mathrm{Vals}^{\gamma}_{X}} & [\![X]\!]_{\gamma} \\
\Big\downarrow{\scriptstyle [\![[\![f]\!]_{\gamma}]\!]_{\gamma}} & & \Big\downarrow{\scriptstyle [\![f]\!]_{\gamma}} \\
[\![|[\![Y]\!]_{\gamma}|]\!]_{\gamma} & \xrightarrow{\mathrm{Vals}^{\gamma}_{Y}} & [\![Y]\!]_{\gamma}
\end{array}$$

$$[\![X]\!]_{\gamma} = [\![X^{\circ}]\!]_{\gamma} \xrightarrow{[\![\mathrm{Nam}^{\gamma}_{X}]\!]_{\gamma}} [\![|[\![X]\!]_{\gamma}|]\!]_{\gamma} \xrightarrow{\mathrm{Vals}^{\gamma}_{X}} [\![X]\!]_{\gamma}$$

(18) follows immediately from the definitions.

$$|\mathbf{M}|^\circ \xrightarrow{\mathrm{Nam}^\gamma_{|\mathbf{M}|}} |[\![|\mathbf{M}|]\!]_\gamma| \xrightarrow{\mathrm{Val}^\gamma_{\mathbf{M}}} |\mathcal{P}\mathbf{M}|$$

■

§25. THE DELTA-TERMINATOR

A) In this section we are going to define the notion "delta-termoid". Delta-termoids are expressions like $\ulcorner 1 \urcorner + \mathtt{f}(\mathtt{g}(\mathtt{c}), \ulcorner 3 \urcorner + \mathtt{g}(\mathtt{d}), \mathtt{c})$. The meaning of the numbers of delta-termoids is similar to their meaning in beta-termoids, but the exact definition is more difficult to state.

Given a structure $\mathbf{M}$, every gamma-termoid with n free variables defines a multivalued function $\mathfrak{t} : |\mathbf{M}|^n \to \mathcal{P}|\mathbf{M}|$. Roughly speaking, μ is a value of the delta-termoid $n + \tau$ if $\mu \in \mathfrak{t}(\nu, \nu_1, \nu_2, \ldots, \nu_k)$ for some $\nu, \nu_1, \nu_2, \ldots, \nu_k$, such that ν is a value of τ and the multivalued function $\mathfrak{t}$ is defined by means of some gamma-termoid whose "height" is less than or equal to n.

B) Let $+$ be a new symbol, different from all operation symbols, parentheses, comma, symbols of the form $\mathtt{f}_i^{-1}$ and $\triangle_\kappa$, or any other formal symbol we use.

When n is a natural number, by $\ulcorner n \urcorner$ we will denote the symbol $\mathrm{nam}_{X,\mathtt{Log}}\, n$, where X is the $\mathtt{Sort}$-indexed set, such that $X_{\mathtt{Log}}$ is the set of the natural numbers and all other components of X are empty sets.[73]

C) **Definition.** Let X be an arbitrary $\mathtt{Sort}$-indexed set. We define the *delta-semitermoids* over X inductively:

(1) If $\mathtt{y} \in X_\kappa$, then $\mathrm{nam}_{X,\kappa}(\mathtt{y})$ is delta-semitermoid of sort κ over X for any algebraic sort κ.

(2) If there exists at least one term of sort κ over X, then $\triangle_\kappa$ is a delta-semitermoid of sort κ over X.

(3) If $\mathtt{f}$ is a functional symbol of type $\langle\langle\kappa_1, \ldots, \kappa_n\rangle, \lambda\rangle$ and $\tau_1, \ldots, \tau_n$ are delta-semitermoids over X of sorts $\kappa_1, \ldots, \kappa_n$, respectively, then the string $\mathtt{f}(\tau_1, \ldots, \tau_n)$ is delta-semitermoid of sort λ over X.

(4) If $\mathtt{f}$ is a functional symbol of type $\langle\langle\kappa_1, \ldots, \kappa_n\rangle, \lambda\rangle$ and τ is a delta-semitermoid over X of sort λ, then the string $\mathtt{f}_i^{-1}(\tau)$ is delta-semitermoid of sort κ_i over X for any $i \in \{1, \ldots, n\}$.

(5) If τ is a delta-semitermoid of sort κ and n is a natural number, then the string $+(\ulcorner n \urcorner, \tau)$ also is a delta-semitermoid of sort κ. In order to

[73] The choice of the sort $\mathtt{Log}$ here is completely arbitrary. What we need is a method to encode the natural numbers with formal symbols. Any sort can be used instead of $\mathtt{Log}$.

improve the readability, we are going to use infix notation and write $\ulcorner n \urcorner + \tau$ instead of $+(\ulcorner n \urcorner, \tau)$.

D) **Corollary.** (1) *If a delta-semitermoid over X does not contain the symbol $+$, then it is a gamma-semitermoid over X of the same sort.*

(2) *Any gamma-semitermoid over X is a delta-semitermoid over X of the same sort.*

Proof. Compare definitions (C) and (24B). ∎

In order to define the semantics of the delta-semitermoids, in the following definition we define the notion “embraces”. Then (in Q) we are going to say that μ is a value of a delta-semitermoid τ if and only if μ is a value of some embraced by τ gamma-semitermoid. In other words, intuitively we can equate a delta-semitermoid with the set of all embraced by it gamma-semitermoids. In future we are going to define also an alternative semantics (see definition 27F).

E) **Definition.** By induction we define the relation “delta-semitermoid τ *embraces* a gamma-semitermoid σ”:

(1) Each gamma-semitermoid embraces itself.

(2) If σ is embraced by τ, then σ is embraced by $\ulcorner 0 \urcorner + \tau$.

(3) If σ is embraced by $\ulcorner n \urcorner + \tau$, then σ is embraced by $\ulcorner n+1 \urcorner + \tau$.

(4) If τ_i embraces σ_i for all $i \in \{1, \ldots, n\}$, then $\mathtt{f}(\tau_1, \ldots, \tau_n)$ embraces $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$.

(5) If τ' embraces σ', then $\mathtt{f}_i^{-1}(\tau')$ embraces $\mathtt{f}_i^{-1}(\sigma')$.

(6) For any functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1, \ldots, \kappa_n\rangle, \lambda\rangle$, if the delta-semitermoid $\ulcorner n+1 \urcorner + \tau$ over X of sort κ_i embraces σ and the delta-semitermoid $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_n})$ embraces σ', then $\ulcorner n+1 \urcorner + \tau$ embraces $\mathtt{f}_i^{-1}(\sigma')$.

F) **Lemma.** *A gamma-semitermoid embraces only itself.*

Proof. By induction on definition (E). According to (E1), a gamma-semitermoid embraces itself. Rules (E2), (E3) and (E6) are talking about gamma-semitermoids embraced by a delta-semitermoid containing at least one symbol $+$, i.e. a delta-semitermoid which is not a gamma-semitermoid.

If we apply (E4) in order to prove that $\mathtt{f}(\tau_1, \ldots, \tau_n)$ embraces $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ and $\mathtt{f}(\tau_1, \ldots, \tau_n)$ is a gamma-semitermoid, then $\tau_1, \ldots, \tau_n$ are gamma-semitermoids, so by induction hypothesis, $\tau_i = \sigma_i$ for any $i \in \{1, \ldots, n\}$, hence $\mathtt{f}(\tau_1, \ldots, \tau_n) = \mathtt{f}(\sigma_1, \ldots, \sigma_n)$.

If we apply (E5) in order to prove that $\mathtt{f}_i^{-1}(\tau')$ embraces $\mathtt{f}_i^{-1}(\sigma')$ and

$\mathtt{f}_i^{-1}(\tau')$ is a gamma-semitermoid, then τ' is a gamma-semitermoid, so by induction hypothesis, $\tau' = \sigma'$, hence $\mathtt{f}_i^{-1}(\tau') = \mathtt{f}_i^{-1}(\sigma')$. ∎

G) For the purpose of the next two lemmas, if τ is a delta-semitermoid, by $\mathfrak{g}\tau$ we will denote the gamma-semitermoid which is obtained from τ by removing all subexpressions of the form "$\ulcorner n \urcorner +$". For example, $\mathfrak{g}(\mathtt{f}(\mathsf{a}, \ulcorner 5 \urcorner + \mathtt{g}(\ulcorner 3 \urcorner + \mathsf{c}))) = \mathtt{f}(\mathsf{a}, \mathtt{g}(\mathsf{c}))$.

H) **Lemma.** *Every delta-semitermoid τ embraces $\mathfrak{g}\tau$.*

Proof. By induction on τ.

If τ is a name or has the form $\triangle_\kappa$, then τ is gamma-semitermoid, hence $\mathfrak{g}\tau = \tau$ and τ is embraced by τ.

If $\tau = \mathtt{f}(\tau_1, \dots, \tau_n)$, then by induction hypothesis, $\mathfrak{g}\tau_i$ is embraced by τ_i for any $i \in \{1, \dots, n\}$, hence $\mathfrak{g}\tau = \mathtt{f}(\mathfrak{g}\tau_1, \dots, \mathfrak{g}\tau_n)$ is embraced by τ.

If $\tau = \mathtt{f}_i^{-1}(\tau')$, then by induction hypothesis $\mathfrak{g}\tau'$ is embraced by τ', hence $\mathfrak{g}\tau = \mathtt{f}_i^{-1}(\mathfrak{g}\tau')$ is embraced by τ.

If $\tau = \ulcorner n \urcorner + \tau'$, then by induction hypothesis, $\mathfrak{g}\tau'$ is embraced by τ', hence by (E2) and (E3), $\mathfrak{g}\tau'$ is embraced by $\ulcorner n \urcorner + \tau'$, so $\mathfrak{g}\tau = \mathfrak{g}\tau'$ is embraced by $\ulcorner n \urcorner + \tau' = \tau$. ∎

I) **Lemma.** *If the delta-semitermoid τ embraces the gamma-semitermoid σ, then for any ρ, ρ is an associated gamma-semitermoid of σ if and only if ρ is an associated gamma-semitermoid of $\mathfrak{g}\tau$.*

Proof. By induction on definition (E).

(1) If τ is a gamma-semitermoid, then $\mathfrak{g}\tau = \tau$ and because of (F), $\sigma = \tau$, hence $\mathfrak{g}\tau = \sigma$, so there is nothing to prove.

(2) σ is embraced by $\ulcorner 0 \urcorner + \tau$, because σ is embraced by τ.

By induction hypothesis, ρ is associated with σ if and only if ρ is associated with $\mathfrak{g}\tau$, if and only if ρ is associated with $\mathfrak{g}(\ulcorner 0 \urcorner + \tau) = \mathfrak{g}\tau$.

(3) σ is embraced by $\ulcorner n+1 \urcorner + \tau$, because σ is embraced by $\ulcorner n \urcorner + \tau$.

By induction hypothesis ρ is associated with σ if and only if ρ is associated with $\mathfrak{g}(\ulcorner n \urcorner + \tau)$, if and only if ρ is associated with $\mathfrak{g}(\ulcorner n+1 \urcorner + \tau) = \mathfrak{g}(\ulcorner n \urcorner + \tau) = \mathfrak{g}\tau$.

(4) $\mathtt{f}(\sigma_1, \dots, \sigma_m)$ is embraced by $\mathtt{f}(\tau_1, \dots, \tau_m)$ because σ_i is embraced by τ_i for all $i \in \{1, \dots, m\}$.

According to definition (24E3), ρ is associated with $\mathtt{f}(\sigma_i, \dots, \sigma_m)$ if and only if $\rho = \mathtt{f}(\rho_1, \dots, \rho_m)$ for some $\rho_1, \dots, \rho_m$, such that ρ_i is associated with σ_i for all $i \in \{1, \dots, m\}$. By induction hypothesis, this is so if and only if $\rho = \mathtt{f}(\rho_1, \dots, \rho_m)$ for some $\rho_1, \dots, \rho_m$, such that ρ_i is associated with $\mathfrak{g}\tau_i$ for all $i \in \{1, \dots, m\}$. According to definition (24E3), this is so if and only if $\rho = \mathtt{f}(\rho_1, \dots, \rho_m)$ is associated with $\mathtt{f}(\mathfrak{g}\tau_1, \dots, \mathfrak{g}\tau_m) = \mathfrak{g}(\mathtt{f}(\tau_1, \dots, \tau_m))$.

(5) $\mathtt{f}_i^{-1}(\tau')$ embraces $\mathtt{f}_i^{-1}(\sigma')$ because τ' embraces σ'.

According to definition (24E4), ρ is associated with $\mathtt{f}_i^{-1}(\sigma')$ if and only if σ' has associated gamma-semitermoid $\mathtt{f}(\rho_1, \dots, \rho_m)$, such that $\rho = \rho_i$. By induction hypothesis, this is so if and only if $\mathfrak{g}\tau'$ has associated gamma-semitermoid $\mathtt{f}(\rho_1, \dots, \rho_m)$, such that $\rho = \rho_i$. According to definition (24E4), this is so if and only if $\rho = \rho_i$ is associated with $\mathtt{f}_i^{-1}(\mathfrak{g}\tau') = \mathfrak{g}(\mathtt{f}_i^{-1}(\tau'))$.

(6) Given a functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1, \dots, \kappa_n\rangle, \lambda\rangle$, $\ulcorner n+1 \urcorner + \tau$ embraces $\mathtt{f}_i^{-1}(\sigma')$ because $\ulcorner n+1 \urcorner + \tau$ embraces σ and $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \dots, \triangle_{\kappa_n})$ embraces σ'.

According to definition (24E4), ρ is associated with $\mathtt{f}_i^{-1}(\sigma')$ if and only if σ' has associated gamma-semitermoid $\mathtt{f}(\rho_1, \dots, \rho_m)$, such that $\rho = \rho_i$. By induction hypothesis,[74] this is so if and only if $\mathfrak{g}(\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \dots, \triangle_{\kappa_n}))$ has associated gamma-semitermoid $\mathtt{f}(\rho_1, \dots, \rho_m)$, such that $\rho = \rho_i$. Considering that $\mathfrak{g}(\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \dots, \triangle_{\kappa_n})) = \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \dots, \triangle_{\kappa_n})$, according to definition (24E3), this is so if and only if $\rho = \rho_i$ is associated gamma-semitermoid of σ. By induction hypothesis,[75] this is so if and only if ρ is associated gamma-semitermoid of $\mathfrak{g}(\ulcorner n+1 \urcorner + \tau) = \mathfrak{g}\tau$. ∎

J) **Definition.** τ is a *delta-termoid* over X of sort κ, if τ is a delta-semitermoid over X of sort κ and τ contains no symbol of the form $\mathtt{f}_i^{-1}$ or $\triangle_\kappa$.

K) **Proposition.** *All embraced by a delta-termoid gamma-semitermoids are gamma-termoids.*

Proof. Suppose that the gamma-semitermoid σ is embraced by the delta-termoid τ. Let $\tau' = \mathfrak{g}\tau$, i.e. τ' is obtained from τ by removing all subexpressions of the form "$\ulcorner n \urcorner +$". Then τ' contains no symbols of the form $\mathtt{f}_i^{-1}$ or $\triangle_\kappa$ (because τ is a delta-termoid) and no symbol "+" (because we have removed the symbols "+" from τ), so τ' is a term and, particularly, a gamma-semitermoid. Therefore, τ' is the only embraced by τ' gamma-semitermoid (see F) and τ' is the associated term of τ' (see 24G2). According to (I), τ' is the associated term of σ, hence σ is a gamma-termoid. ∎

L) **Definition.** For any $\mathtt{Sort}$-indexed set X, let $[\![X]\!]_\delta$ be the algebra, such that:

(1) The algebraic carrier of sort κ of $[\![X]\!]_\delta$ is the set of all delta-termoids over X of sort κ.

[74] About $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \dots, \triangle_{\kappa_n})$ embraces σ'.

[75] About $\ulcorner n+1 \urcorner + \tau$ embraces σ.

(2) For any functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1,\dots,\kappa_n\rangle,\lambda\rangle$ and delta-termoids $\tau_1,\dots,\tau_n$ of sorts $\kappa_1,\dots,\kappa_n$, respectively, let

$$\mathtt{f}^{[\![X]\!]_\delta}\langle\tau_1,\dots,\tau_n\rangle = \mathtt{f}(\tau_1,\dots,\tau_n)$$

where on the right side of the equality sign stays a formal expression.

This definition is correct because:

First, the elements of the algebraic carriers of $[\![X]\!]_\delta$ are exactly the delta-termoids, hence $\mathtt{f}(\tau_1,\dots,\tau_n)$ belongs to the carrier of sort λ of $[\![X]\!]_\delta$.

Second, any algebra is uniquely determined by its algebraic carriers and the interpretation of the functional symbols (see 12Q1).

M) **Definition.** Given $\mathtt{Sort}$-indexed sets X and Y and a $\mathtt{Sort}$-indexed function $f : X \to Y$, let $[\![f]\!]_\delta : [\![X]\!]_\delta \to [\![Y]\!]_\delta$ be the homomorphism, who when applied to a delta-termoid τ, replaces all occurrences of names $\mathrm{nam}_{X,\lambda}(\mathtt{z})$ in τ with $\mathrm{nam}_{Y,\lambda}(f_\lambda \mathtt{z})$ (i.e. $[\![f]\!]_\delta$ replaces all occurrences of $\ulcorner\mathtt{z}\urcorner$ with $\ulcorner f\mathtt{z}\urcorner$).

We are going to use postfix notation for this homomorphism. Thus $\tau[\![f]\!]_\delta$ means to apply $[\![f]\!]_\delta$ to τ. As an extension of the notation, we are going to write $\tau[\![f]\!]_\delta$ even when τ is an arbitrary delta-semitermoid, i.e. not necessarily a delta-termoid. Let $\tau[\![f]\!]_\delta$ be the expression which is obtained from τ by replacing all occurrences of names $\mathrm{nam}_{X,\lambda}(\mathtt{z})$ in τ with $\mathrm{nam}_{Y,\lambda}(f_\lambda \mathtt{z})$ (i.e. in $\tau[\![f]\!]_\delta$ all occurrences of $\ulcorner\mathtt{z}\urcorner$ in τ are replaced with $\ulcorner f\mathtt{z}\urcorner$).

The following proposition shows that the above definition is correct:

N) **Proposition.** *Let $f : X \to Y$ be a $\mathtt{Sort}$-indexed function. Then:*

(1) *If τ is a delta-semitermoid over X of sort κ, then $\tau[\![f]\!]_\delta$ is a delta-semitermoid over Y of sort κ.*

(2) *If τ is a delta-termoid over X of sort κ, then $\tau[\![f]\!]_\delta$ is a delta-termoid over Y of sort κ.*

(3) *There exists unique homomorphism from $[\![X]\!]_\delta$ to $[\![Y]\!]_\delta$, such that the result of its application to any delta-termoid τ is equal to $\tau[\![f]\!]_\delta$.*

Proof. **(1)** can be proved by a simple induction on definition (C).

(2) follows immediately from (1) and definitions (J) and (L).

(3) follows from (2) and (12Q2). We only have to notice that for any functional symbol $\mathtt{f}$ and delta-termoids $\tau_1,\dots,\tau_n$ of suitable sorts, $(\mathtt{f}^{[\![X]\!]_\delta}\langle\tau_1,\dots,\tau_n\rangle)[\![f]\!]_\delta = (\mathtt{f}(\tau_1,\dots,\tau_n))[\![f]\!]_\delta = \mathtt{f}(\tau_1[\![f]\!]_\delta,\dots,\tau_n[\![f]\!]_\delta) = \mathtt{f}^{[\![Y]\!]_\delta}\langle\tau_1[\![f]\!]_\delta,\dots,\tau_n[\![f]\!]_\delta\rangle$. ∎

O) **Lemma.** *Given a $\mathtt{Sort}$-indexed function $f : X \to Y$, a gamma-semitermoid σ over X and a delta-semitermoid τ over X, if σ is embraced* *by τ, then $\sigma[\![f]\!]_\gamma$ is embraced by $\tau[\![f]\!]_\delta$.*

Proof. By induction on definition (E).

(1) If $\tau = \sigma$ and both are gamma-semitermoids, then $\tau[\![f]\!]_\delta = \sigma[\![f]\!]_\gamma$ and both are gamma-semitermoids, hence $\tau[\![f]\!]_\delta$ embraces $\sigma[\![f]\!]_\gamma$.

(2) If σ is embraced by $\ulcorner 0 \urcorner + \tau$ because σ is embraced by τ, then by induction hypothesis, $\sigma[\![f]\!]_\gamma$ is embraced by $\tau[\![f]\!]_\delta$, hence $\sigma[\![f]\!]_\gamma$ is embraced by $\ulcorner 0 \urcorner + \tau[\![f]\!]_\delta = (\ulcorner 0 \urcorner + \tau)[\![f]\!]_\delta$.

(3) If σ is embraced by $\ulcorner n+1 \urcorner + \tau$ because σ is embraced by $\ulcorner n \urcorner + \tau$, then by induction hypothesis, σ is embraced by $(\ulcorner n \urcorner + \tau)[\![f]\!]_\delta = \ulcorner n \urcorner + \tau[\![f]\!]_\delta$, hence σ is embraced by $\ulcorner n+1 \urcorner + \tau[\![f]\!]_\delta = (\ulcorner n+1 \urcorner + \tau)[\![f]\!]_\delta$.

(4) If $\mathtt{f}(\sigma_1, \dots, \sigma_n)$ is embraced by $\mathtt{f}(\tau_1, \dots, \tau_n)$, because σ_i is embraced by τ_i for all $i \in \{1, \dots, n\}$, then by induction hypothesis, $\sigma_i[\![f]\!]_\gamma$ is embraced by $\tau_i[\![f]\!]_\delta$ for all $i \in \{1, \dots, n\}$, hence $\mathtt{f}(\sigma_1[\![f]\!]_\gamma, \dots, \sigma_n[\![f]\!]_\gamma) = (\mathtt{f}(\sigma_1, \dots, \sigma_n))[\![f]\!]_\gamma$ is embraced by $\mathtt{f}(\tau_1[\![f]\!]_\delta, \dots, \tau_n[\![f]\!]_\delta) = (\mathtt{f}(\tau_1, \dots, \tau_n))[\![f]\!]_\delta$.

(5) If $\mathtt{f}_i^{-1}(\sigma')$ is embraced by $\mathtt{f}_i^{-1}(\tau')$ because σ' is embraced by τ', then by induction hypothesis, $\sigma'[\![f]\!]_\gamma$ is embraced by $\tau'[\![f]\!]_\delta$, hence $(\mathtt{f}_i^{-1}(\sigma'))[\![f]\!]_\gamma = \mathtt{f}_i^{-1}(\sigma'[\![f]\!]_\gamma)$ is embraced by $(\mathtt{f}_i^{-1}(\tau'))[\![f]\!]_\delta = \mathtt{f}_i^{-1}(\tau'[\![f]\!]_\delta)$.

(6) Given a functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1, \dots, \kappa_n\rangle, \lambda\rangle$, suppose that $\ulcorner n+1 \urcorner + \tau$ embraces $\mathtt{f}_i^{-1}(\sigma')$ because $\ulcorner n+1 \urcorner + \tau$ embraces σ and $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \dots, \triangle_{\kappa_n})$ embraces σ'. By induction hypothesis, $(\ulcorner n+1 \urcorner + \tau)[\![f]\!]_\delta = \ulcorner n+1 \urcorner + \tau[\![f]\!]_\delta$ embraces $\sigma[\![f]\!]_\gamma$. Also by induction hypothesis,

$$
\begin{aligned}
(\ulcorner n \urcorner + &\mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \dots, \triangle_{\kappa_n}))[\![f]\!]_\delta = \\
&= \ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{i-1}}, \sigma[\![f]\!]_\delta, \triangle_{\kappa_{i+1}}, \dots, \triangle_{\kappa_n}) \\
&= \ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{i-1}}, \sigma[\![f]\!]_\gamma, \triangle_{\kappa_{i+1}}, \dots, \triangle_{\kappa_n})
\end{aligned}
$$

embraces $\sigma'[\![f]\!]_\gamma$. Consequently, $\ulcorner n+1 \urcorner + \tau[\![f]\!]_\delta = (\ulcorner n+1 \urcorner + \tau)[\![f]\!]_\delta$ embraces $\mathtt{f}_i^{-1}(\sigma'[\![f]\!]_\gamma) = (\mathtt{f}_i^{-1}(\sigma'))[\![f]\!]_\gamma$. ∎

The following Lemma is close to the opposite direction of the previous Lemma.

P) **Lemma.** *Given a* Sort*-indexed function $f : X \to Y$, a gamma-semitermoid σ over Y and a delta-semitermoid τ over X, if σ is embraced by $\tau[\![f]\!]_\delta$, then there exists a gamma-semitermoid ρ, such that ρ is embraced by τ and $\rho[\![f]\!]_\gamma = \sigma$.*

Proof. We are going to prove the following statement: for any gamma-semitermoid σ and delta-semitermoid $\tau^{\%}$, both over Y, if σ is embraced by $\tau^{\%}$, then for any delta-semitermoid τ over X, if $\tau[\![f]\!]_\delta = \tau^{\%}$, then there exists a gamma-semitermoid ρ, such that ρ is embraced by τ and $\rho[\![f]\!]_\gamma = \sigma$.

We are going to prove this by induction on definition (E) regarding the "embracement" of σ by $\tau^{\%}$. In order to simplify the presentation, we are going to implicitly use that the homomorphisms $[\![f]\!]_\gamma$ and $[\![f]\!]_\delta$ have several "injective" properties, such as: if $(\ulcorner m' \urcorner + \rho')[\![f]\!]_\delta = (\ulcorner m'' \urcorner + \rho'')[\![f]\!]_\delta$, then $m' = m''$ and $\rho'[\![f]\!]_\delta = \rho''[\![f]\!]_\delta$, if $(\mathtt{f}(\rho'_1, \ldots, \rho'_n))[\![f]\!]_\gamma = (\mathtt{f}(\rho''_1, \ldots, \rho''_n))[\![f]\!]_\gamma$, then $\rho'_i[\![f]\!]_\gamma = \rho'_i[\![f]\!]_\gamma$ for any $i \in \{1, \ldots, n\}$, etc.

(1) If $\sigma = \tau[\![f]\!]_\delta$ and both are gamma-semitermoids, then τ is a gamma-semitermoid, so τ is embraced by τ, hence we can take $\rho = \tau$.

(2) If σ is embraced by $(\ulcorner 0 \urcorner + \tau)[\![f]\!]_\delta = \ulcorner 0 \urcorner + \tau[\![f]\!]_\delta$ because σ is embraced by $\tau[\![f]\!]_\delta$, then by induction hypothesis, there exists an embraced by τ gamma-semitermoid ρ, such that $\rho[\![f]\!]_\gamma = \sigma$. It only remains to notice that ρ will be also embraced by $\ulcorner 0 \urcorner + \tau$.

(3) If σ is embraced by $(\ulcorner n+1 \urcorner + \tau)[\![f]\!]_\delta = \ulcorner n+1 \urcorner + \tau[\![f]\!]_\delta$ because σ is embraced by $(\ulcorner n \urcorner + \tau)[\![f]\!]_\delta = \ulcorner n \urcorner + \tau[\![f]\!]_\delta$, then by induction hypothesis, there exists an embraced by $\ulcorner n \urcorner + \tau$ gamma-semitermoid ρ, such that $\rho[\![f]\!]_\gamma = \sigma$. It only remains to notice that ρ will be also embraced by $\ulcorner n+1 \urcorner + \tau$.

(4) If $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ is embraced by $(\mathtt{f}(\tau_1, \ldots, \tau_n))[\![f]\!]_\delta = \mathtt{f}(\tau_1[\![f]\!]_\delta, \ldots, \tau_n[\![f]\!]_\delta)$, because σ_i is embraced by $\tau_i[\![f]\!]_\delta$ for all $i \in \{1, \ldots, n\}$, then by induction hypothesis, there exist gamma-semitermoids $\rho_1, \ldots, \rho_n$ embraced respectively by $\tau_1, \ldots, \tau_n$, such that $\rho_i[\![f]\!]_\gamma = \sigma_i$ for all $i \in \{1, \ldots, n\}$. It only remains to notice that $\mathtt{f}(\rho_1, \ldots, \rho_n)$ will be embraced by $\mathtt{f}(\tau_1, \ldots, \tau_n)$ and $(\mathtt{f}(\rho_1, \ldots, \rho_n))[\![f]\!]_\gamma = \mathtt{f}(\rho_1[\![f]\!]_\gamma, \ldots, \rho_n[\![f]\!]_\gamma) = \mathtt{f}(\sigma_1, \ldots, \sigma_n)$.

(5) If $\mathtt{f}_i^{-1}(\sigma')$ is embraced by $(\mathtt{f}_i^{-1}(\tau'))[\![f]\!]_\delta = \mathtt{f}_i^{-1}(\tau'[\![f]\!]_\delta)$ because σ' is embraced by $\tau'[\![f]\!]_\delta$, then by induction hypothesis, there exists an embraced by τ' gamma-semitermoid ρ, such that $\rho[\![f]\!]_\gamma = \sigma'$. It only remains to notice that $\mathtt{f}_i^{-1}(\rho)$ will be embraced by $\mathtt{f}_i^{-1}(\tau')$ and $(\mathtt{f}_i^{-1}(\rho))[\![f]\!]_\gamma = \mathtt{f}_i^{-1}(\rho[\![f]\!]_\gamma) = \mathtt{f}_i^{-1}(\sigma')$.

(6) Given a functional symbol $\mathtt{f}$ of type $\langle\langle \kappa_1, \ldots, \kappa_n \rangle, \lambda\rangle$, suppose that $(\ulcorner n+1 \urcorner + \tau)[\![f]\!]_\delta = \ulcorner n+1 \urcorner + \tau[\![f]\!]_\delta$ embraces $\mathtt{f}_i^{-1}(\sigma')$ because $(\ulcorner n+1 \urcorner + \tau)[\![f]\!]_\delta = \ulcorner n+1 \urcorner + \tau[\![f]\!]_\delta$ embraces σ and $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_n})$ embraces σ'. By induction hypothesis, there exists an embraced by $\ulcorner n+1 \urcorner + \tau$ gamma-semitermoid ρ, such that $\rho[\![f]\!]_\gamma = \sigma$.

Since $\rho[\![f]\!]_\gamma = \sigma$, $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \rho[\![f]\!]_\gamma, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_n})$ embraces σ' and is equal to $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_n})$. Therefore, we are permitted to use the induction hypothesis again. We obtain that there exists an embraced by $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \rho, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_n})$

gamma-semitermoid ρ', such that $\rho'[\![f]\!]_\gamma = \sigma'$.

Since $\ulcorner n+1 \urcorner + \tau$ embraces ρ and $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{i-1}}, \rho, \triangle_{\kappa_{i+1}}, \dots, \triangle_{\kappa_n})$ embraces ρ', from (E6) we obtain that $\ulcorner n+1 \urcorner + \tau$ embraces ρ'. Now it only remains to notice that $\rho'[\![f]\!]_\gamma = \sigma'$ implies that $(\mathtt{f}_i^{-1}(\rho'))[\![f]\!]_\gamma = \mathtt{f}_i^{-1}(\sigma')$. ■

Q) **Definition.** Given a structure $\mathbf{M}$, let $\mathrm{Val}^\delta_{\mathbf{M}} : [\![|\mathbf{M}|]\!]_\delta \to \mathcal{P}\mathbf{M}$ be the only homomorphism, such that for any delta-termoid τ over $|\mathbf{M}|$, $\mathrm{Val}^\delta_{\mathbf{M}}\,\tau$ is equal to the union of the sets of the values in $\mathbf{M}$ of all embraced by τ gamma-termoids.

As an extension of the notation, we are going to write $\mathrm{Val}^\delta_{\mathbf{M}}\,\tau$ even when τ is an arbitrary delta-semitermoid over $|\mathbf{M}|$ (i.e. not necessarily a delta-termoid). In this case too, let $\mathrm{Val}^\delta_{\mathbf{M}}\,\tau$ be equal to the union of the sets of the values in $\mathbf{M}$ of all embraced by τ gamma-semitermoids.

According to (12Q2), in order to prove that $\mathrm{Val}^\delta_{\mathbf{M}}$ is actually a homomorphism, we only have to prove that for any functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1, \dots, \kappa_n\rangle, \lambda\rangle$ and any termoids $\tau_1 \dots, \tau_n$ of sorts $\kappa_1, \dots, \kappa_n$, we have $\mathrm{Val}^\delta_{\mathbf{M}}(\mathtt{f}^{[\![|\mathbf{M}|]\!]_\delta}\langle\tau_1, \dots, \tau_n\rangle) = \mathtt{f}^{\mathcal{P}\mathbf{M}}\langle \mathrm{Val}^\delta_{\mathbf{M}}\,\tau_1, \dots, \mathrm{Val}^\delta_{\mathbf{M}}\,\tau_n\rangle$.

For any delta-semitermoid τ, let $\mathrm{Emb}\,\tau$ be the set of all embraced by τ gamma-semitermoids. Notice that from definition (E) it follows that $\mathrm{Emb}(\mathtt{f}(\tau_1, \dots, \tau_n)) = \{\mathtt{f}(\sigma_1, \dots, \sigma_n) : \sigma_1 \in \mathrm{Emb}\,\tau_1, \dots, \sigma_n \in \mathrm{Emb}\,\tau_n\}$ and that we already know that $\mathrm{Val}^\gamma_{\mathbf{M}}$ is a homomorphism. Therefore:

$$\begin{aligned}
&\mathrm{Val}^\delta_{\mathbf{M}}(\mathtt{f}^{[\![|\mathbf{M}|]\!]_\delta}\langle\tau_1, \dots, \tau_n\rangle) = \\
&\quad = \mathrm{Val}^\delta_{\mathbf{M}}(\mathtt{f}(\tau_1, \dots, \tau_n)) && \text{from (L2)} \\
&\quad = \cup\,\{\mathrm{Val}^\gamma_{\mathbf{M}}\,\sigma : \sigma \in \mathrm{Emb}(\mathtt{f}(\tau_1, \dots, \tau_n))\} && \text{from (Q)} \\
&\quad = \cup\,\{\mathrm{Val}^\gamma_{\mathbf{M}}(\mathtt{f}(\sigma_1, \dots, \sigma_n)) : \sigma_1 \in \mathrm{Emb}\,\tau_1, \dots, \sigma_n \in \mathrm{Emb}\,\tau_n\} && \text{from (E)} \\
&\quad = \cup\,\{\mathrm{Val}^\gamma_{\mathbf{M}}(\mathtt{f}^{[\![|\mathbf{M}|]\!]_\gamma}\langle\sigma_1, \dots, \sigma_n\rangle) : \sigma_1 \in \mathrm{Emb}\,\tau_1, \dots, \sigma_n \in \mathrm{Emb}\,\tau_n\} \\
&\quad = \cup\,\{\mathtt{f}^{\mathcal{P}\mathbf{M}}\langle\mathrm{Val}^\gamma_{\mathbf{M}}\,\sigma_1, \dots, \mathrm{Val}^\gamma_{\mathbf{M}}\,\sigma_n\rangle : \sigma_1 \in \mathrm{Emb}\,\tau_1, \dots, \sigma_n \in \mathrm{Emb}\,\tau_n\} \\
&\quad = \{\mathtt{f}^{\mathbf{M}}\langle\mu_1, \dots, \mu_n\rangle : \mu_1 \in \mathrm{Val}^\delta_{\mathbf{M}}\,\tau_1, \dots, \mu_n \in \mathrm{Val}^\delta_{\mathbf{M}}\,\tau_n\} && \text{from (Q)} \\
&\quad = \mathtt{f}^{\mathcal{P}\mathbf{M}}\langle\mathrm{Val}^\delta_{\mathbf{M}}\,\tau_1, \dots, \mu_n \in \mathrm{Val}^\delta_{\mathbf{M}}\,\tau_n\rangle
\end{aligned}$$

R) **Proposition.** (1) *Given a structure* $\mathbf{M}$ *and a delta-termoid* τ *over* $|\mathbf{M}|$*, if the term* σ *is obtained from* τ *by removing all occurrences of substrings of the form "*$\ulcorner n \urcorner +$*", then* $\sigma^{\mathbf{M}} \in \mathrm{Val}^\delta_{\mathbf{M}}\,\tau$.

(2) *In addition, if* $\mathbf{M}$ *is a structure of terms, then* $\{\sigma^{\mathbf{M}}\} = \mathrm{Val}^\delta_{\mathbf{M}}\,\tau$.

<u>Proof.</u> **(1)** According to (H), σ is embraced by τ, so the required follows from (24I2) and definition (Q).

(2) According to (H), σ is embraced by τ. According to (I), ρ is associated gamma-semitermoid of τ if and only if ρ is associated gamma-semitermoid of σ. According to (24G2), this is so if and only if $\rho = \sigma$. Consequently, ρ is associated gamma-semitermoid of some embraced by τ gamma-semitermoid if and only if $\rho = \sigma$. According to (24P), $\sigma^{\mathbf{M}}$ is the only value in $\mathbf{M}$ of any embraced by τ gamma-semitermoid. ∎

S) **Definition.** Given a $\mathtt{Sort}$-indexed set X, let $\mathrm{Vals}^{\delta}_{X} : [\![|[\![X]\!]_{\delta}|]\!]_{\delta} \to [\![X]\!]_{\delta}$ be the only homomorphism, such that for any delta-termoid τ over $|[\![X]\!]_{\delta}|$, $\mathrm{Vals}^{\delta}_{X}\,\tau$ is the result of the replacement in τ of all names $\mathrm{nam}_{|[\![X]\!]_{\delta}|}\,\sigma$ with σ.

As an extension of the notation, we are going to write $\mathrm{Vals}^{\delta}_{X}\,\tau$ even when τ is only a delta-semitermoid over $|[\![X]\!]_{\delta}|$, not necessarily a delta-termoid. In this case too, let $\mathrm{Val}^{\delta}_{X}\,\tau$ be the result of the replacement in τ of all names $\mathrm{nam}_{|[\![X]\!]_{\delta}|}\,\sigma$ with σ.

It is not difficult to see that when $\mathrm{Vals}^{\delta}_{X}$ is applied to a delta-termoid over $|[\![X]\!]_{\delta}|$, the result is a delta-termoid over X of the same sort. Therefore, according to (12Q2), in order to see that the above definition is correct, it remains to notice that for any functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1,\ldots,\kappa_n\rangle,\lambda\rangle$ and any delta-termoids $\tau_1\ldots,\tau_n$ over $|[\![X]\!]_{\delta}|$ of sorts $\kappa_1,\ldots,\kappa_n$, we have $\mathrm{Vals}^{\delta}_{X}(\mathtt{f}^{[\![|[\![X]\!]_{\delta}|]\!]_{\delta}}\langle\tau_1,\ldots,\tau_n\rangle) = \mathtt{f}^{[\![X]\!]_{\delta}}\langle\mathrm{Vals}^{\delta}_{X}\,\tau_1,\ldots,\mathrm{Vals}^{\delta}_{X}\,\tau_n\rangle$. This is so because anything $\mathrm{Vals}^{\delta}_{X}$ replaces is a name and $\mathtt{f}$ is not a name, hence $\mathrm{Vals}^{\delta}_{X}(\mathtt{f}^{[\![|[\![X]\!]_{\delta}|]\!]_{\delta}}\langle\tau_1,\ldots,\tau_n\rangle) = \mathrm{Vals}^{\delta}_{X}(\mathtt{f}(\tau_1,\ldots,\tau_n)) = \mathtt{f}(\mathrm{Vals}^{\delta}_{X}\,\tau_1,\ldots,\mathrm{Vals}^{\delta}_{X}\,\tau_n) = \mathtt{f}^{[\![X]\!]_{\delta}}\langle\mathrm{Vals}^{\delta}_{X}\,\tau_1,\ldots,\mathrm{Vals}^{\delta}_{X}\,\tau_n\rangle$.

T) **Lemma.** *Let the* $\mathtt{Sort}$*-indexed functions* $g : X \to |[\![Y]\!]_{\gamma}|$ *and* $d : X \to |[\![Y]\!]_{\delta}|$ *be such that the gamma-termoid* $g\xi$ *is embraced by the delta-termoid* $d\xi$ *for any* ξ *belonging to an algebraic component of* X*. Then for any gamma-semitermoid* τ *over* X*, the gamma-semitermoid* $\mathrm{Vals}^{\gamma}_{X}(\tau[\![g]\!]_{\gamma})$ *is embraced by the delta-semitermoid* $\mathrm{Vals}^{\delta}_{X}(\tau[\![d]\!]_{\delta})$*.*

Proof. By induction on τ.

If τ is a name, i.e. $\tau = \ulcorner\xi\urcorner$ for some $\xi \in X$, then $\mathrm{Vals}^{\gamma}_{X}(\tau[\![g]\!]_{\gamma}) = \mathrm{Vals}^{\gamma}_{X}(\ulcorner\xi\urcorner[\![g]\!]_{\gamma}) = \mathrm{Vals}^{\gamma}_{X}(\ulcorner g\xi\urcorner) = g\xi$, which is embraced by $d\xi = \mathrm{Vals}^{\delta}_{X}(\ulcorner d\xi\urcorner) = \mathrm{Vals}^{\delta}_{X}(\ulcorner\xi\urcorner[\![d]\!]_{\delta}) = \mathrm{Vals}^{\delta}_{X}(\tau[\![d]\!]_{\delta})$.

If $\tau = \triangle_{\kappa}$, then $\mathrm{Vals}^{\gamma}_{X}(\tau[\![g]\!]_{\gamma}) = \triangle_{\kappa}$ is embraced by $\mathrm{Vals}^{\delta}_{X}(\tau[\![d]\!]_{\delta}) = \triangle_{\kappa}$.

If $\tau = \mathtt{f}(\tau_1,\ldots,\tau_n)$, then $\mathrm{Vals}^{\gamma}_{X}(\tau[\![g]\!]_{\gamma}) = \mathrm{Vals}^{\gamma}_{X}((\mathtt{f}(\tau_1,\ldots,\tau_n))[\![g]\!]_{\gamma}) = \mathrm{Vals}^{\gamma}_{X}(\mathtt{f}(\tau_1[\![g]\!]_{\gamma},\ldots,\tau_n[\![g]\!]_{\gamma})) = \mathtt{f}(\mathrm{Vals}^{\gamma}_{X}(\tau_1[\![g]\!]_{\gamma}),\ldots,\mathrm{Vals}^{\gamma}_{X}(\tau_n[\![g]\!]_{\gamma}))$. By induction hypothesis, $\mathrm{Vals}^{\gamma}_{X}(\tau_i[\![g]\!]_{\gamma})$ is embraced by $\mathrm{Vals}^{\delta}_{X}(\tau_i[\![d]\!]_{\delta})$ for any $i \in \{1,\ldots,n\}$, so (E4) implies that the above gamma-semitermoid is embraced by $\mathtt{f}(\mathrm{Vals}^{\delta}_{X}(\tau_1[\![d]\!]_{\delta}),\ldots,\mathrm{Vals}^{\delta}_{X}(\tau_n[\![d]\!]_{\delta})) = \mathrm{Vals}^{\delta}_{X}(\mathtt{f}(\tau_1[\![d]\!]_{\delta},\ldots,\tau_n[\![d]\!]_{\delta})) = \mathrm{Vals}^{\delta}_{X}((\mathtt{f}(\tau_1,\ldots,\tau_n))[\![d]\!]_{\delta}) = \mathrm{Vals}^{\delta}_{X}(\tau[\![d]\!]_{\delta})$.

If $\tau = \mathtt{f}_i^{-1}(\tau')$, then $\mathrm{Vals}^{\gamma}_{X}(\tau[\![g]\!]_{\gamma}) = \mathrm{Vals}^{\gamma}_{X}((\mathtt{f}_i^{-1}(\tau'))[\![g]\!]_{\gamma}) =$

$\mathrm{Vals}_X^\gamma(\mathtt{f}_i^{-1}(\tau'[\![g]\!]_\gamma)) = \mathtt{f}_i^{-1}(\mathrm{Vals}_X^\gamma(\tau'[\![g]\!]_\gamma))$. By induction hypothesis, $\mathrm{Vals}_X^\gamma(\tau'[\![g]\!]_\gamma)$ is embraced by $\mathrm{Vals}_X^\delta(\tau'[\![d]\!]_\delta)$, so (E5) implies that the above gamma-semitermoid is embraced by $\mathtt{f}_i^{-1}(\mathrm{Vals}_X^\delta(\tau'[\![d]\!]_\delta)) = \mathrm{Vals}_X^\delta(\mathtt{f}_i^{-1}(\tau'[\![d]\!]_\delta)) = \mathrm{Vals}_X^\delta((\mathtt{f}_i^{-1}(\tau'))[\![d]\!]_\delta) = \mathrm{Vals}_X^\delta(\tau[\![d]\!]_\delta)$. ■

U) **Lemma.** *Let the* `Sort`*-indexed functions* $g : X \to |[\![Y]\!]_\gamma|$ *and* $d : X \to |[\![Y]\!]_\delta|$ *be such that the gamma-termoid* $g\xi$ *is embraced by the delta-termoid* $d\xi$ *for any* ξ *belonging to an algebraic component of* X*. If* σ *is a gamma-termoid over* X *embraced by* τ*, a delta-termoid over* X*, then* $\mathrm{Vals}_X^\gamma(\sigma[\![g]\!]_\gamma)$ *is embraced by* $\mathrm{Vals}_X^\delta(\tau[\![d]\!]_\delta)$*.*

Proof. By induction on definition (E) regarding the "embracement" of σ by τ.

(1) If $\tau = \sigma$ and both are gamma-termoids, then (T) implies that $\mathrm{Vals}_X^\gamma(\sigma[\![g]\!]_\gamma)$ is embraced by $\mathrm{Vals}_X^\delta(\tau[\![d]\!]_\delta)$.

(2) If σ is embraced by $\ulcorner 0 \urcorner + \tau$ because σ is embraced by τ, then by induction hypothesis, $\mathrm{Vals}_X^\gamma(\sigma[\![g]\!]_\gamma)$ is embraced by $\mathrm{Vals}_X^\delta(\tau[\![d]\!]_\delta)$, hence (E2) implies that it is also embraced by $\ulcorner 0 \urcorner + \mathrm{Vals}_X^\delta(\tau[\![d]\!]_\delta) = \mathrm{Vals}_X^\delta(\ulcorner 0 \urcorner + \tau[\![d]\!]_\delta) = \mathrm{Vals}_X^\delta((\ulcorner 0 \urcorner + \tau)[\![d]\!]_\delta)$.

(3) If σ is embraced by $\ulcorner n+1 \urcorner + \tau$ because σ is embraced by $\ulcorner n \urcorner + \tau$, then by induction hypothesis, $\mathrm{Vals}_X^\gamma(\sigma[\![g]\!]_\gamma)$ is embraced by $\mathrm{Vals}_X^\delta((\ulcorner n \urcorner + \tau)[\![d]\!]_\delta) = \mathrm{Vals}_X^\delta(\ulcorner n \urcorner + \tau[\![d]\!]_\delta) = \ulcorner n \urcorner + \mathrm{Vals}_X^\delta(\tau[\![d]\!]_\delta)$, hence (E3) implies that it is also embraced by $\ulcorner n+1 \urcorner + \mathrm{Vals}_X^\delta(\tau[\![d]\!]_\delta) = \mathrm{Vals}_X^\delta(\ulcorner n+1 \urcorner + \tau[\![d]\!]_\delta) = \mathrm{Vals}_X^\delta((\ulcorner n+1 \urcorner + \tau)[\![d]\!]_\delta)$.

(4) If $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ is embraced by $\mathtt{f}(\tau_1, \ldots, \tau_n)$ because σ_i is embraced by τ_i for all $i \in \{1, \ldots, n\}$, then $\mathrm{Vals}_X^\gamma((\mathtt{f}(\sigma_1, \ldots, \sigma_n))[\![g]\!]_\gamma) = \mathrm{Vals}_X^\gamma(\mathtt{f}(\sigma_1[\![g]\!]_\gamma, \ldots, \sigma_n[\![g]\!]_\gamma)) = \mathtt{f}(\mathrm{Vals}_X^\gamma(\sigma_1[\![g]\!]_\gamma), \ldots, \mathrm{Vals}_X^\gamma(\sigma_n[\![g]\!]_\gamma))$. By induction hypothesis, $\mathrm{Vals}_X^\gamma(\sigma_i[\![g]\!]_\gamma)$ is embraced by $\mathrm{Vals}_X^\delta(\tau_i[\![d]\!]_\delta)$ for all $i \in \{1, \ldots, n\}$, hence (E4) implies that the above gamma-termoid is embraced by $\mathtt{f}(\mathrm{Vals}_X^\delta(\tau_1[\![d]\!]_\delta), \ldots, \mathrm{Vals}_X^\delta(\tau_n[\![d]\!]_\delta)) = \mathrm{Vals}_X^\delta(\mathtt{f}(\tau_1[\![d]\!]_\delta, \ldots, \tau_n[\![d]\!]_\delta)) = \mathrm{Vals}_X^\delta((\mathtt{f}(\tau_1, \ldots, \tau_n))[\![d]\!]_\delta)$.

(5) If $\mathtt{f}_i^{-1}(\sigma')$ is embraced by $\mathtt{f}_i^{-1}(\tau')$ because σ' is embraced by τ', then $\mathrm{Vals}_X^\gamma((\mathtt{f}_i^{-1}(\sigma'))[\![g]\!]_\gamma) = \mathrm{Vals}_X^\gamma(\mathtt{f}_i^{-1}(\sigma'[\![g]\!]_\gamma)) = \mathtt{f}_i^{-1}(\mathrm{Vals}_X^\gamma(\sigma'[\![g]\!]_\gamma))$. By induction hypothesis, $\mathrm{Vals}_X^\gamma(\sigma'[\![g]\!]_\gamma)$ is embraced by $\mathrm{Vals}_X^\delta(\tau'[\![d]\!]_\delta)$, hence (E5) implies that the above gamma-termoid is embraced by $\mathtt{f}_i^{-1}(\mathrm{Vals}_X^\delta(\tau'[\![d]\!]_\delta)) = \mathrm{Vals}_X^\delta(\mathtt{f}_i^{-1}(\tau'[\![d]\!]_\delta)) = \mathrm{Vals}_X^\delta((\mathtt{f}_i^{-1}(\tau'))[\![d]\!]_\delta)$.

(6) Given a functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1, \ldots, \kappa_n\rangle, \lambda\rangle$, suppose that $\ulcorner n+1 \urcorner + \tau$ embraces $\mathtt{f}_i^{-1}(\sigma')$ because $\ulcorner n+1 \urcorner + \tau$ embraces σ and $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_n})$ embraces σ'.

By induction hypothesis, $\mathrm{Vals}_X^\gamma(\sigma'[\![g]\!]_\gamma)$ is embraced by $\mathrm{Vals}_X^\delta((\ulcorner n+1 \urcorner + \tau)[\![d]\!]_\delta) = \mathrm{Vals}_X^\delta(\ulcorner n+1 \urcorner + \tau[\![d]\!]_\delta) =$

$\ulcorner n+1\urcorner + \mathrm{Vals}^{\delta}_{X}(\tau[\![d]\!]_\delta)$.

Also by induction hypothesis, $\mathrm{Vals}^{\gamma}_{X}(\sigma'[\![g]\!]_\gamma)$ is embraced by

$$\begin{aligned}&\mathrm{Vals}^{\delta}_{X}((\ulcorner n\urcorner + \mathtt{f}(\triangle_{\kappa_1},\dots,\triangle_{\kappa_{i-1}},\sigma,\triangle_{\kappa_{i+1}},\dots,\triangle_{\kappa_n}))[\![d]\!]_\delta)\\ &\quad=\mathrm{Vals}^{\delta}_{X}(\ulcorner n\urcorner + \mathtt{f}(\triangle_{\kappa_1},\dots,\triangle_{\kappa_{i-1}},\sigma[\![d]\!]_\delta,\triangle_{\kappa_{i+1}},\dots,\triangle_{\kappa_n}))\\ &\quad=\ulcorner n\urcorner + \mathtt{f}(\triangle_{\kappa_1},\dots,\triangle_{\kappa_{i-1}},\mathrm{Vals}^{\delta}_{X}(\sigma[\![d]\!])_\delta,\triangle_{\kappa_{i+1}},\dots,\triangle_{\kappa_n})\end{aligned}$$

These two facts combined with (E6) imply that $\mathtt{f}_i^{-1}(\mathrm{Vals}^{\gamma}_{X}(\sigma'[\![g]\!]_\gamma)) = \mathrm{Vals}^{\gamma}_{X}(\mathtt{f}_i^{-1}(\sigma'[\![g]\!]_\gamma)) = \mathrm{Vals}^{\gamma}_{X}((\mathtt{f}_i^{-1}(\sigma'))[\![g]\!]_\gamma)$ is embraced by $\ulcorner n+1\urcorner + \mathrm{Vals}^{\delta}_{X}(\tau[\![d]\!]_\delta) = \mathrm{Vals}^{\delta}_{X}(\ulcorner n+1\urcorner + \tau[\![d]\!]_\delta) = \mathrm{Vals}^{\delta}_{X}((\ulcorner n+1\urcorner + \tau)[\![d]\!]_\delta)$. ■

V) **Definition.** Given a `Sort`-indexed set X, let $\mathrm{Nam}^{\delta}_{X} : X^\circ \to |[\![X]\!]_\delta|$ be the `Sort`-indexed function, such that for any $\mathtt{x} \in X^\circ$

$$\mathrm{Nam}^{\delta}_{X}(\mathtt{x}) = \mathrm{nam}_X(\mathtt{x})$$

W) **Definition.** The quadruple $\langle [\![.]\!]_\delta, \mathrm{Val}^\delta, \mathrm{Vals}^\delta, \mathrm{Nam}^\delta\rangle$ is the *delta-terminator*.

In order to avoid ambiguities, the termoidal expressions and the formuloids defined by this terminator will be called “delta-termoidal expressions” and “delta-formuloids”. Because of (L), the notion of termoid corresponding to this terminator (see definition 14J) is identical with the notion of delta-termoid, as defined in (J).

We have to prove that the above definition is correct.

Proof. We are going to prove the requirements of definition (14I) one by one.

(1) According to definition (L), $[\![X]\!]_\delta$ is an algebra.

(2) According to definition (M), $[\![f]\!]_\delta$ is a homomorphism from $[\![X]\!]_\delta$ to $[\![Y]\!]_\delta$ for any `Sort`-indexed function $f : X \to Y$.

(3) A trivial induction on the definition of delta-semitermoid (C) can be used to prove that if τ is a delta-semitermoid over X and at the same time a delta-semitermoid over Y, then τ is a delta-semitermoid over $X \cap Y$ as well. From this and definition (J) we obtain that if τ is a delta-termoid over X and at the same time a delta-termoid over Y, then τ is a delta-termoid over $X \cap Y$ as well. Since any delta-termoid over $X \cap Y$ is obviously a delta-termoid over X and a delta-termoid over Y, we obtain that the identity $|[\![X]\!]_\delta| \cap |[\![Y]\!]_\delta| = |[\![X \cap Y]\!]_\delta|$ is true with respect to the algebraic components of the `Sort`-indexed sets. According to definition (12C2), the elements of the logical carrier of $|[\![X]\!]_\delta|$ are exactly the relational formulae over $|[\![X]\!]_\delta|$, the elements of the logical carrier of $|[\![Y]\!]_\delta|$ are exactly the relational formulae over $|[\![Y]\!]_\delta|$ and the elements of the logical carrier of $|[\![X \cap Y]\!]_\delta|$ are exactly the relational formulae over $|[\![X \cap Y]\!]_\delta|$. No formula may contain names of logical sort, hence the mentioned identity is true for the logical carrier as well.

(4) Definition (M) implies that if the values of $f' : X' \to Y'$ and $f'' : X'' \to Y''$ are equal over the objects whose names occur in the delta-termoid τ, then $\tau[\![f']\!]_\delta = \tau[\![f'']\!]_\delta$. Consequently, the homomorphisms $[\![f']\!]_\delta$ and $[\![f'']\!]_\delta \upharpoonright [\![X']\!]_\delta$ map identically over the algebraic carriers, hence according to $(12\mathsf{H}2)$ they map identically over all carriers. On the other hand, definition (M) trivially implies that $\tau[\![f'']\!]_\delta = \tau([\![f'']\!]_\delta \upharpoonright [\![X']\!]_\delta)$ for any $\tau \in |[\![X']\!]|$. Consequently, the homomorphisms $[\![f']\!]_\delta$ and $[\![f'']\!]_\delta$ map identically any element of $|[\![X']\!]_\delta|$.

(5) The definition of delta-semitermoids over X (C) does not refer to the elements of $X_{\mathtt{Log}}$ in any way, so $[\![X]\!]_\delta = [\![X^\circ]\!]_\delta$. Delta-termoids contain no names of logical sort, hence immediately from definition (M) it follows that $[\![f]\!]_\delta = [\![f^\circ]\!]_\delta$.

(6) and **(7)** follow from the definitions.

(8) follows from definition (V).

(9) Given a `Sort`-indexed function $f : X \to Y$, for any $\xi \in X^\circ$ we have $([\![f]\!]_\delta \circ \mathrm{Nam}^\delta_X)\xi = (\mathrm{Nam}^\delta_X\,\xi)[\![f]\!]_\delta = \ulcorner\xi\urcorner[\![f]\!]_\delta = \ulcorner f\xi\urcorner = \ulcorner f^\circ\xi\urcorner = \mathrm{Nam}^\delta_X\,(f^\circ\xi) = (\mathrm{Nam}^\delta_X \circ f^\circ)\xi$.

$$\begin{array}{ccc} X^\circ & \xrightarrow{\mathrm{Nam}^\delta_X} & |[\![X]\!]_\delta| \\ {\scriptstyle f^\circ}\downarrow & & \downarrow{\scriptstyle [\![f]\!]_\delta} \\ Y^\circ & \xrightarrow{\mathrm{Nam}^\delta_Y} & |[\![Y]\!]_\delta| \end{array}$$

(10) According to definition (Q), Val^δ is a homomorphism, hence it also is a quasimorphism.

(11) Definition (E) shows that the set of the embraced by a delta-termoid τ gamma-semitermoids is independent on whether we regard τ as a delta-termoid over $|\mathbf{M}|$, or as a delta-termoid over $|\partial\mathbf{M}|$. On the other hand, from $(14\mathsf{I}11)$ applied about the gamma-terminator it follows that the set of the values of a gamma-termoid is independent on whether we regard it as a gamma-termoid over $|\mathbf{M}|$, or as a gamma-termoid over $|\partial\mathbf{M}|$. This and definition (Q) imply that $\mathrm{Val}^\delta_{\mathbf{M}}\,\tau = \mathrm{Val}^\delta_{\partial\mathbf{M}}\,\tau$ for any delta-termoid τ.

(12) Due to $(14\mathsf{E})$, we only have to prove that the algebraic components of $\mathrm{Val}^\delta_{\mathbf{M}}$ map to non-empty sets. Let τ be an arbitrary delta-termoid over $|\mathbf{M}|$. Let τ' be obtained from τ by removing from τ all subexpressions of the form "$\ulcorner n \urcorner +$" (consult G). According to (H), τ' is embraced by τ. Since τ, being a delta-termoid, does not contain symbols of the form $\mathtt{f}_i^{-1}$ or $\triangle_\kappa$, τ' is a term. Therefore, τ' is associated term of τ'. According to (24|2), the value of the term τ' in $\mathbf{M}$ belongs to the set of the values of the gamma-termoid τ' in $\mathbf{M}$. On the other hand, according to (Q), the set of the values of the embraced by τ gamma-semitermoid τ' is a subset of $\mathrm{Val}^\delta_{\mathbf{M}}\,\tau$. Consequently, $\mathrm{Val}^\delta_{\mathbf{M}}\,\tau$ is a non-empty set.

Due to (14F), we only have to prove that the algebraic components of $\mathrm{Val}^\delta_{[X]}$ map to one-element sets. Let τ be an arbitrary delta-termoid over $|[X]|$ and let τ' be defined as above. Since τ' is the associated term of τ' and the gamma-termoid τ' is embraced by τ, from (I) it follows that τ' is the associated term of any embraced by τ gamma-termoid. Now (24P) implies that the value of the term τ' in $[X]$ is the only element of the set of the values of any embraced by τ gamma-termoid. Consequently, by definition (Q), the value of the term τ' in $[X]$ is the only element of $\mathrm{Val}^\delta_{[X]}\,\tau$.

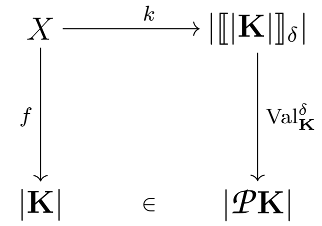
(13) Let $h : \mathbf{M} \to \mathbf{K}$ be a homomorphism. Due to (14G), we only have to prove that $(h^{\mathcal{P}} \circ \mathrm{Val}^\delta_{\mathbf{M}})\tau \subseteq (\mathrm{Val}^\delta_{\mathbf{K}} \circ [\![h]\!]_\delta)\tau$ for any delta-termoid τ over $|\mathbf{M}|$.

Suppose that $\varkappa \in (h^{\mathcal{P}} \circ \mathrm{Val}^\delta_{\mathbf{M}})\tau$. Then there exists some $\mu \in |\mathbf{M}|$, such that $\varkappa = h\mu$ and μ belongs to the set of the values in $\mathbf{M}$ of some embraced by τ gamma-termoid σ. From (14|13) applied about the gamma-terminator, it follows that $h\mu$ belongs to $\mathrm{Val}^\gamma_{\mathbf{K}}(\sigma[\![h]\!]_\gamma)$, i.e. $h\mu$ belongs to the set of the values of $\sigma[\![h]\!]_\gamma$. On the other hand, from (O) it follows that $\sigma[\![h]\!]_\gamma$ is embraced by $\tau[\![h]\!]_\delta$. Consequently, $\varkappa = h\mu$ belongs to $\mathrm{Val}^\delta_{\mathbf{K}}(\tau[\![h]\!]_\delta)$, i.e. $\varkappa \in (\mathrm{Val}^\delta_{\mathbf{K}} \circ [\![h]\!]_\delta)\tau$.

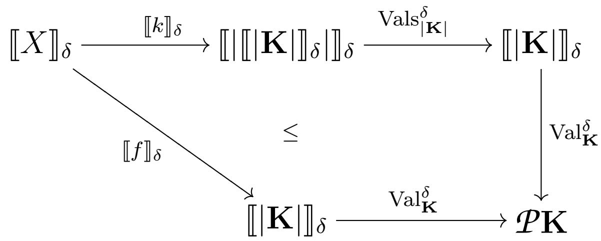
$$\begin{array}{ccc}
[\![|\mathbf{M}|]\!]_\delta & \xrightarrow{\mathrm{Val}^\delta_{\mathbf{M}}} & \mathcal{P}\mathbf{M} \\
{\scriptstyle [\![h]\!]_\delta}\downarrow & \geq & \downarrow{\scriptstyle h^{\mathcal{P}}} \\
[\![|\mathbf{K}|]\!]_\delta & \xrightarrow{\mathrm{Val}^\delta_{\mathbf{K}}} & \mathcal{P}\mathbf{K}
\end{array}$$

(14) According to definition (S), Vals^δ is a homomorphism.

(15) Given a `Sort`-indexed set X, a structure $\mathbf{K}$, a `Sort`-indexed function $k : X \to |[\![|\mathbf{K}|]\!]_\delta|$ and a `Sort`-indexed function $f : X \to |\mathbf{K}|$, suppose that $f \ll \mathrm{Val}^{\delta}_{\mathbf{K}} \circ k$.



Due to (14G), in order to prove that $(\mathrm{Val}^{\delta}_{\mathbf{K}} \circ [\![f]\!]_\delta)\tau \subseteq (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ \mathrm{Vals}^{\delta}_{|\mathbf{K}|} \circ [\![k]\!]_\delta)\tau$ for any $\tau \in |[\![X]\!]_\delta|$, it is enough to consider only the algebraic carriers, i.e. it is enough to consider only the case when τ is a delta-termoid over X.

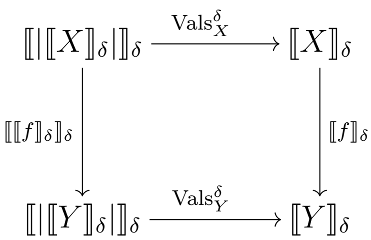


Suppose that $\varkappa \in (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ [\![f]\!]_\delta)\tau = \mathrm{Val}^{\delta}_{\mathbf{K}}(\tau[\![f]\!]_\delta)$. According to (Q), there exists some embraced by $\tau[\![f]\!]_\delta$ gamma-termoid σ', such that $\varkappa \in \mathrm{Val}^{\gamma}_{\mathbf{K}}\,\sigma'$. Since σ' is embraced by $\tau[\![f]\!]_\delta$, (P) implies that there exists an embraced by τ gamma-termoid ρ, such that $\rho[\![f]\!]_\gamma = \sigma'$. Hence $\varkappa \in \mathrm{Val}^{\gamma}_{\mathbf{K}}(\rho[\![f]\!]_\gamma) = (\mathrm{Val}^{\gamma}_{\mathbf{K}} \circ [\![f]\!]_\gamma)\rho$.

Since the `Sort`-indexed function $k : X \to |[\![|\mathbf{K}|]\!]_\delta|$ is such that $f\xi \in (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ k)\xi = \mathrm{Val}^{\delta}_{\mathbf{K}}(k\xi)$ for any $\xi \in X$, definition (Q) implies that there exists a `Sort`-indexed function $k' : X \to |[\![|\mathbf{K}|]\!]_\gamma|$, such that $k'\xi$ is embraced by $k\xi$ for any $\xi \in X$ and $f\xi \in (\mathrm{Val}^{\gamma}_{\mathbf{K}} \circ k')\xi = \mathrm{Val}^{\gamma}_{\mathbf{K}}(k'\xi)$ for any $\xi \in X$. From this and (14I15) applied about the gamma-terminator we obtain that $(\mathrm{Val}^{\gamma}_{\mathbf{K}} \circ [\![f]\!]_\gamma)\rho \subseteq (\mathrm{Val}^{\gamma}_{\mathbf{K}} \circ \mathrm{Vals}^{\gamma}_{|\mathbf{K}|} \circ [\![k']\!]_\gamma)\rho$, so $\varkappa \in \mathrm{Val}^{\gamma}_{\mathbf{K}}(\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}(\rho[\![k']\!]_\gamma))$.

Since $k'\xi$ is embraced by $k\xi$ for any $\xi \in X$ and ρ is embraced by τ, from (U) it follows that $\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}(\rho[\![k']\!]_\gamma)$ is embraced by $\mathrm{Vals}^{\delta}_{|\mathbf{K}|}(\tau[\![k]\!]_\delta)$. But $\varkappa \in \mathrm{Val}^{\gamma}_{\mathbf{K}}(\mathrm{Vals}^{\gamma}_{|\mathbf{K}|}(\rho[\![k']\!]_\gamma))$, so $\varkappa \in \mathrm{Val}^{\delta}_{\mathbf{K}}(\mathrm{Vals}^{\delta}_{|\mathbf{K}|}(\tau[\![k]\!]_\delta)) = (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ \mathrm{Vals}^{\delta}_{|\mathbf{K}|} \circ [\![k]\!]_\delta)\tau$.

(16) and **(17)** follow immediately from the definitions and (12H2).



$$[\![X]\!]_\delta = [\![X^\circ]\!]_\delta \xrightarrow{[\![\mathrm{Nam}^\delta_X]\!]_\delta} [\![|[\![X]\!]_\delta|]\!]_\delta \xrightarrow{\mathrm{Vals}^\delta_X} [\![X]\!]_\delta$$

(18) Considering that $\mathrm{Nam}^\delta_{|\mathbf{M}|}\,\mu = \mathrm{Nam}^\gamma_{|\mathbf{M}|}\,\mu = \mathrm{nam}_{|\mathbf{M}|}\,\mu$, from (14I18) applied for the gamma-terminator we obtain that $\mathrm{Val}^\gamma_{\mathbf{M}}(\mathrm{Nam}^\delta_{|\mathbf{M}|}\,\mu) = \{\mu\}$. On the other hand, from (F) it follows that $\mathrm{Nam}^\delta_{|\mathbf{M}|}\,\mu$ is the only embraced by $\mathrm{Nam}^\delta_{|\mathbf{M}|}\,\mu$ gamma-semitermoid. Therefore, from this and definition (Q) we obtain that $\mathrm{Val}^\delta_{\mathbf{M}}(\mathrm{Nam}^\delta_{|\mathbf{M}|}\,\mu) = \{\mu\}$.

$$|\mathbf{M}|^\circ \xrightarrow{\mathrm{Nam}^\delta_{|\mathbf{M}|}} |[\![|\mathbf{M}|]\!]_\delta| \xrightarrow{\mathrm{Val}^\delta_{\mathbf{M}}} |\mathcal{P}\mathbf{M}|$$

■

X) **Proposition.** *Given a delta-termoid τ over X, if the term σ is obtained from τ by removing all occurrences of substrings of the form "$\ulcorner n \urcorner +$", then $\tau[\![\mathrm{nam}_X]\!]^{[X]}_\delta = \sigma$.*

Proof. Since $\sigma[\mathrm{nam}_X]$ is the result of the removal of all substrings of the form "$\ulcorner n \urcorner +$" from $\tau[\![\mathrm{nam}_X]\!]_\delta$, from (R2) applied about $\tau[\![\mathrm{nam}_X]\!]_\delta$ we can conclude that $\mathrm{Valt}^\delta_{[X]}(\tau[\![\mathrm{nam}_X]\!]_\delta) = \sigma[\mathrm{nam}_X]^{[X]}$, that is $\tau[\![\mathrm{nam}_X]\!]^{[X]}_\delta = \sigma[\mathrm{nam}_X]^{[X]}$. According to (11V1), the last expression is equal to σ. ■

§26. THE EPSILON-TERMINATOR

A) **Definition.** Given a `Sort`-indexed set X, we define by induction the notion *epsilon-regular* delta-termoid over X:

(1) If $\mathtt{y} \in X_\kappa$, then $\ulcorner n \urcorner + \mathrm{nam}_{X,\kappa}\,\mathtt{y}$ is an epsilon-regular delta-termoid for any natural number n.

(2) If $\mathtt{f}$ is a functional symbol of type $\langle\langle\kappa_1,\dots,\kappa_n\rangle,\lambda\rangle$, $k \geq 1$ and $\ulcorner k \urcorner + \tau_1, \dots, \ulcorner k \urcorner + \tau_n$ are epsilon-regular delta-termoids over X of sorts $\kappa_1,\dots,\kappa_n$, respectively, then the string $\ulcorner k-1 \urcorner + \mathtt{f}(\ulcorner k \urcorner + \tau_1, \dots, \ulcorner k \urcorner + \tau_n)$ is an epsilon-regular delta-termoid of sort λ over X.

Obviously, any epsilon-regular delta-termoid is a delta-termoid.

B) **Definition.** If n is a whole number and τ is an epsilon-regular delta-termoid, by $n \oplus \tau$ we will denote the expression which is obtained from τ by replacing each occurrence of $\ulcorner k \urcorner$ in τ (for any natural number k) with $\ulcorner n+k \urcorner$.

C) **Proposition.** *If τ is an epsilon-regular delta-termoid over X, then for any natural number n, $n \oplus \tau$ also is an epsilon-regular delta-termoid over X of the same sort as the sort of τ.*

Proof. By induction on τ. If $\tau = \ulcorner k \urcorner + \ulcorner \xi \urcorner$, where $\xi \in X$, then $n \oplus \tau = \ulcorner k + n \urcorner + \ulcorner \xi \urcorner$ which is an epsilon-regular delta-termoid.

If $\tau = \ulcorner k - 1 \urcorner + \mathtt{f}(\ulcorner k \urcorner + \tau_1, \ldots, \ulcorner k \urcorner + \tau_m)$, where $\ulcorner k \urcorner + \tau_i$ are epsilon-regular delta-termoids for $i \in \{1, \ldots, m\}$, then $n \oplus \tau$ is equal to $\ulcorner n + k - 1 \urcorner + \mathtt{f}(\ulcorner n + k \urcorner + (n \oplus \tau_1), \ldots, \ulcorner n + k \urcorner + (n \oplus \tau_m))$, which is an epsilon-regular delta-termoid, because, on one hand, $\ulcorner n + k \urcorner + (n \oplus \tau_i) = n \oplus (\ulcorner k \urcorner + \tau_i)$ for any i and on the other hand, by induction hypothesis, $n \oplus (\ulcorner k \urcorner + \tau_i)$ are epsilon-regular delta-termoids. ∎

D) **Definition.** For any delta-termoid τ over X, we define by induction the minimal covering τ epsilon-regular delta-termoid, denoted $\mathfrak{c}(\tau)$:

(1) If $\tau = \ulcorner \xi \urcorner$ is a name, then $\mathfrak{c}(\tau) = \ulcorner 0 \urcorner + \tau$.

(2) If $\mathtt{f}$ is a functional symbol of type $\langle\langle \kappa_1, \ldots, \kappa_n \rangle, \lambda \rangle$, $\tau_1, \ldots, \tau_n$ are delta-termoids of sorts $\kappa_1, \ldots, \kappa_n$, respectively, $\mathfrak{c}(\tau_i) = \ulcorner k_1 \urcorner + \sigma_i$ for any $i \in \{1, \ldots, n\}$ and $k = \max\{1, k_1, \ldots, k_n\}$, then $\mathfrak{c}(\mathtt{f}(\tau_1, \ldots, \tau_n))$ is equal to $\ulcorner k - 1 \urcorner + \mathtt{f}((k - k_1) \oplus (\ulcorner k_1 \urcorner + \sigma_1), \ldots, (k - k_n) \oplus (\ulcorner k_n \urcorner + \sigma_n))$.[76]

(3) If $\mathfrak{c}(\tau) = \ulcorner n \urcorner + \sigma$ and $m = \max\{n, k\}$, then $\mathfrak{c}(\ulcorner k \urcorner + \tau) = (m - n) \oplus (\ulcorner n \urcorner + \sigma)$.

A simple induction can be used to prove that for any delta-termoid τ, $\mathfrak{c}(\tau)$ is an epsilon-regular delta-termoid.

E) **Lemma.** (1) *The sort of* $\mathfrak{c}(\tau)$ *is the same as the sort of* τ.

(2) $\mathfrak{c}(\tau) = \tau$ *for any epsilon-regular delta-termoid* τ.

(3) $\mathfrak{c}(\mathfrak{c}(\tau)) = \mathfrak{c}(\tau)$ *for any delta-termoid* τ.

(4) $\mathfrak{c}(\mathtt{f}(\tau_1, \ldots, \tau_n)) = \mathfrak{c}(\mathtt{f}(\mathfrak{c}(\tau_1), \ldots, \mathfrak{c}(\tau_n)))$ *for any n-ary functional symbol* $\mathtt{f}$ *and delta-termoids* $\tau_1, \ldots, \tau_n$ *of suitable sorts.*

(5) $\mathfrak{c}(n + \tau) = \mathfrak{c}(n + \mathfrak{c}(\tau))$ *for any natural number n and delta-termoid* τ.

Proof. **(1)** follows immediately from the definition.

(2) By induction on τ (according to A).

If $\tau = \ulcorner n \urcorner + \ulcorner \xi \urcorner$, then $\mathfrak{c}(\tau) = \mathfrak{c}(\ulcorner n \urcorner + \ulcorner \xi \urcorner) = \ulcorner n \urcorner + \ulcorner \xi \urcorner$ because $\mathfrak{c}(\ulcorner \xi \urcorner) = \ulcorner 0 \urcorner + \ulcorner \xi \urcorner$.

Let $\tau = \ulcorner n \urcorner + \mathtt{f}(\ulcorner n + 1 \urcorner + \tau_1, \ldots, \ulcorner n + 1 \urcorner + \tau_m)$, where $m \neq 0$ and $\ulcorner n + 1 \urcorner + \tau_i$ is an epsilon-regular delta-termoid for any i. By induction hypothesis, $\mathfrak{c}(\ulcorner n + 1 \urcorner + \tau_i) = \ulcorner n + 1 \urcorner + \tau_i$, so

$$\begin{aligned}
&\mathfrak{c}(\mathtt{f}(\ulcorner n + 1 \urcorner + \tau_1, \ldots, \ulcorner n + 1 \urcorner + \tau_m)) = \\
&\quad = \ulcorner n \urcorner + \mathtt{f}(0 \oplus (\ulcorner n + 1 \urcorner + \tau_1), \ldots, 0 \oplus (\ulcorner n + 1 \urcorner + \tau_m)) \\
&\quad = \ulcorner n \urcorner + \mathtt{f}(\ulcorner n + 1 \urcorner + \tau_1, \ldots, \ulcorner n + 1 \urcorner + \tau_m)
\end{aligned}$$

[76] Notice that this definition is not ambiguous when $n = 0$.

hence

$$\begin{aligned}\mathfrak{c}(\tau) &= \mathfrak{c}(\ulcorner n\urcorner + \mathtt{f}(\ulcorner n+1\urcorner + \tau_1, \dots, \ulcorner n+1\urcorner + \tau_m)) \\ &= 0 \oplus (\ulcorner n\urcorner + \mathtt{f}(\ulcorner n+1\urcorner + \tau_1, \dots, \ulcorner n+1\urcorner + \tau_m)) \\ &= \ulcorner n\urcorner + \mathtt{f}(\ulcorner n+1\urcorner + \tau_1, \dots, \ulcorner n+1\urcorner + \tau_m) \\ &= \tau\end{aligned}$$

Finally, if $\tau = \ulcorner n\urcorner + \mathtt{f}()$ where $\mathtt{f}$ is a nullary functional symbol, then $\mathfrak{c}(\tau) = \mathfrak{c}(\ulcorner n\urcorner + \mathtt{f}()) = \ulcorner n\urcorner + \mathtt{f}() = \tau$, because $\mathfrak{c}(\mathtt{f}()) = \ulcorner 0\urcorner + \mathtt{f}()$.

(3) follows from (2) because $\mathfrak{c}(\tau)$ is epsilon-regular.

(4) Let $\mathfrak{c}(\tau_i) = \ulcorner k_i\urcorner + \sigma_i$ for all i and $k = \max\{1, k_1, \dots, k_n\}$. Then according to definition (D2), $\mathfrak{c}(\mathtt{f}(\tau_1, \dots, \tau_n))$ is equal to $\ulcorner k-1\urcorner + \mathtt{f}((k-k_1) \oplus (\ulcorner k_1\urcorner + \sigma_i), \dots, (k-k_1) \oplus (\ulcorner k_1\urcorner + \sigma_i))$. On the other hand, (3) implies that $\mathfrak{c}(\mathfrak{c}(\tau_i)) = \mathfrak{c}(\tau_i) = \ulcorner k_i\urcorner + \sigma_i$, hence according to definition (D2), $\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\tau_1), \dots, \mathfrak{c}(\tau_n)))$ is equal to the same expression.

(5) Let $\mathfrak{c}(\tau) = \ulcorner k\urcorner + \sigma$. Then according to (D3), $\mathfrak{c}(\ulcorner n\urcorner + \tau) = \max\{k, n\} \oplus (\ulcorner k\urcorner + \sigma)$. On the other hand, (3) implies that $\mathfrak{c}(\mathfrak{c}(\tau)) = \mathfrak{c}(\tau) = \ulcorner k\urcorner + \sigma$, so (D3) implies that $\mathfrak{c}(\ulcorner n\urcorner + \mathfrak{c}(\tau))$ is equal to the same thing. ∎

F) **Lemma.** *If the gamma-termoid σ is embraced by the delta-termoid τ, then σ is embraced by $\mathfrak{c}(\tau)$.*

Proof. Given a delta-semitermoid τ, we are going to say that the delta-semitermoid τ' is a *lifting* of τ, if τ' can be obtained from τ by applying modifications of the following two kinds:

1. Replace some occurrences of symbols of the form $\ulcorner k\urcorner$ in τ with symbols $\ulcorner k'\urcorner$, such that $k' > k$.
2. Insert subexpressions of the form "$\ulcorner k\urcorner +$" in arbitrary places in τ.

Given a delta-semitermoid τ, a *trivial lifting* of τ is a delta-semitermoid of the form $\ulcorner k_1\urcorner + (\ulcorner k_2\urcorner + (\dots + (\ulcorner k_n\urcorner + \tau)))$.

Notice that (25E2) and (25E3) imply that for any σ and τ if σ is embraced by τ, then σ is embraced by any trivial lifting of τ.

By induction on definition (25E) we are going to prove that if the gamma-semitermoid σ is embraced by the delta-semitermoid τ, then σ is embraced by any lifting τ' of τ. Since $\mathfrak{c}(\tau)$ is a lifting of τ, this will prove the lemma.

(1) If $\sigma = \tau$ and both are gamma-semitermoids, then τ contains no symbols of the form $\ulcorner k\urcorner$, so τ' is obtained from τ through inserting of subexpressions of the form $\ulcorner k\urcorner +$, hence (25H) implies that $\sigma = \tau$ is embraced by τ'.

(2) If σ is embraced by $\ulcorner 0 \urcorner + \tau$ because σ is embraced by τ, then any lifting of $\ulcorner 0 \urcorner + \tau$ is a trivial lifting of a delta-semitermoid of the form $\ulcorner k \urcorner + \tau'$, where τ' is a lifting of τ. By induction hypothesis, σ is embraced by τ', hence it is embraced by any trivial lifting of τ', and in particular by any trivial lifting of $\ulcorner k \urcorner + \tau'$.

(3) If σ is embraced by $\ulcorner n+1 \urcorner + \tau$ because σ is embraced by $\ulcorner n \urcorner + \tau$, then any lifting of $\ulcorner n+1 \urcorner + \tau$ is a trivial lifting of a delta-semitermoid of the form $\ulcorner k+1 \urcorner + \tau'$, where $k \geq n$ and τ' is a lifting of τ. Therefore, $\ulcorner k \urcorner + \tau'$ is a lifting of $\ulcorner n \urcorner + \tau$, so by induction hypothesis, σ is embraced by $\ulcorner k \urcorner + \tau'$, hence it is embraced by $\ulcorner k+1 \urcorner + \tau'$ and by any trivial lifting of $\ulcorner k+1 \urcorner + \tau'$.

(4) If $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ is embraced by $\mathtt{f}(\tau_1, \ldots, \tau_n)$ because σ_i is embraced by τ_i for any $i \in \{1, \ldots, n\}$, then any lifting of $\mathtt{f}(\tau_1, \ldots, \tau_n)$ is a trivial lifting of delta-semitermoid of the form $\mathtt{f}(\tau'_1, \ldots, \tau'_n)$, where τ'_i is a lifting of τ_i for any $i \in \{1, \ldots, n\}$. By induction hypothesis, σ_i is embraced by τ'_i for any $i \in \{1, \ldots, n\}$, so $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ is embraced by $\mathtt{f}(\tau'_1, \ldots, \tau'_n)$ and any trivial lifting of $\mathtt{f}(\tau'_1, \ldots, \tau'_n)$.

(5) If $\mathtt{f}_i^{-1}(\sigma)$ is embraced by $\mathtt{f}_i^{-1}(\tau)$ because σ is embraced by τ, then any lifting of $\mathtt{f}_i^{-1}(\tau)$ is a trivial lifting of a delta-semitermoid of the form $\mathtt{f}_i^{-1}(\tau')$, where τ' is a lifting of τ. By induction hypothesis, σ is embraced by τ', hence $\mathtt{f}_i^{-1}(\sigma')$ is embraced by $\mathtt{f}_i^{-1}(\tau')$ and by any trivial lifting of $\mathtt{f}_i^{-1}(\tau')$.

(6) Given a functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1, \ldots, \kappa_l\rangle, \lambda\rangle$, suppose that $\ulcorner n+1 \urcorner + \tau$ embraces $\mathtt{f}_i^{-1}(\sigma)$ because $\ulcorner n+1 \urcorner + \tau$ embraces ρ and $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \rho, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_l})$ embraces σ.

Any lifting of $\ulcorner n+1 \urcorner + \tau$ is a trivial lifting of a delta-semitermoid of the form $\ulcorner m+1 \urcorner + \tau'$, where $m \geq n$ and τ' is a lifting of τ. Since $\ulcorner m+1 \urcorner + \tau'$ is a lifting of $\ulcorner n+1 \urcorner + \tau$, by induction hypothesis, $\ulcorner m+1 \urcorner + \tau'$ embraces ρ.

Since $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \rho, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_l})$ embraces σ and $m \geq n$, from (25E3) we obtain that $\ulcorner m \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \rho, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_l})$ embraces σ. Now (25E6) implies that $\mathtt{f}_i^{-1}(\sigma)$ is embraced by $\ulcorner m+1 \urcorner + \tau'$ and by any trivial lifting of $\ulcorner m+1 \urcorner + \tau'$. ∎

G) **Lemma.** *Given a structure* $\mathbf{M}$*, for any delta-termoid* τ *over* $|\mathbf{M}|$*,* $\mathrm{Val}^{\delta}_{\mathbf{M}}\,\tau \subseteq \mathrm{Val}^{\delta}_{\mathbf{M}}(\mathfrak{c}(\tau))$.

Proof. From (F) it follows that every embraced by τ gamma-termoid is embraced by $\mathfrak{c}(\tau)$ too. ∎

H) **Lemma.** *Given a structure* $\mathbf{M}$ *and* μ *belonging to an algebraic carrier of* $\mathbf{M}$*,* $\mathrm{Val}^{\delta}_{\mathbf{M}}\ulcorner \mu \urcorner = \mathrm{Val}^{\delta}_{\mathbf{M}}(\mathfrak{c}(\ulcorner \mu \urcorner))$.

Proof. A quick inspection of definition (25E) reveals that (25E2) is the only item of this definition capable of deducing that a delta-semitermoid of the form $\ulcorner 0 \urcorner + \ulcorner \xi \urcorner$ embraces a gamma-semitermoid. If we apply (25E2) to $\ulcorner 0 \urcorner + \ulcorner \mu \urcorner$ we will obtain that a gamma-semitermoid is embraced by $\ulcorner 0 \urcorner + \ulcorner \mu \urcorner$ if and only if it is embraced by $\ulcorner \mu \urcorner$. From this result and definition (25Q) we obtain that $\mathrm{Val}^{\delta}_{\mathbf{M}}(\ulcorner 0 \urcorner + \ulcorner \mu \urcorner) = \mathrm{Val}^{\delta}_{\mathbf{M}} \ulcorner \mu \urcorner$. According to definition (D1), $\mathrm{Val}^{\delta}_{\mathbf{M}}(\mathfrak{c}(\ulcorner \mu \urcorner)) = \mathrm{Val}^{\delta}_{\mathbf{M}}(\ulcorner 0 \urcorner + \ulcorner \mu \urcorner)$, so we obtain the required. ∎

I) **Definition.** (1) Two delta-termoids τ and σ are $\mathfrak{c}$-*equivalent* if $\mathfrak{c}(\tau) = \mathfrak{c}(\sigma)$.

(2) Two epsilon-regular delta-termoids τ and σ are *similar* if there exists a natural number n, such that $\tau = n \oplus \sigma$ or $\sigma = n \oplus \tau$.

(3) If an epsilon-regular delta-termoid starts with "$\ulcorner n \urcorner +$", the natural number n is its *radius*.

J) **Lemma.** (1) *The relation "$\mathfrak{c}$-equivalent" is reflexive, symmetric and transitive.*

(2) *Two epsilon-regular delta-termoids τ and σ are similar if and only if there exists a whole number n, such that $\tau = n \oplus \sigma$.*

(3) *The relation "similar" is reflexive, symmetric and transitive.*

(4) *Any epsilon-regular delta-termoid has a radius.*

(5) *Given a functional symbol $\mathtt{f}$ of type $\langle\langle \kappa_1, \dots, \kappa_m \rangle, \lambda\rangle$, delta-termoids $\tau_1, \dots, \tau_m$ of respective sorts $\kappa_1, \dots, \kappa_m$ and delta-termoids $\sigma_1, \dots, \sigma_m$ of the same sorts, if τ_i is $\mathfrak{c}$-equivalent with σ_i for any i, then $\mathtt{f}(\tau_1, \dots, \tau_m)$ is $\mathfrak{c}$-equivalent with $\mathtt{f}(\sigma_1, \dots, \sigma_m)$.*

(6) *If τ is $\mathfrak{c}$-equivalent with σ, then $\ulcorner k \urcorner + \tau$ is $\mathfrak{c}$-equivalent with $\ulcorner k \urcorner + \sigma$ for any natural number k.*

(7) *τ and $\ulcorner 0 \urcorner + \tau$ are $\mathfrak{c}$-equivalent.*

(8) *Two similar epsilon-regular delta-termoids with equal radii are equal.*

(9) *Given a functional symbol $\mathtt{f}$ of type $\langle\langle \kappa_1, \dots, \kappa_m \rangle, \lambda\rangle$, epsilon-regular delta-termoids $\tau_1, \dots, \tau_m$ of respective sorts $\kappa_1, \dots, \kappa_m$ and epsilon-regular delta-termoids $\sigma_1, \dots, \sigma_m$ of the same sorts, if the termoids $\tau_1, \dots, \tau_m$ are similar to $\sigma_1, \dots, \sigma_m$, respectively, then $\mathfrak{c}(\mathtt{f}(\tau_1, \dots, \tau_m))$ is similar to $\mathfrak{c}(\mathtt{f}(\sigma_1, \dots, \sigma_m))$.*

(10) *$\mathfrak{c}(\ulcorner n \urcorner + \tau)$ is similar to $\mathfrak{c}(\tau)$.*

(11) *$\mathfrak{c}(n \oplus \tau)$ is similar to $\mathfrak{c}(\tau)$.*

(12) *Given a $\mathtt{Sort}$-indexed set X and two epsilon-regular delta-termoids τ and σ over $|[\![X]\!]_\delta|$, if τ is similar to σ, then $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\, \tau)$ is similar to $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\, \sigma)$.*

(13) *Given a functional symbol $\mathtt{f}$ and delta-termoids $\tau_1, \dots, \tau_n$ of suitable sorts, if $k_1, \dots, k_n$ are the respective radii of $\mathfrak{c}(\tau_1), \dots, \mathfrak{c}(\tau_n)$, then the*

radius of $\mathfrak{c}(\mathtt{f}(\tau_1,\dots,\tau_n))$ *is* $\max\{1,k_1,\dots,k_n\}-1$.

(14) *Given a delta-termoid* τ*, if* k *is the radius of* $\mathfrak{c}(\tau)$*, then the radius of* $\mathfrak{c}(\ulcorner n\urcorner+\tau)$ *is* $\max\{k,n\}$.

(15) *If* τ *is an epsilon-regular delta-termoid with radius* n*, then the radius of* $k\oplus\tau$ *is* $k+n$.

(16) *Given a* Sort*-indexed set* X *and delta-termoids* τ *and* σ *over* $|[\![X]\!]_\delta|$*, if* τ *is* $\mathfrak{c}$*-equivalent with* σ*, then* $\mathfrak{c}(\mathrm{Vals}^\delta_X\,\tau)$ *is similar to* $\mathfrak{c}(\mathrm{Vals}^\delta_X\,\sigma)$.

(17) *Given a natural number* n*, a* Sort*-indexed set* X *and a delta-termoid* τ *over* $|[\![X]\!]_\delta|$*, if* k' *is the radius of* $\mathfrak{c}(\mathrm{Vals}^\delta_X\,\tau)$ *and* k'' *is the radius of* $\mathfrak{c}(n\oplus\tau)$*, then the radius of* $\mathfrak{c}(\mathrm{Vals}^\delta_X(n\oplus\tau))$ *is* $\max\{k',k''\}$.

(18) *The radius of* $\mathfrak{c}(\mathrm{Vals}^\delta_X\,\tau)$ *is greater than or equal to the radius of* $\mathfrak{c}(\tau)$ *for any delta-termoid* τ.

(19) *Given a* Sort*-indexed set* X *and delta-termoids* τ *and* σ *over* $|[\![X]\!]_\delta|$*, if* τ *is* $\mathfrak{c}$*-equivalent with* σ*, then* $\mathrm{Vals}^\delta_X\,\tau$ *is* $\mathfrak{c}$*-equivalent with* $\mathrm{Vals}^\delta_X\,\sigma$.

(20) *Given a* Sort*-indexed set* X *and a delta-termoid* τ *over* $|[\![X]\!]_\delta|$*,* $\mathfrak{c}(\mathrm{Vals}^\delta_X\,\tau)=\mathfrak{c}(\mathrm{Vals}^\delta_X\,\mathfrak{c}(\tau))$.

(21) *Given a* Sort*-indexed set* X *and a delta-termoid* τ *over* $|[\![X]\!]_\delta|$*,* $\mathfrak{c}(\mathrm{Vals}^\delta_X\,\tau)=\mathfrak{c}(\mathrm{Vals}^\delta_X(\tau[\![\mathfrak{c}]\!]_\delta))$*. Equivalently,* $(\mathfrak{c}\circ\mathrm{Vals}^\delta_X)\tau=(\mathfrak{c}\circ\mathrm{Vals}^\delta_X\circ[\![\mathfrak{c}]\!]_\delta)\tau$.

Proof. **(1)** is obvious, **(2)** is obvious and **(3)** obviously follows from (2).

(4) Definition (A) clearly shows that all epsilon-regular delta-termoids have the form "$\ulcorner n\urcorner+\tau$" for some n and τ. Therefore, any epsilon-regular delta-termoid has a radius.

(5) $\mathfrak{c}(\mathtt{f}(\tau_1,\dots,\tau_n))$ is equal to $\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\tau_1),\dots,\mathfrak{c}(\tau_n)))$ because of (E4), which is equal to $\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\sigma_1),\dots,\mathfrak{c}(\sigma_n)))$ because τ_i and σ_i are $\mathfrak{c}$-equivalent, which is equal to $\mathfrak{c}(\mathtt{f}(\sigma_1,\dots,\sigma_n))$ because of (E4).

(6) Let n be the radius of $\mathfrak{c}(\tau)=\mathfrak{c}(\sigma)$. Then according to definition (D3), $\mathfrak{c}(\ulcorner k\urcorner+\tau)=(\max\{n,k\}-k)\oplus\mathfrak{c}(\tau)=(\max\{n,k\}-k)\oplus\mathfrak{c}(\sigma)=\mathfrak{c}(\ulcorner k\urcorner+\sigma)$.

(7) Let n be the radius of $\mathfrak{c}(\tau)$. Then according to definition (D3), $\mathfrak{c}(\ulcorner 0\urcorner+\tau)=(\max\{0,n\}-n)\oplus\mathfrak{c}(\tau)=0\oplus\mathfrak{c}(\tau)=\mathfrak{c}(\tau)$.

(8) Suppose that $\ulcorner k'\urcorner+\tau'$ and $\ulcorner k''\urcorner+\tau''$ are similar and have equal radii. Because of the similarity, there exists a whole number n, such that $\ulcorner k'\urcorner+\tau'=n\oplus(\ulcorner k''\urcorner+\tau'')$, so $k'=k''+n$. But the radii of these termoids are equal, so $k'=k''$, hence $n=0$, so $\ulcorner k'\urcorner+\tau'=0\oplus(\ulcorner k''\urcorner+\tau'')=\ulcorner k''\urcorner+\tau''$.

(9) Because of the similarity of $\tau_1,\dots,\tau_m$ and $\sigma_1,\dots,\sigma_m$ there exist whole numbers $n_1,\dots,n_m$, such that $\tau_i=n_i\oplus\sigma_i$ for any $i\in\{1,\dots,m\}$. Let $k_1,\dots,k_m$ be the respective raidii of $\sigma_1,\dots,\sigma_m$. Then $k_1+n_1,\dots,k_m+n_m$ will be the respective radii of $\tau_1,\dots,\tau_m$. Let $l=$

$\max\{k_1+n_1,\dots,k_m+n_m\}$ and $j=\max\{k_1,\dots,k_m\}$. Then

$$\begin{aligned}
&\mathfrak{c}(\mathtt{f}(\tau_1,\dots,\tau_m)) = \\
&\quad = \ulcorner l-1\urcorner + \mathtt{f}((l-(k_1+n_1))\oplus\tau_1,\dots,(l-(k_m+n_m))\oplus\tau_m) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{from (D2) and (E2)} \\
&\quad = \ulcorner l-1\urcorner + \mathtt{f}((l-k_1-n_1)\oplus n_1\oplus\sigma_1,\dots,(l-k_m-n_m)\oplus n_m\oplus\sigma_m) \\
&\quad = \ulcorner l-1\urcorner + \mathtt{f}((l-k_1)\oplus\sigma_1,\dots,(l-k_m)\oplus\sigma_m) \\
&\quad = (l-j)\oplus(\ulcorner j-1\urcorner + \mathtt{f}((j-k_1)\oplus\sigma_1,\dots,(j-k_m)\oplus\sigma_m)) \\
&\quad = (l-j)\oplus\mathfrak{c}(\mathtt{f}(\sigma_1,\dots,\sigma_m)) \qquad\qquad \text{from (D2) and (E2)}
\end{aligned}$$

(10) Let k be the radius of $\mathfrak{c}(\tau)$. Then according to (D3), $\mathfrak{c}(\ulcorner n\urcorner+\tau) = (\max\{k,n\}-k)\oplus\mathfrak{c}(\tau)$.

(11) By induction on τ. If $\tau=\ulcorner\xi\urcorner$ is a name, then $\mathfrak{c}(n\oplus\tau)=\mathfrak{c}(n\oplus\ulcorner\xi\urcorner) = \mathfrak{c}(\ulcorner\xi\urcorner)=\mathfrak{c}(\tau)$.

If $\tau=\mathtt{f}(\tau_1,\dots,\tau_k)$, then

$$\begin{aligned}
\mathfrak{c}(n\oplus\tau) &= \mathfrak{c}(n\oplus\mathtt{f}(\tau_1,\dots,\tau_k)) \\
&= \mathfrak{c}(\mathtt{f}(n\oplus\tau_1,\dots,n\oplus\tau_k)) \\
&= \mathfrak{c}(\mathtt{f}(\mathfrak{c}(n\oplus\tau_1),\dots,\mathfrak{c}(n\oplus\tau_k))) && \text{from (E4)}
\end{aligned}$$

By induction hypothesis, $\mathfrak{c}(n\oplus\tau_i)$ is similar to $\mathfrak{c}(\tau_i)$ for any $i\in\{1,\dots,k\}$, hence according to (9), the above expression is similar to

$$\begin{aligned}
\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\tau_1),\dots,\mathfrak{c}(\tau_k))) &= \mathfrak{c}(\mathtt{f}(\tau_1,\dots,\tau_k)) && \text{from (E4)} \\
&= \mathfrak{c}(\tau)
\end{aligned}$$

If $\tau=\ulcorner k\urcorner+\sigma$, then

$$\begin{aligned}
\mathfrak{c}(n\oplus\tau) &= \mathfrak{c}(n\oplus(\ulcorner k\urcorner+\sigma)) \\
&= \mathfrak{c}(\ulcorner n+k\urcorner+(n\oplus\sigma))
\end{aligned}$$

According to (10), this is similar to $\mathfrak{c}(n\oplus\sigma)$, which by induction hypothesis is similar to $\mathfrak{c}(\sigma)$, which according to (10) is similar to $\mathfrak{c}(\ulcorner k\urcorner+\sigma)$, i.e. to $\mathfrak{c}(\tau)$.

(12) By induction on the epsilon-regular delta-termoid τ over $|[\![X]\!]_\delta|$ we are going to prove that for any natural number n, $\mathfrak{c}(\mathrm{Vals}^\delta_X\,\tau)$ is similar to $\mathfrak{c}(\mathrm{Vals}^\delta_X(n\oplus\tau))$.

If $\tau=\ulcorner k\urcorner+\ulcorner\sigma\urcorner$, where σ is a delta-termoid over X (i.e. an element an algebraic carrier of $[\![X]\!]_\delta$), then $\mathfrak{c}(\mathrm{Vals}^\delta_X(n\oplus\tau)) = \mathfrak{c}(\mathrm{Vals}^\delta_X(\ulcorner n+k\urcorner+\ulcorner\sigma\urcorner)) = \mathfrak{c}(\ulcorner n+k\urcorner+\sigma)$. Because of (10), this is similar to $\mathfrak{c}(\sigma)$, which, on its part, is similar to $\mathfrak{c}(\ulcorner k\urcorner+\sigma)=\mathfrak{c}(\mathrm{Vals}^\delta_X\,\tau)$.

Let $\tau = \ulcorner k \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m)$. Then

$$\begin{aligned}
&\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau)) = \\
&\qquad = \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus (\ulcorner k \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m)))) \\
&\qquad = \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\ulcorner n + k \urcorner + \mathtt{f}(n \oplus \tau_1, \ldots, n \oplus \tau_m))) \\
&\qquad = \mathfrak{c}(\ulcorner n + k \urcorner + \mathtt{f}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_1), \ldots, \mathrm{Vals}^{\delta}_{X}(n \oplus \tau_m))) \\
&\qquad = \mathfrak{c}(\ulcorner n + k \urcorner + \mathfrak{c}(\mathtt{f}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_1), \ldots, \mathrm{Vals}^{\delta}_{X}(n \oplus \tau_m)))) && \text{from (E5)} \\
&\qquad = \mathfrak{c}(\ulcorner n + k \urcorner + \mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_1)), \ldots, \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_m))))) && \text{from (E4)}
\end{aligned}$$

By induction hypothesis, $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_i))$ is similar to $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\, \tau_i)$ for $i \in \{1, \ldots, m\}$, hence (9) implies that $\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_1)), \ldots, \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_m))))$ is similar to $\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\, \tau_1), \ldots, \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\, \tau_m)))$, hence (10), (3) and (E2) imply that the last expression in the above sequence of equalities is similar to

$$\begin{aligned}
&\mathfrak{c}(\ulcorner k \urcorner + \mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\, \tau_1), \ldots, \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\, \tau_m)))) = \\
&\qquad = \mathfrak{c}(\ulcorner k \urcorner + \mathfrak{c}(\mathtt{f}(\mathrm{Vals}^{\delta}_{X}\, \tau_1, \ldots, \mathrm{Vals}^{\delta}_{X}\, \tau_m))) && \text{from (E4)} \\
&\qquad = \mathfrak{c}(\ulcorner k \urcorner + \mathtt{f}(\mathrm{Vals}^{\delta}_{X}\, \tau_1, \ldots, \mathrm{Vals}^{\delta}_{X}\, \tau_m)) && \text{from (E5)} \\
&\qquad = \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\ulcorner k \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m))) \\
&\qquad = \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\tau))
\end{aligned}$$

(13) and **(14)** follow immediately from the definitions.

(15) Since the radius of τ is n, $\tau = \ulcorner n \urcorner + \sigma$ for some σ. Therefore, $k \oplus \tau = k \oplus (\ulcorner n \urcorner + \sigma) = \ulcorner k + n \urcorner + (k \oplus \sigma)$, so the radius of $k \oplus \tau$ is equal to $k + n$.

(16) By induction on τ we are going to prove that $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\mathfrak{c}(\tau)))$ is similar to $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\, \tau)$.

If $\tau = \ulcorner \sigma \urcorner$ is a name, then

$$\begin{aligned}
\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\mathfrak{c}(\tau))) &= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\mathfrak{c}(\ulcorner \sigma \urcorner))) \\
&= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\ulcorner 0 \urcorner + \ulcorner \sigma \urcorner)) && \text{from (D1)} \\
&= \mathfrak{c}(\ulcorner 0 \urcorner + \sigma) && \text{from (25S)} \\
&= \mathfrak{c}(\sigma) && \text{from (7)} \\
&= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X} \ulcorner \sigma \urcorner) && \text{from (25S)} \\
&= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\, \tau)
\end{aligned}$$

Let $\tau = \mathtt{f}(\tau_1, \ldots, \tau_n)$ and $k_1, \ldots, k_n$ be the respective radii of $\mathfrak{c}(\tau_1), \ldots, \mathfrak{c}(\tau_n)$. Let $k = \max\{1, k_1, \ldots, k_n\}$. Then

$$\begin{aligned}
&\mathfrak{c}(\mathrm{Vals}^\delta_X(\mathfrak{c}(\tau))) = \\
&= \mathfrak{c}(\mathrm{Vals}^\delta_X(\mathfrak{c}(\mathtt{f}(\tau_1, \ldots, \tau_n)))) \\
&= \mathfrak{c}(\mathrm{Vals}^\delta_X(\ulcorner k-1 \urcorner + \mathtt{f}((k-k_1) \oplus \mathfrak{c}(\tau_1), \ldots, (k-k_n) \oplus \mathfrak{c}(\tau_n)))) && \text{from (D2)} \\
&= \mathfrak{c}(\ulcorner k-1 \urcorner + \mathtt{f}(\mathrm{Vals}^\delta_X((k-k_1) \oplus \mathfrak{c}(\tau_1)), \ldots, \mathrm{Vals}^\delta_X((k-k_n) \oplus \mathfrak{c}(\tau_n)))) \\
&= \mathfrak{c}(\ulcorner k-1 \urcorner + \mathfrak{c}(\mathtt{f}(\mathrm{Vals}^\delta_X((k-k_1) \oplus \mathfrak{c}(\tau_1)), \ldots, \mathrm{Vals}^\delta_X((k-k_n) \oplus \mathfrak{c}(\tau_n))))) && \text{from (E5)} \\
&= \mathfrak{c}(\ulcorner k-1 \urcorner + \mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}^\delta_X((k-k_1) \oplus \mathfrak{c}(\tau_1))), \ldots, \mathfrak{c}(\mathrm{Vals}^\delta_X((k-k_n) \oplus \mathfrak{c}(\tau_n)))))) && \text{from (E4)}
\end{aligned}$$

According to (12), $\mathfrak{c}(\mathrm{Vals}^\delta_X((k-k_i) \oplus \mathfrak{c}(\tau_i)))$ is similar to $\mathfrak{c}(\mathrm{Vals}^\delta_X(\mathfrak{c}(\tau_i)))$ for any $i \in \{1, \ldots, n\}$, which, by induction hypothesis, is similar to $\mathfrak{c}(\mathrm{Vals}^\delta_X\, \tau_i)$. Therefore, (10), (E2) and (9) imply that the last expression in the above sequence of equalities is similar to

$$\begin{aligned}
&\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}^\delta_X\, \tau_1), \ldots, \mathfrak{c}(\mathrm{Vals}^\delta_X\, \tau_n))) = \\
&= \mathfrak{c}(\mathtt{f}(\mathrm{Vals}^\delta_X\, \tau_1, \ldots, \mathrm{Vals}^\delta_X\, \tau_n)) && \text{from (E4)} \\
&= \mathfrak{c}(\mathrm{Vals}^\delta_X(\mathtt{f}(\tau_1, \ldots, \tau_n))) \\
&= \mathfrak{c}(\mathrm{Vals}^\delta_X\, \tau)
\end{aligned}$$

Let $\tau = \ulcorner n \urcorner + \sigma$ and k be the radius of $\mathfrak{c}(\sigma)$. Then

$$\begin{aligned}
\mathfrak{c}(\mathrm{Vals}^\delta_X(\mathfrak{c}(\tau))) &= \mathfrak{c}(\mathrm{Vals}^\delta_X(\mathfrak{c}(\ulcorner n \urcorner + \sigma))) \\
&= \mathfrak{c}(\mathrm{Vals}^\delta_X((\max\{n,k\} - k) \oplus \mathfrak{c}(\sigma))) && \text{from (D3)}
\end{aligned}$$

According to (12), this is similar to $\mathfrak{c}(\mathrm{Vals}^\delta_X(\mathfrak{c}(\sigma)))$, which by induction hypothesis, is similar to $\mathfrak{c}(\mathrm{Vals}^\delta_X\, \sigma)$, which according to (10), is similar to $\mathfrak{c}(\ulcorner n \urcorner + \mathrm{Vals}^\delta_X\, \sigma)$, which is equal to $\mathfrak{c}(\mathrm{Vals}^\delta_X(\ulcorner n \urcorner + \sigma))$, i.e. to $\mathfrak{c}(\mathrm{Vals}^\delta_X\, \tau)$.

(17) For any epsilon-regular delta-termoid τ, by $\mathfrak{rad}(\tau)$ we will denote the radius of τ. By induction on τ we are going to prove that for any natural number n, $\mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^\delta_X(n \oplus \tau)))$ is equal to $\max\{\mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^\delta_X\, \tau)), \mathfrak{rad}(\mathfrak{c}(n \oplus \tau))\}$.

If $\tau = \ulcorner \sigma \urcorner$ is a name, then

$$\begin{aligned}
\mathfrak{rad}(\mathfrak{c}(n \oplus \tau)) &= \mathfrak{rad}(\mathfrak{c}(n \oplus \ulcorner \sigma \urcorner)) \\
&= \mathfrak{rad}(\mathfrak{c}(\ulcorner \sigma \urcorner)) \\
&= \mathfrak{rad}(\ulcorner 0 \urcorner + \ulcorner \sigma \urcorner) \\
&= 0
\end{aligned}$$

Consequently,

$$\begin{aligned}\mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau))) &= \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \ulcorner\sigma\urcorner)))\\ &= \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\ulcorner\sigma\urcorner))\\ &= \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau))\\ &= \max\{\mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau)), 0\}\\ &= \max\{\mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau)), \mathfrak{rad}(\mathfrak{c}(n \oplus \tau))\}\end{aligned}$$

Let $\tau = \mathtt{f}(\tau_1, \ldots, \tau_k)$. Then

$$\begin{aligned}\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau)) &= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \mathtt{f}(\tau_1, \ldots, \tau_k)))\\ &= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\mathtt{f}(n \oplus \tau_1, \ldots, n \oplus \tau_k)))\\ &= \mathfrak{c}(\mathtt{f}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_1), \ldots, \mathrm{Vals}^{\delta}_{X}(n \oplus \tau_k)))\\ &= \mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_1)), \ldots, \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_k)))) \quad \text{from } (\mathsf{E4})\end{aligned}$$

By induction hypothesis, for any $i \in \{1, \ldots, k\}$ the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau_i))$ is equal to $\max\{\mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_i)), \mathfrak{rad}(\mathfrak{c}(n \oplus \tau_i))\}$, hence according to (13), the radius of the above expression is equal to

$$\begin{aligned}&\max\{1, \max\{\mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_1)), \mathfrak{rad}(\mathfrak{c}(n \oplus \tau_1))\}, \ldots,\\ &\qquad \max\{\mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_k)), \mathfrak{rad}(\mathfrak{c}(n \oplus \tau_k))\}\} - 1 =\\ &= \max\{1, \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_1)), \ldots, \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_k)),\\ &\qquad \mathfrak{rad}(\mathfrak{c}(n \oplus \tau_1)), \ldots, \mathfrak{rad}(\mathfrak{c}(n \oplus \tau_k))\} - 1\\ &= \max\{\max\{1, \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_1)), \ldots, \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_k))\} - 1,\\ &\qquad \max\{1, \mathfrak{rad}(\mathfrak{c}(n \oplus \tau_1)), \ldots, \mathfrak{rad}(\mathfrak{c}(n \oplus \tau_k))\} - 1\}\end{aligned}$$

Consequently, for the case when $\tau = \mathtt{f}(\tau_1, \ldots, \tau_k)$ it only remains to see that the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\tau))$ is equal to $\max\{1, \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_1)), \ldots, \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_k))\} - 1$ and the radius of $\mathfrak{c}(n \oplus \tau)$ is equal to $\max\{1, \mathfrak{rad}(\mathfrak{c}(n \oplus \tau_1)), \ldots, \mathfrak{rad}(\mathfrak{c}(n \oplus \tau_k))\} - 1$. This follows from (13) and the following two sequences of equalities:

$$\begin{aligned}\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\tau)) &= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\mathtt{f}(\tau_1, \ldots, \tau_k)))\\ &= \mathfrak{c}(\mathtt{f}(\mathrm{Vals}^{\delta}_{X}\,\tau_1, \ldots, \mathrm{Vals}^{\delta}_{X}\,\tau_k))\\ &= \mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_1), \ldots, \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_k))) \qquad \text{from } (\mathsf{E4})\end{aligned}$$

and

$$\begin{aligned}\mathfrak{c}(n \oplus \tau) &= \mathfrak{c}(n \oplus \mathtt{f}(\tau_1, \ldots, \tau_k))\\ &= \mathfrak{c}(\mathtt{f}(n \oplus \tau_1, \ldots, n \oplus \tau_k))\\ &= \mathfrak{c}(\mathtt{f}(\mathfrak{c}(n \oplus \tau_1), \ldots, \mathfrak{c}(n \oplus \tau_k))) \qquad \text{from } (\mathsf{E4})\end{aligned}$$

Let $\tau = \ulcorner k \urcorner + \sigma$. Then

$$\begin{aligned}\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \tau)) &= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus (\ulcorner k \urcorner + \sigma)))\\ &= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\ulcorner n + k \urcorner + (n \oplus \sigma)))\\ &= \mathfrak{c}(\ulcorner n + k \urcorner + \mathrm{Vals}^{\delta}_{X}(n \oplus \sigma))\\ &= \mathfrak{c}(\ulcorner n + k \urcorner + \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \sigma))) \qquad \text{from } (\mathsf{E}5)\end{aligned}$$

By the induction hypothesis, the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(n \oplus \sigma))$ is equal to $\max\{\mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\sigma)), \mathfrak{rad}(\mathfrak{c}(n \oplus \sigma))\}$, hence according to (14), the radius of the above expression is equal to

$$\begin{aligned}&\max\{n + k, \max\{\mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\sigma)), \mathfrak{rad}(\mathfrak{c}(n \oplus \sigma))\}\} =\\ &\quad = \max\{n + k, \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\sigma)), \mathfrak{rad}(\mathfrak{c}(n \oplus \sigma))\}\\ &\quad = \max\{\max\{k, \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\sigma))\}, \max\{n + k, \mathfrak{rad}(\mathfrak{c}(n \oplus \sigma))\}\}\end{aligned}$$

Consequently, for the case when $\tau = \ulcorner k \urcorner + \sigma$ it only remains to see that the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\tau))$ is equal to $\max\{k, \mathfrak{rad}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\sigma))\}$ and the radius of $\mathfrak{c}(n \oplus \tau)$ is equal to $\max\{n + k, \mathfrak{rad}(\mathfrak{c}(n \oplus \sigma))\}$. This follows from (13) and the following two sequences of equalities:

$$\begin{aligned}\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau) &= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\ulcorner k \urcorner + \sigma))\\ &= \mathfrak{c}(\ulcorner k \urcorner + \mathrm{Vals}^{\delta}_{X}\,\sigma)\\ &= \mathfrak{c}(\ulcorner k \urcorner + \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\sigma)) \qquad \text{from } (\mathsf{E}5)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{c}(n \oplus \tau) &= \mathfrak{c}(n \oplus (\ulcorner k \urcorner + \sigma))\\ &= \mathfrak{c}(\ulcorner n + k \urcorner + (n \oplus \sigma))\\ &= \mathfrak{c}(\ulcorner n + k \urcorner + \mathfrak{c}(n \oplus \sigma)) \qquad \text{from } (\mathsf{E}5)\end{aligned}$$

(18) can be proved by simple induction on τ. If $\tau = \ulcorner \sigma \urcorner$ is a name, then $\mathfrak{c}(\tau) = \ulcorner 0 \urcorner + \ulcorner \sigma \urcorner$, hence the radius of $\mathfrak{c}(\tau)$ is 0, so it has to be smaller than or equal to the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau)$.

If $\tau = \mathtt{f}(\tau_1, \dots, \tau_n)$, then $\mathrm{Vals}^{\delta}_{X}\,\tau = \mathtt{f}(\mathrm{Vals}^{\delta}_{X}\,\tau_1, \dots, \mathrm{Vals}^{\delta}_{X}\,\tau_n)$. By induction hypothesis, the radii of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_1), \dots, \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_n)$ are greater than or equal to the radii of $\mathfrak{c}(\tau_1), \dots, \mathfrak{c}(\tau_n)$, respectively. From this and (13) we obtain that the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau)$ is greater than or equal to the radius of $\mathfrak{c}(\tau)$.

If $\tau = \ulcorner n \urcorner + \sigma$, then $\mathrm{Vals}^{\delta}_{X}\,\tau = \ulcorner n \urcorner + \mathrm{Vals}^{\delta}_{X}\,\sigma$. By induction hypothesis, the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\sigma)$ is greater than or equal to the radius of $\mathfrak{c}(\sigma)$. From this and (14) we obtain that the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau)$ is greater than or equal to the radius of $\mathfrak{c}(\tau)$.

(19) Because of (16) and (8) it suffices to prove that the radii of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau)$ and $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\sigma)$ are equal. In order to see that this is true, we are going to prove by induction on τ that for any delta-termoid τ over $|[\![X]\!]_{\delta}|$, the radii of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\mathfrak{c}(\tau))$ and $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau)$ are equal.

If $\tau = \ulcorner\sigma\urcorner$ is a name, then $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\mathfrak{c}(\tau)) = \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\mathfrak{c}(\ulcorner\sigma\urcorner)) = \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\ulcorner 0\urcorner + \ulcorner\sigma\urcorner)) = \mathfrak{c}(\ulcorner 0\urcorner + \sigma)$. Because of (7), this is equal to $\mathfrak{c}(\sigma) = \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\ulcorner\sigma\urcorner) = \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau)$.

Let $\tau = \mathtt{f}(\tau_1, \ldots, \tau_k)$. Let $k_1, \ldots, k_n$ be the respective radii of $\mathfrak{c}(\tau_1), \ldots, \mathfrak{c}(\tau_n)$, $k = \max\{1, k_1, \ldots, k_n\}$, $m_1, \ldots, m_n$ be the respective radii of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_1), \ldots, \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_m)$ and $m = \max\{1, m_1, \ldots, m_n\}$. Then

$$\begin{aligned}
&\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\mathfrak{c}(\tau)) = \\
&= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\mathfrak{c}(\mathtt{f}(\tau_1, \ldots, \tau_n))) \\
&= \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\ulcorner k-1\urcorner + \mathtt{f}((k-k_1)\oplus\mathfrak{c}(\tau_1), \ldots, (k-k_n)\oplus\mathfrak{c}(\tau_n)))) \\
&\hfill \text{from } (\mathsf{D2}) \\
&= \mathfrak{c}(\ulcorner k-1\urcorner + \mathtt{f}(\mathrm{Vals}^{\delta}_{X}((k-k_1)\oplus\mathfrak{c}(\tau_1)), \ldots, \mathrm{Vals}^{\delta}_{X}((k-k_n)\oplus\mathfrak{c}(\tau_n)))) \\
&= \mathfrak{c}(\ulcorner k-1\urcorner + \mathfrak{c}(\mathtt{f}(\mathrm{Vals}^{\delta}_{X}((k-k_1)\oplus\mathfrak{c}(\tau_1)), \ldots, \mathrm{Vals}^{\delta}_{X}((k-k_n)\oplus\mathfrak{c}(\tau_n))))) \\
&\hfill \text{from } (\mathsf{E5}) \\
&= \mathfrak{c}(\ulcorner k-1\urcorner + \mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}((k-k_1)\oplus\mathfrak{c}(\tau_1))), \ldots, \mathfrak{c}(\mathrm{Vals}^{\delta}_{X}((k-k_n)\oplus\mathfrak{c}(\tau_n)))))) \\
&\hfill \text{from } (\mathsf{E4})
\end{aligned}$$

According to (17), for any $i \in \{1, \ldots, n\}$, the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}((k-k_i)\oplus\mathfrak{c}(\tau_i)))$ is equal to the greater number among the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\mathfrak{c}(\tau_i)))$ and the radius of $\mathfrak{c}((k-k_i)\oplus\mathfrak{c}(\tau_i))$. But according to (15), the radius of $\mathfrak{c}((k-k_i)\oplus\mathfrak{c}(\tau_i)) = (k-k_i)\oplus\mathfrak{c}(\tau_i)$ is equal to $(k-k_i)+k_i$, i.e. to k. On the other hand, by induction hypothesis, the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}(\mathfrak{c}(\tau_i)))$ is equal to the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}\,\tau_i)$, i.e. to m_i. Consequently, the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_{X}((k-k_i)\oplus\mathfrak{c}(\tau_i)))$ is equal to $\max\{k, m_i\}$. From this and (13) and (14) we obtain that the radius of the last expression of the above sequence of equalities is equal to

$$\begin{aligned}
&\max\{k-1, \max\{1, \max\{k, m_1\}, \ldots, \max\{k, m_n\}\} - 1\} = \\
&\quad = \max\{k, \max\{1, \max\{k, m_1\}, \ldots, \max\{k, m_n\}\}\} - 1 \\
&\quad = \max\{1, k, m_1, \ldots, m_n\} - 1 \\
&\quad = \max\{1, \max\{1, k_1, \ldots, k_n\}, m_1, \ldots, m_n\} - 1 \\
&\quad = \max\{1, k_1, \ldots, k_n, m_1, \ldots, m_n\} - 1 \\
&\quad = \max\{1, m_1, \ldots, m_n\} - 1 && \text{from (18)} \\
&\quad = m - 1
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \tau) &= \mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \mathtt{f}(\tau_1, \ldots, \tau_n)) \\
&= \mathfrak{c}(\mathtt{f}(\mathrm{Vals}^{\delta}_X\, \tau_1, \ldots, \mathrm{Vals}^{\delta}_X\, \tau_n)) \\
&= \mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \tau_1), \ldots, \mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \tau_n))) && \text{from (E4)}
\end{aligned}$$

According to (13), the radius of the above expression is equal to $\max\{1, m_1, \ldots, m_n\} - 1$, i.e. to $m - 1$.

Now, let $\tau = \ulcorner n \urcorner + \sigma$. Let k be the radius of $\mathfrak{c}(\sigma)$ and m be the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \sigma)$. Then

$$\begin{aligned}
\mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \mathfrak{c}(\tau)) &= \mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \mathfrak{c}(\ulcorner n \urcorner + \sigma)) \\
&= \mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \mathfrak{c}(\ulcorner n \urcorner + \mathfrak{c}(\sigma))) && \text{from (E5)} \\
&= \mathfrak{c}(\mathrm{Vals}^{\delta}_X((\max\{n, k\} - k) \oplus \mathfrak{c}(\sigma))) && \text{from (D3)}
\end{aligned}$$

From (17) it follows that the radius of the above expression is equal to the greater number among the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_X(\mathfrak{c}(\sigma)))$ and the radius of $\mathfrak{c}((\max\{n, k\} - k) \oplus \mathfrak{c}(\sigma))$. By the induction hypothesis, the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_X(\mathfrak{c}(\sigma)))$ is equal to the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \sigma)$, i.e. to m. In addition, from (15) we obtain that the radius of $\mathfrak{c}((\max\{n, k\} - k) \oplus \mathfrak{c}(\sigma)) = (\max\{n, k\} - k) \oplus \mathfrak{c}(\sigma)$ is equal to $\max\{n, k\}$. Consequently, the radius of $\mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \mathfrak{c}(\tau))$ is equal to $\max\{m, n, k\}$, which is equal to $\max\{n, m\}$, because (18) implies that $m \geq k$.

On the other hand,

$$\begin{aligned}
\mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \tau) &= \mathfrak{c}(\mathrm{Vals}^{\delta}_X(\ulcorner n \urcorner + \sigma)) \\
&= \mathfrak{c}(\ulcorner n \urcorner + \mathrm{Vals}^{\delta}_X\, \sigma) \\
&= \mathfrak{c}(\ulcorner n \urcorner + \mathfrak{c}(\mathrm{Vals}^{\delta}_X\, \sigma)) && \text{from (E5)}
\end{aligned}$$

From (14) it follows that the radius of the above expression is equal to $\max\{n, m\}$

(20) follows from (19).

(21) By induction on τ.

If $\tau = \ulcorner\sigma\urcorner$ is a name, then $\mathfrak{c}(\mathrm{Vals}_X^\delta(\tau[\![\mathfrak{c}]\!]_\delta)) = \mathfrak{c}(\mathrm{Vals}_X^\delta(\ulcorner\sigma\urcorner[\![\mathfrak{c}]\!]_\delta)) = \mathfrak{c}(\mathrm{Vals}_X^\delta(\ulcorner\mathfrak{c}(\sigma)\urcorner)) = \mathfrak{c}(\mathfrak{c}(\sigma)) = \mathfrak{c}(\sigma) = \mathfrak{c}(\mathrm{Vals}_X^\delta\ulcorner\sigma\urcorner) = \mathfrak{c}(\mathrm{Vals}_X^\delta\,\tau)$.

If $\tau = \mathtt{f}(\tau_1, \ldots, \tau_n)$, then $\mathfrak{c}(\mathrm{Vals}_X^\delta\,\tau) = \mathfrak{c}(\mathrm{Vals}_X^\delta\,\mathtt{f}(\tau_1, \ldots, \tau_n)) = \mathfrak{c}(\mathtt{f}(\mathrm{Vals}_X^\delta\,\tau_1, \ldots, \mathrm{Vals}_X^\delta\,\tau_n))$. According to (E4), this is equal to $\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}_X^\delta\,\tau_1), \ldots, \mathfrak{c}(\mathrm{Vals}_X^\delta\,\tau_n)))$, which, by the induction hypothesis, is equal to $\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}_X^\delta(\tau_1[\![\mathfrak{c}]\!]_\delta)), \ldots, \mathfrak{c}(\mathrm{Vals}_X^\delta(\tau_n[\![\mathfrak{c}]\!]_\delta))))$. Again, because of (E4), this is equal to $\mathfrak{c}(\mathtt{f}(\mathrm{Vals}_X^\delta(\tau_1[\![\mathfrak{c}]\!]_\delta), \ldots, \mathrm{Vals}_X^\delta(\tau_n[\![\mathfrak{c}]\!]_\delta))) = \mathfrak{c}(\mathrm{Vals}_X^\delta\,\mathtt{f}(\tau_1[\![\mathfrak{c}]\!]_\delta, \ldots, \tau_n[\![\mathfrak{c}]\!]_\delta)) = \mathfrak{c}(\mathrm{Vals}_X^\delta(\mathtt{f}(\tau_1, \ldots, \tau_n)[\![\mathfrak{c}]\!]_\delta))$.

If $\tau = \ulcorner n\urcorner + \sigma$, then $\mathfrak{c}(\mathrm{Vals}_X^\delta\,\tau) = \mathfrak{c}(\mathrm{Vals}_X^\delta(\ulcorner n\urcorner + \sigma)) = \mathfrak{c}(\ulcorner n\urcorner + \mathrm{Vals}_X^\delta\,\sigma)$. Because of (E5), this is equal to $\mathfrak{c}(\ulcorner n\urcorner + \mathfrak{c}(\mathrm{Vals}_X^\delta\,\sigma))$ which, by the induction hypothesis, is equal to $\mathfrak{c}(\ulcorner n\urcorner + \mathfrak{c}(\mathrm{Vals}_X^\delta(\sigma[\![\mathfrak{c}]\!]_\delta)))$. Again, because of (E5), this is equal to $\mathfrak{c}(\ulcorner n\urcorner + \mathrm{Vals}_X^\delta(\sigma[\![\mathfrak{c}]\!]_\delta)) = \mathfrak{c}(\mathrm{Vals}_X^\delta(\ulcorner n\urcorner + (\sigma[\![\mathfrak{c}]\!]_\delta))) = \mathfrak{c}(\mathrm{Vals}_X^\delta((\ulcorner n\urcorner + \sigma)[\![\mathfrak{c}]\!]_\delta))$ ■

K) **Definition.** *Epsilon-termoid* over X of sort κ is an expression of the form $\ulcorner n\urcorner + \tau$, where n is an arbitrary natural number and τ is a term over X of sort κ.

Obviously any epsilon-termoid also is a delta-termoid. However, the intended interpretation of the epsilon-termoids is different. If τ is an epsilon-termoid, then, intuitively, we should thing of it as representing the delta-termoid $\mathfrak{c}(\tau)$.

L) **Notation.** Let $\ulcorner n\urcorner + \tau$ be an epsilon-regular delta-termoid. Let τ' be obtained from τ by removing all subexpressions of the form "$\ulcorner k\urcorner +$". Since τ' is a term, $\ulcorner n\urcorner + \tau'$ is an epsilon-termoid. We are going to denote this epsilon-termoid by $\mathfrak{c}^{-1}(\ulcorner n\urcorner + \tau)$.

M) **Lemma.** (1) *The sort of $\mathfrak{c}^{-1}(\tau)$ is the same as the sort of τ.*

(2) *If τ is a term, then $\mathfrak{c}(\tau)$ has the form $\ulcorner 0\urcorner + \sigma$ for some σ.*

(3) *If $\ulcorner n\urcorner + \tau$ is an epsilon-termoid, then $\mathfrak{c}(\ulcorner n\urcorner + \tau)$ has the form $\ulcorner n\urcorner + \sigma$ for some σ.*

(4) *If $\mathfrak{c}(\tau') = \mathfrak{c}(\tau'')$ for some epsilon-termoids τ' and τ'', then $\tau' = \tau''$.*

(5) *$\mathfrak{c}(\mathfrak{c}^{-1}(\tau)) = \tau$ for any epsilon-regular delta-termoid τ.*

(6) *$\mathfrak{c}(\mathfrak{c}^{-1}(\mathfrak{c}(\tau))) = \mathfrak{c}(\tau)$ for any delta-termoid τ.*

(7) *$\mathfrak{c}^{-1}(\mathfrak{c}(\tau)) = \tau$ for any epsilon-termoid τ.*

(8) *$\mathfrak{c}(\ulcorner n\urcorner + \tau) = n \oplus \mathfrak{c}(\tau)$ for any term τ.*

(9) *$\mathfrak{c}(\ulcorner 0\urcorner + \tau) = \mathfrak{c}(\tau)$ for any term τ.*

Proof. **(1)** can be considered obvious. Alternatively, one can prove it by simple induction on τ.

(2) can be proved by simple induction on τ as well.

If $\tau = \ulcorner\xi\urcorner$ is a name, then $\mathfrak{c}(\tau) = \ulcorner 0\urcorner + \ulcorner\xi\urcorner$.

If $\tau = \mathtt{f}(\tau_1, \ldots, \tau_n)$, then by induction hypothesis for any i, $\mathfrak{c}(\tau_i) = \ulcorner 0\urcorner + \sigma_i$ for some σ_i, hence by definition (D2), $\mathfrak{c}(\tau) = \mathfrak{c}(\mathtt{f}(\tau_1, \ldots, \tau_n)) = \ulcorner 0\urcorner + \mathtt{f}(1 \oplus \mathfrak{c}(\tau_1), \ldots, 1 \oplus \mathfrak{c}(\tau_n))$.

(3) Since $\ulcorner n\urcorner + \tau$ is an epsilon-termoid, τ is a term, so (2) implies that $\mathfrak{c}(\tau) = \ulcorner 0\urcorner + \sigma$ for some σ, hence by (D3), $\mathfrak{c}(\ulcorner n\urcorner + \tau) = n \oplus (\ulcorner 0\urcorner + \sigma) = \ulcorner n\urcorner + (n \oplus \sigma)$.

(4) Definition (D) implies that for any τ, $\mathfrak{c}(\tau)$ is obtained from τ by applying modifications of the following two kinds:

1. Replacing some occurrences of symbols of the form $\ulcorner k\urcorner$ in τ with symbols $\ulcorner k'\urcorner$, such that $k' > k$.
2. Inserting subexpressions of the form "$\ulcorner k\urcorner +$" in arbitrary places in τ.

Consequently, for any τ, if we remove all occurrences of subexpressions of the form "$\ulcorner k\urcorner +$" from both τ and $\mathfrak{c}(\tau)$, we will obtain one and the same term.

In particular, since we know that $\mathfrak{c}(\tau') = \mathfrak{c}(\tau'')$, this implies that if remove all such symbols from τ' and τ'', we will obtain one and the same term.

Both τ' and τ'' are epsilon-termoids, so $\tau' = \ulcorner n'\urcorner + \sigma$ and $\tau'' = \ulcorner n''\urcorner + \sigma$ for some term σ. In addition, (3) implies that $n' = n''$.

(5) Let for any τ, $\mathfrak{d}(\tau)$ be the term which is obtained from τ by removing all subexpressions of the form "$\ulcorner n\urcorner +$".

We are going to prove that $\mathfrak{c}(\mathfrak{c}^{-1}(\tau)) = \tau$ for any epsilon-regular delta-termoid τ by induction on τ.

If $\tau = \ulcorner n\urcorner + \ulcorner\xi\urcorner$, then $\mathfrak{c}(\mathfrak{c}^{-1}(\tau)) = \mathfrak{c}(\mathfrak{c}^{-1}(\ulcorner n\urcorner + \ulcorner\xi\urcorner)) = \mathfrak{c}(\ulcorner n\urcorner + \ulcorner\xi\urcorner) = \ulcorner n\urcorner + \ulcorner\xi\urcorner = \tau$, because $\mathfrak{c}(\ulcorner\xi\urcorner) = \ulcorner 0\urcorner + \ulcorner\xi\urcorner$.

Let $\tau = \ulcorner n\urcorner + \mathtt{f}(\ulcorner n+1\urcorner + \tau_1, \ldots, \ulcorner n+1\urcorner + \tau_m)$, where $\ulcorner n+1\urcorner + \tau_i$ is an epsilon-regular delta-termoid for any i. Then

$$\begin{aligned}\mathfrak{c}(\mathfrak{c}^{-1}(\tau)) &= \mathfrak{c}(\mathfrak{c}^{-1}(\ulcorner n\urcorner + \mathtt{f}(\ulcorner n+1\urcorner + \tau_1, \ldots, \ulcorner n+1\urcorner + \tau_m))) \\ &= \mathfrak{c}(\ulcorner n\urcorner + \mathtt{f}(\mathfrak{d}(\tau_1), \ldots, \mathfrak{d}(\tau_m)))\end{aligned}$$

Since $\mathtt{f}(\mathfrak{d}(\tau_1), \ldots, \mathfrak{d}(\tau_m)))$ is a term, (2) implies that $\mathfrak{c}(\mathtt{f}(\mathfrak{d}(\tau_1), \ldots, \mathfrak{d}(\tau_m)))$ begins with $\ulcorner 0\urcorner +$, so the above expression is equal to $n \oplus \mathfrak{c}(\mathtt{f}(\mathfrak{d}(\tau_1), \ldots, \mathfrak{d}(\tau_m)))$. Since $\mathfrak{c}(\mathfrak{d}(\tau_i))$ also begins with $\ulcorner 0\urcorner +$, this is equal to

$$\begin{aligned} & n \oplus (\ulcorner 0 \urcorner + \mathtt{f}(1 \oplus \mathfrak{c}(\mathfrak{d}(\tau_1)), \dots, 1 \oplus \mathfrak{c}(\mathfrak{d}(\tau_m)))) = \\ & \quad = \ulcorner n \urcorner + \mathtt{f}((n+1) \oplus \mathfrak{c}(\mathfrak{d}(\tau_1)), \dots, (n+1) \oplus \mathfrak{c}(\mathfrak{d}(\tau_m))) \\ & \quad = \ulcorner n \urcorner + \mathtt{f}(\mathfrak{c}(\ulcorner n+1 \urcorner + \mathfrak{d}(\tau_1)), \dots, \mathfrak{c}(\ulcorner n+1 \urcorner + \mathfrak{d}(\tau_m))) \\ & \quad = \ulcorner n \urcorner + \mathtt{f}(\mathfrak{c}(\mathfrak{c}^{-1}(\ulcorner n+1 \urcorner + \tau_1)), \dots, \mathfrak{c}(\mathfrak{c}^{-1}(\ulcorner n+1 \urcorner + \tau_m))) \end{aligned}$$

By the induction hypothesis, the above expression is equal to $\ulcorner n \urcorner + \mathtt{f}(\ulcorner n+1 \urcorner + \tau_1, \dots, \ulcorner n+1 \urcorner + \tau_m) = \tau$.

(6) follows from (5) since $\mathfrak{c}(\tau)$ is an epsilon-regular delta-termoid.

(7) From (6) it follows that $\mathfrak{c}(\mathfrak{c}^{-1}(\mathfrak{c}(\tau))) = \mathfrak{c}(\tau)$, hence from (4) we obtain that $\mathfrak{c}^{-1}(\mathfrak{c}(\tau)) = \tau$.

(8) According to (J10) and (J11), $\mathfrak{c}(\ulcorner n \urcorner + \tau)$ is similar to $n \oplus \mathfrak{c}(\tau)$. According to (3), the radius of $\mathfrak{c}(\ulcorner n \urcorner + \tau)$ is equal to n. According to (J15) and (2), the radius of $n \oplus \mathfrak{c}(\tau)$ also is equal to n.

(9) follows from (8). ■

N) **Definition.** For any `Sort`-indexed set X, let $[\![X]\!]_\varepsilon$ be the algebra, such that:

(1) The algebraic carrier of sort κ of $[\![X]\!]_\varepsilon$ is the set of all epsilon-termoids of sort κ over X.

(2) For any functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1, \dots, \kappa_n\rangle, \lambda\rangle$ and epsilon-termoids $\tau_1, \dots, \tau_n$ of respective sorts $\kappa_1, \dots, \kappa_n$, let

$$\mathtt{f}^{[\![X]\!]_\varepsilon}\langle\tau_1, \dots, \tau_n\rangle = \mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\tau_1, \dots, \tau_n)))$$

This definition is correct because:

First, the elements of the algebraic carriers of $[\![X]\!]_\varepsilon$ are exactly the epsilon-termoids and $\mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\tau_1, \dots, \tau_n)))$ is an epsilon-termoid of sort λ.

Second, any algebra is uniquely determined by its algebraic carriers and the interpretation of the functional symbols (see 12Q1).

O) **Definition.** Given `Sort`-indexed sets X and Y and a `Sort`-indexed function $f : X \to Y$, let $[\![f]\!]_\varepsilon : [\![X]\!]_\varepsilon \to [\![Y]\!]_\varepsilon$ be the homomorphism, who when applied to an epsilon-termoid τ, replaces all occurrences of names $\mathrm{nam}_{X,\lambda}(\mathtt{z})$ in τ with $\mathrm{nam}_{Y,\lambda}(f_\lambda \mathtt{z})$ (i.e. $[\![f]\!]_\varepsilon$ replaces all occurrences of $\ulcorner \mathtt{z} \urcorner$ with $\ulcorner f\mathtt{z} \urcorner$).

We are going to use postfix notation for this homomorphism. Thus $\tau[\![f]\!]_\varepsilon$ means to apply $[\![f]\!]_\varepsilon$ to τ.

The following proposition shows that the above definition is correct:

P) **Proposition.** *Let $f : X \to Y$ be a* `Sort`*-indexed function. Then:*

(1) *If* τ *is an epsilon-termoid of sort* κ *over* X*, then* $\tau[\![f]\!]_\varepsilon$ *is an epsilon-termoid of sort* κ *over* Y*.*

(2) *If* τ *is a delta-termoid over* X*, then* $\mathfrak{c}(\tau[\![f]\!]_\delta) = (\mathfrak{c}(\tau))[\![f]\!]_\delta$*. If* τ *is a epsilon-termoid over* X*, then* $\mathfrak{c}(\tau[\![f]\!]_\varepsilon) = (\mathfrak{c}(\tau))[\![f]\!]_\delta$*.*

(3) *If* τ *is an epsilon-regular delta-termoid over* X*, then* $\tau[\![f]\!]_\delta$ *is an epsilon-regular delta-termoid over* Y *and* $\mathfrak{c}^{-1}(\tau[\![f]\!]_\delta) = (\mathfrak{c}^{-1}(\tau))[\![f]\!]_\delta$*.*

(4) *There exists unique homomorphism from* $[\![X]\!]_\varepsilon$ *to* $[\![Y]\!]_\varepsilon$*, such that the result of its application to any epsilon-termoid* τ *is equal to* $\tau[\![f]\!]_\varepsilon$*.*

Proof. **(1)** Since τ is an epsilon-termoid over X of sort κ, $\tau = \ulcorner n \urcorner + \tau'$ for some natural number n and term τ' over X of sort κ. Therefore, $\tau[\![f]\!]_\varepsilon = (\ulcorner n \urcorner + \tau')[\![f]\!]_\varepsilon = \ulcorner n \urcorner + \tau'[f]$. Since $\tau'[f]$ is a term over Y, $\ulcorner n \urcorner + \tau'[f]$ is an epsilon-termoid over Y.

(2) We are going give a proof for the case of $[\![.]\!]_\delta$. From this the case of $[\![.]\!]_\varepsilon$ follows, because according to definitions (25M) and (O), $\tau[\![f]\!]_\delta = \tau[\![f]\!]_\varepsilon$ for any epsilon-termoid τ over X.

By induction of τ.

If $\tau = \ulcorner \xi \urcorner$ is a name, then $\mathfrak{c}(\tau[\![f]\!]_\delta) = \mathfrak{c}(\ulcorner \xi \urcorner[\![f]\!]_\delta) = \mathfrak{c}(\ulcorner f\xi \urcorner) = \ulcorner 0 \urcorner + \ulcorner f\xi \urcorner = \ulcorner 0 \urcorner + \ulcorner \xi \urcorner[\![f]\!]_\delta = (\ulcorner 0 \urcorner + \ulcorner \xi \urcorner)[\![f]\!]_\delta = (\mathfrak{c}(\ulcorner \xi \urcorner))[\![f]\!]_\delta = (\mathfrak{c}(\tau))[\![f]\!]_\delta$.

If $\tau = \mathtt{f}(\tau_1, \dots, \tau_n)$, let $\mathfrak{c}(\tau_i) = \ulcorner k_i \urcorner + \sigma_i$ for any $i \in \{1, \dots, n\}$ and $m = \max\{1, k_1, \dots, k_n\}$. By induction hypothesis, $\mathfrak{c}(\tau_i[\![f]\!]_\delta) = (\mathfrak{c}(\tau_i))[\![f]\!]_\delta = (\ulcorner k_i \urcorner + \sigma_i)[\![f]\!]_\delta = \ulcorner k_i \urcorner + \sigma_i[\![f]\!]_\delta$. Therefore, $\mathfrak{c}(\tau[\![f]\!]_\delta) = \mathfrak{c}((\mathtt{f}(\tau_1, \dots, \tau_n))[\![f]\!]_\delta) = \mathfrak{c}(\mathtt{f}(\tau_1[\![f]\!]_\delta, \dots, \tau_n[\![f]\!]_\delta)) = \ulcorner k-1 \urcorner + \mathtt{f}((k-k_1) \oplus (\ulcorner k_1 \urcorner + \sigma_1[\![f]\!]_\delta), \dots, (k-k_n) \oplus (\ulcorner k_n \urcorner + \sigma_n[\![f]\!]_\delta)) = (\ulcorner k-1 \urcorner + \mathtt{f}((k-k_1) \oplus (\ulcorner k_1 \urcorner + \sigma_1), \dots, (k-k_n) \oplus (\ulcorner k_n \urcorner + \sigma_n)))[\![f]\!]_\delta = (\mathfrak{c}(\mathtt{f}(\tau_1, \dots, \tau_n)))[\![f]\!]_\delta = (\mathfrak{c}(\tau))[\![f]\!]_\delta$.

If $\tau = \ulcorner k \urcorner + \tau'$, let $\mathfrak{c}(\tau') = \ulcorner n \urcorner + \sigma$ and $m = \max\{n, k\}$. By induction hypothesis, $\mathfrak{c}(\tau'[\![f]\!]_\delta) = (\mathfrak{c}(\tau'))[\![f]\!]_\delta = (\ulcorner n \urcorner + \sigma)[\![f]\!]_\delta = \ulcorner n \urcorner + \sigma[\![f]\!]_\delta$. Therefore, $\mathfrak{c}(\tau[\![f]\!]_\delta) = \mathfrak{c}((\ulcorner k \urcorner + \tau')[\![f]\!]_\delta) = \mathfrak{c}(\ulcorner k \urcorner + \tau'[\![f]\!]_\delta) = (m-n) \oplus (\ulcorner n \urcorner + \sigma[\![f]\!]_\delta) = ((m-n) \oplus (\ulcorner n \urcorner + \sigma))[\![f]\!]_\delta = (\mathfrak{c}(\ulcorner k \urcorner + \tau'))[\![f]\!]_\delta. = (\mathfrak{c}(\tau))[\![f]\!]_\delta$.

(3) (E2) implies that $\mathfrak{c}(\tau) = \tau$, so from (2) it follows that $\tau[\![f]\!]_\delta = (\mathfrak{c}(\tau))[\![f]\!]_\delta = \mathfrak{c}(\tau[\![f]\!]_\delta)$. Therefore, $\tau[\![f]\!]_\delta$ is an epsilon-regular delta-termoid. It only remains to notice that the equality $\mathfrak{c}^{-1}(\tau[\![f]\!]_\delta) = (\mathfrak{c}^{-1}(\tau))[\![f]\!]_\delta$ follows from the definitions.

(4) follows from (1) and (12Q2). It only remains to notice that (2) and (3) imply that for any functional symbols $\mathtt{f}$ and epsilon-termoids $\tau_1, \dots, \tau_n$ of suitable sorts, $(\mathtt{f}^{[\![X]\!]_\varepsilon}\langle \tau_1, \dots, \tau_n \rangle)[\![f]\!]_\varepsilon = (\mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\tau_1, \dots, \tau_n))))[\![f]\!]_\varepsilon = \mathfrak{c}^{-1}(\mathfrak{c}((\mathtt{f}(\tau_1, \dots, \tau_n))[\![f]\!]_\varepsilon)) = \mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\tau_1[\![f]\!]_\varepsilon, \dots, \tau_n[\![f]\!]_\varepsilon))) = \mathtt{f}^{[\![Y]\!]_\varepsilon}\langle \tau_1[\![f]\!]_\varepsilon, \dots, \tau_n[\![f]\!]_\varepsilon \rangle$. ■

Q) **Definition.** Given a `Sort`-indexed set X, let

$\mathrm{Vals}_X^\varepsilon : [\![|[\![X]\!]_\varepsilon|]\!]_\varepsilon \to [\![X]\!]_\varepsilon$, be the only homomorphism, such that for any epsilon-termoid τ over $|[\![X]\!]_\varepsilon|$, $\mathrm{Vals}_X^\varepsilon \tau = \mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta \tau))$.

If τ is an epsilon-termoid over $|[\![X]\!]_\varepsilon|$, then τ also is a delta-termoid over $|[\![X]\!]_\delta|$, hence $\mathrm{Vals}_X^\delta \tau$ is a delta-termoid over X, so $\mathfrak{c}(\mathrm{Vals}_X^\delta \tau)$ is an epsilon-regular delta-termoid over X, hence $\mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta \tau))$ is an epsilon-termoid over X. Therefore, according to (12Q2), in order to see that the above definition is correct it only remains to notice that for any functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1,\ldots,\kappa_n\rangle,\lambda\rangle$ and any epsilon-termoids $\tau_1\ldots,\tau_n$ over $|[\![X]\!]_\varepsilon|$ of sorts $\kappa_1,\ldots,\kappa_n$,

$$\begin{aligned}
&\mathrm{Vals}_X^\varepsilon(\mathtt{f}^{[\![|[\![X]\!]_\varepsilon|]\!]_\varepsilon}\langle\tau_1,\ldots,\tau_n\rangle) = \\
&\quad= \mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta(\mathtt{f}^{[\![|[\![X]\!]_\varepsilon|]\!]_\varepsilon}\langle\tau_1,\ldots,\tau_n\rangle))) && \text{from (Q)}\\
&\quad= \mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta(\mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\tau_1,\ldots,\tau_n)))))) && \text{from (N2)}\\
&\quad= \mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta(\mathfrak{c}(\mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\tau_1,\ldots,\tau_n))))))) && \text{from (J20)}\\
&\quad= \mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta(\mathfrak{c}(\mathtt{f}(\tau_1,\ldots,\tau_n))))) && \text{from (M6)}\\
&\quad= \mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta(\mathtt{f}(\tau_1,\ldots,\tau_n)))) && \text{from (J20)}\\
&\quad= \mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\mathrm{Vals}_X^\delta \tau_1,\ldots,\mathrm{Vals}_X^\delta \tau_n))) && \text{from (25S)}\\
&\quad= \mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathrm{Vals}_X^\delta \tau_1),\ldots,\mathfrak{c}(\mathrm{Vals}_X^\delta \tau_n)))) && \text{from (E4)}\\
&\quad= \mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta \tau_1))),\ldots,\mathfrak{c}(\mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta \tau_n)))))) \\
& && \text{from (M6)}\\
&\quad= \mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta \tau_1)),\ldots,\mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta \tau_n))))) && \text{from (E4)}\\
&\quad= \mathtt{f}^{[\![X]\!]_\varepsilon}\langle\mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta \tau_1)),\ldots,\mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{Vals}_X^\delta \tau_n))\rangle && \text{from (N2)}\\
&\quad= \mathtt{f}^{[\![X]\!]_\varepsilon}\langle\mathrm{Vals}_X^\varepsilon \tau_1,\ldots,\mathrm{Vals}_X^\varepsilon \tau_n\rangle && \text{from (Q)}
\end{aligned}$$

R) **Definition.** Given a structure $\mathbf{M}$, let $\mathrm{Val}_{\mathbf{M}}^\varepsilon : [\![|\mathbf{M}|]\!]_\varepsilon \to \mathcal{P}\mathbf{M}$, be the only quasimorphism, such that for any epsilon-termoid τ over $|\mathbf{M}|$, $\mathrm{Val}_{\mathbf{M}}^\varepsilon \tau = \mathrm{Val}_{\mathbf{M}}^\delta \mathfrak{c}(\tau)$.

In order to see that the above definition is correct, we have to prove that there exists unique quasimorphism $\mathrm{Val}_{\mathbf{M}}^\varepsilon$ with the specified property.

First, we are going to prove the uniqueness of $\mathrm{Val}_{\mathbf{M}}^\varepsilon$. Definition (R) already specifies uniquely the value of $\mathrm{Val}_{\mathbf{M}}^\varepsilon \tau$ when τ belongs to an algebraic carrier of $[\![|\mathbf{M}|]\!]_\varepsilon$, i.e. when τ is an epsilon-termoid. Let φ be an arbitrary element of the logical carrier of $[\![|\mathbf{M}|]\!]_\varepsilon$. According to (12C2), φ is a relational formula over $|[\![|\mathbf{M}|]\!]_\varepsilon|$.

If φ is an atomic formula, then $\varphi = \mathtt{p}(\ulcorner\tau_1\urcorner,\ldots,\ulcorner\tau_n\urcorner)$ for some epsilon-termoids $\tau_1,\ldots,\tau_n$. Then according to (12C2) and (14C),

$\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\varphi = \mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\mathtt{p}(\ulcorner\tau_1\urcorner,\dots,\ulcorner\tau_n\urcorner) = \mathrm{Val}^{\varepsilon}_{\mathbf{M}}(\mathtt{p}^{[\![|\mathbf{M}|]\!]_\varepsilon}\langle\tau_1,\dots,\tau_n\rangle) = \mathtt{p}^{\mathcal{P}\mathbf{M}}\langle\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\tau_1,\dots,\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\tau_n\rangle$, so the value of $\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\varphi$ is uniquely determined. In addition, this definition of $\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\varphi$ ensures that

$$\mathtt{p}^{\mathcal{P}\mathbf{M}}\langle\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\tau_1,\dots,\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\tau_n\rangle = \mathrm{Val}^{\varepsilon}_{\mathbf{M}}(\mathtt{p}^{[\![|\mathbf{M}|]\!]_\varepsilon}\langle\tau_1,\dots,\tau_n\rangle) \qquad (\sharp)$$

for any predicate symbol $\mathtt{p}$.

If φ is not an atomic formula, then $\varphi = \mathtt{d}(\varphi_1,\dots,\varphi_n)$ for some elements $\varphi_1,\dots,\varphi_n$ of the logical carrier of $[\![|\mathbf{M}|]\!]_\varepsilon$. Then according to (12C2) and (14C), $\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\varphi = \mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\mathtt{d}(\varphi_1,\dots,\varphi_n) = \mathrm{Val}^{\varepsilon}_{\mathbf{M}}(\mathtt{d}^{[\![|\mathbf{M}|]\!]_\varepsilon}\langle\varphi_1,\dots,\varphi_n\rangle) = \mathtt{d}^{\mathcal{P}\mathbf{M}}\langle\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\varphi_1,\dots,\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\varphi_n\rangle$, so again, the value of $\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\varphi$ is uniquely determined by recursion on φ. In addition, this definition of $\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\varphi$ ensures that

$$\mathtt{d}^{\mathcal{P}\mathbf{M}}\langle\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\varphi_1,\dots,\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\varphi_n\rangle = \mathrm{Val}^{\varepsilon}_{\mathbf{M}}(\mathtt{d}^{[\![|\mathbf{M}|]\!]_\varepsilon}\langle\varphi_1,\dots,\varphi_n\rangle) \qquad (\flat)$$

for any logical symbol $\mathtt{d}$.

It remains to see that the uniquely determined `Sort`-indexed function $\mathrm{Val}^{\varepsilon}_{\mathbf{M}}$ is indeed a quasimorphism. Because of ($\sharp$) and ($\flat$), in order to see this, it only remains to notice that for any functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1,\dots,\kappa_n\rangle,\lambda\rangle$ and any epsilon-termoids $\tau_1\dots,\tau_n$ over $|\mathbf{M}|$ of sorts $\kappa_1,\dots,\kappa_n$, we have

$$\begin{aligned}
&\mathtt{f}^{\mathcal{P}\mathbf{M}}\langle\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\tau_1,\dots,\mathrm{Val}^{\varepsilon}_{\mathbf{M}}\,\tau_n\rangle = \\
&\quad = \mathtt{f}^{\mathcal{P}\mathbf{M}}\langle\mathrm{Val}^{\delta}_{\mathbf{M}}\,\mathfrak{c}(\tau_1),\dots,\mathrm{Val}^{\delta}_{\mathbf{M}}\,\mathfrak{c}(\tau_n)\rangle && \text{from (R)}\\
&\quad = \mathrm{Val}^{\delta}_{\mathbf{M}}(\mathtt{f}^{[\![|\mathbf{M}|]\!]_\delta}\langle\mathfrak{c}(\tau_1),\dots,\mathfrak{c}(\tau_n)\rangle) && \text{homomorphism}\\
&\quad = \mathrm{Val}^{\delta}_{\mathbf{M}}\,\mathtt{f}(\mathfrak{c}(\tau_1),\dots,\mathfrak{c}(\tau_n)) && \text{from (25L2)}\\
&\quad \subseteq \mathrm{Val}^{\delta}_{\mathbf{M}}\,\mathfrak{c}(\mathtt{f}(\mathfrak{c}(\tau_1),\dots,\mathfrak{c}(\tau_n))) && \text{from (G)}\\
&\quad = \mathrm{Val}^{\delta}_{\mathbf{M}}\,\mathfrak{c}(\mathtt{f}(\tau_1,\dots,\tau_n)) && \text{from (E4)}\\
&\quad = \mathrm{Val}^{\delta}_{\mathbf{M}}\,\mathfrak{c}(\mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\tau_1,\dots,\tau_n)))) && \text{from (M6)}\\
&\quad = \mathrm{Val}^{\delta}_{\mathbf{M}}\,\mathfrak{c}(\mathtt{f}^{[\![|\mathbf{M}|]\!]_\varepsilon}\langle\tau_1,\dots,\tau_n\rangle) && \text{from (N2)}\\
&\quad = \mathrm{Val}^{\varepsilon}_{\mathbf{M}}(\mathtt{f}^{[\![|\mathbf{M}|]\!]_\varepsilon}\langle\tau_1,\dots,\tau_n\rangle) && \text{from (R)}
\end{aligned}$$

S) **Definition.** Given a `Sort`-indexed set X, let $\mathrm{Nam}^{\varepsilon}_X : X^\circ \to |[\![X]\!]_\varepsilon|$ be the `Sort`-indexed function, such that for any $\mathtt{x} \in X^\circ$

$$\mathrm{Nam}^{\varepsilon}_X(\mathtt{x}) = \mathfrak{c}^{-1}(\mathfrak{c}(\mathrm{nam}_X(\mathtt{x})))$$

Equivalently,

$$\mathrm{Nam}^{\varepsilon}_X(\mathtt{x}) = \mathfrak{c}^{-1}(\mathfrak{c}(\ulcorner\mathtt{x}\urcorner)) = \mathfrak{c}^{-1}(\ulcorner 0\urcorner + \ulcorner\mathtt{x}\urcorner) = \ulcorner 0\urcorner + \ulcorner\mathtt{x}\urcorner$$

Notice that while nam_X, Nam^{α}_X, Nam^{γ}_X and Nam^{δ}_X are defined more or less identically, $\mathrm{Nam}^{\varepsilon}_X$ is defined differently. Because of this and in order to avoid ambiguities we are never going to use the short notation $\ulcorner\xi\urcorner$ in place of $\mathrm{Nam}^{\varepsilon}_X\,\xi$.

T) **Definition.** The quadruple $\langle [\![.]\!]_{\varepsilon}, \mathrm{Val}^{\varepsilon}, \mathrm{Vals}^{\varepsilon}, \mathrm{Nam}^{\varepsilon}\rangle$ is the *epsilon-terminator*.

In order to avoid ambiguities, the termoidal expressions and the formuloids defined by this terminator will be called "epsilon-termoidal expressions" and "epsilon-formuloids". Because of (N), the notion of termoid corresponding to this terminator (see definition 14J) is identical with the notion of epsilon-termoid, as defined in (K).

We have to prove that the above definition is correct.

Proof. We are going to prove the requirements of definition (14I) one by one.

(1) According to definition (N), $[\![X]\!]_{\varepsilon}$ is an algebra.

(2) According to definition (O), $[\![f]\!]_{\varepsilon}$ is a homomorphism from $[\![X]\!]_{\varepsilon}$ to $[\![Y]\!]_{\varepsilon}$ for any `Sort`-indexed function $f : X \to Y$.

(3) Definitions (N) and (K) immediately imply that the identity $|[\![X]\!]_{\varepsilon}| \cap |[\![Y]\!]_{\varepsilon}| = |[\![X \cap Y]\!]_{\varepsilon}|$ is true with respect to the algebraic components of the `Sort`-indexed sets. According to definition (12C2), the elements of the logical carrier of $|[\![X]\!]_{\varepsilon}|$ are exactly the relational formulae over $|[\![X]\!]_{\varepsilon}|$, the elements of the logical carrier of $|[\![Y]\!]_{\varepsilon}|$ are exactly the relational formulae over $|[\![Y]\!]_{\varepsilon}|$ and the elements of the logical carrier of $|[\![X \cap Y]\!]_{\varepsilon}|$ are exactly the relational formulae over $|[\![X \cap Y]\!]_{\varepsilon}|$. No formula may contain names of logical sort, hence the mentioned identity is true for the logical carrier as well.

(4) Definition (O) implies that if the values of $f' : X' \to Y'$ and $f'' : X'' \to Y''$ are equal over the objects whose names occur in the epsilon-termoid τ, then $\tau[\![f']\!]_{\varepsilon} = \tau[\![f'']\!]_{\varepsilon}$. Consequently, the homomorphisms $[\![f']\!]_{\varepsilon}$ and $[\![f'']\!]_{\varepsilon} \restriction [\![X']\!]_{\varepsilon}$ map identically over the algebraic carriers, hence according to (12H2) they map identically over all carriers. On the other hand, definition (O) trivially implies that $\tau[\![f'']\!]_{\varepsilon} = \tau([\![f'']\!]_{\varepsilon} \restriction [\![X']\!]_{\varepsilon})$ for any $\tau \in |[\![X']\!]|$. Consequently, the homomorphisms $[\![f']\!]_{\varepsilon}$ and $[\![f'']\!]_{\varepsilon}$ map identically any element of $|[\![X']\!]_{\varepsilon}|$.

(5) Definition (K) does not refer to the elements of $X_{\mathtt{Log}}$ in any way, so $[\![X]\!]_{\varepsilon} = [\![X^{\circ}]\!]_{\varepsilon}$. Epsilon-termoids contain no names of logical sort, hence immediately from definition (O) it follows that $[\![f]\!]_{\varepsilon} = [\![f^{\circ}]\!]_{\varepsilon}$.

(6) and **(7)** follow from the definitions.

(8) follows from definition (S).

(9) Given a `Sort`-indexed function $f : X \to Y$, for any $\xi \in X^\circ$ we have $(\llbracket f \rrbracket_\varepsilon \circ \mathrm{Nam}^\varepsilon_X)\xi = (\mathrm{Nam}^\varepsilon_X\, \xi)\llbracket f \rrbracket_\varepsilon = (\ulcorner 0 \urcorner + \ulcorner \xi \urcorner)\llbracket f \rrbracket_\varepsilon = \ulcorner 0 \urcorner + \ulcorner \xi \urcorner \llbracket f \rrbracket_\varepsilon = \ulcorner 0 \urcorner + \ulcorner f\xi \urcorner = \ulcorner 0 \urcorner + \ulcorner f^\circ \xi \urcorner = \mathrm{Nam}^\varepsilon_X\, (f^\circ \xi) = (\mathrm{Nam}^\varepsilon_X \circ f^\circ)\xi$.

$$\begin{array}{ccc} X^\circ & \xrightarrow{\mathrm{Nam}^\varepsilon_X} & |\llbracket X \rrbracket_\varepsilon| \\ {\scriptstyle f^\circ}\downarrow & & \downarrow{\scriptstyle \llbracket f \rrbracket_\varepsilon} \\ Y^\circ & \xrightarrow{\mathrm{Nam}^\varepsilon_Y} & |\llbracket Y \rrbracket_\varepsilon| \end{array}$$

(10) According to definition (R), Val^ε is a quasimorphism.

(11) According to (14I11) applied for the delta-terminator, $\mathrm{Val}^\delta_{\mathbf{M}}\, \tau = \mathrm{Val}^\delta_{\partial\mathbf{M}}\, \tau$. Consequently, definition (R) immediately implies that $\mathrm{Val}^\varepsilon_{\mathbf{M}}\, \tau = \mathrm{Val}^\varepsilon_{\partial\mathbf{M}}\, \tau$.

(12) Due to (14E), we only have to prove that the algebraic components of $\mathrm{Val}^\varepsilon_{\mathbf{M}}$ map to non-empty sets. This follows immediately from definition (R) and the fact that according to (14I12), applied for the delta-terminator, $\mathrm{Val}^\delta_{\mathbf{M}}$ maps to non-empty sets.

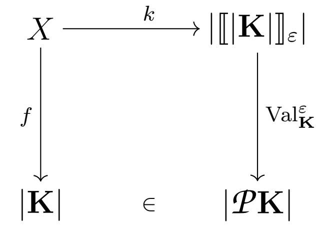
Due to (14F), we only have to prove that the algebraic components of $\mathrm{Val}^\varepsilon_{[X]}$ map to one-element sets. This follows immediately from definition (R) and the fact that according to (14I12), applied for the delta-terminator, $\mathrm{Val}^\delta_{[X]}$ maps to one-element sets.

(13) Let $h : \mathbf{M} \to \mathbf{K}$ be a homomorphism. Due to (14G), we only have to prove that $(h^{\mathcal{P}} \circ \mathrm{Val}^\varepsilon_{\mathbf{M}})\tau \subseteq (\mathrm{Val}^\varepsilon_{\mathbf{K}} \circ \llbracket h \rrbracket_\varepsilon)\tau$ for any epsilon-termoid τ over $|\mathbf{M}|$. From (14I13), applied for the delta-terminator, it follows that $h^{\mathcal{P}} \circ \mathrm{Val}^\delta_{\mathbf{M}} \leq \mathrm{Val}^\delta_{\mathbf{K}} \circ \llbracket h \rrbracket_\delta$, so from (14B1) we obtain $(h^{\mathcal{P}} \circ \mathrm{Val}^\delta_{\mathbf{M}} \circ \mathfrak{c})\tau \subseteq (\mathrm{Val}^\delta_{\mathbf{K}} \circ \llbracket h \rrbracket_\delta \circ \mathfrak{c})\tau$ for any epsilon-termoid τ over $|\mathbf{M}|$. According to (P2), $(\llbracket h \rrbracket_\delta \circ \mathfrak{c})\tau = (\mathfrak{c} \circ \llbracket h \rrbracket_\delta)\tau$, so $(h^{\mathcal{P}} \circ \mathrm{Val}^\delta_{\mathbf{M}} \circ \mathfrak{c})\tau \subseteq (\mathrm{Val}^\delta_{\mathbf{K}} \circ \mathfrak{c} \circ \llbracket h \rrbracket_\delta)\tau$. But $\mathrm{Val}^\delta_{\mathbf{M}} \circ \mathfrak{c} = \mathrm{Val}^\varepsilon_{\mathbf{M}}$ and $\mathrm{Val}^\delta_{\mathbf{K}} \circ \mathfrak{c} = \mathrm{Val}^\varepsilon_{\mathbf{K}}$, so we obtain the required.

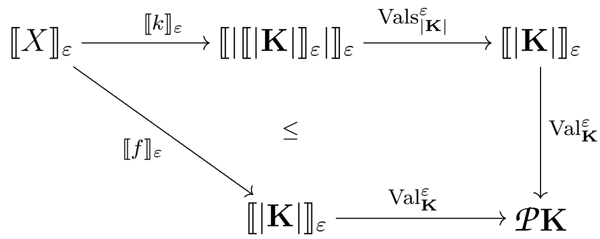
$$\begin{array}{ccc} \llbracket |\mathbf{M}| \rrbracket_\varepsilon & \xrightarrow{\mathrm{Val}^\varepsilon_{\mathbf{M}}} & \mathcal{P}\mathbf{M} \\ {\scriptstyle \llbracket h \rrbracket_\varepsilon}\downarrow & \geq & \downarrow{\scriptstyle h^{\mathcal{P}}} \\ \llbracket |\mathbf{K}| \rrbracket_\varepsilon & \xrightarrow{\mathrm{Val}^\varepsilon_{\mathbf{K}}} & \mathcal{P}\mathbf{K} \end{array}$$

(14) According to definition (Q), $\mathrm{Vals}^\varepsilon$ is a homomorphism.

(15) Given a `Sort`-indexed set X, a structure $\mathbf{K}$, a `Sort`-indexed function $k : X \to |\llbracket |\mathbf{K}| \rrbracket_\varepsilon|$ and a `Sort`-indexed function $f : X \to |\mathbf{K}|$, suppose that $f \ll \mathrm{Val}^{\varepsilon}_{\mathbf{K}} \circ k$.



Due to (14G), in order to prove that $(\mathrm{Val}^{\varepsilon}_{\mathbf{K}} \circ [\![f]\!]_{\varepsilon})\tau \subseteq (\mathrm{Val}^{\varepsilon}_{\mathbf{K}} \circ \mathrm{Vals}^{\varepsilon}_{|\mathbf{K}|} \circ [\![k]\!]_{\varepsilon})\tau$ for any $\tau \in |[\![X]\!]_{\varepsilon}|$, it is enough to consider only the algebraic carriers, i.e. it is enough to consider only the case when τ is a epsilon-termoid over X.



First, notice that $f \ll \mathrm{Val}^{\varepsilon}_{\mathbf{K}} \circ k$ and (R) imply that $f \ll \mathrm{Val}^{\delta}_{\mathbf{K}} \circ (\mathfrak{c} \circ k)$. Consequently,

$$
\begin{aligned}
(\mathrm{Val}^{\varepsilon}_{\mathbf{K}} \circ [\![f]\!]_{\varepsilon})\tau &= (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ \mathfrak{c} \circ [\![f]\!]_{\varepsilon})\tau && \text{from (Q)}\\
&= (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ [\![f]\!]_{\varepsilon} \circ \mathfrak{c})\tau && \text{from (P2)}\\
&= (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ [\![f]\!]_{\varepsilon})\mathfrak{c}(\tau) &&\\
&\subseteq (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ \mathrm{Vals}^{\delta}_{|\mathbf{K}|} \circ [\![\mathfrak{c} \circ k]\!]_{\delta})\mathfrak{c}(\tau) &&\\
&\qquad \text{from (I15), applied for the delta-terminator} &&\\
&= (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ \mathrm{Vals}^{\delta}_{|\mathbf{K}|} \circ [\![\mathfrak{c}]\!]_{\delta} \circ [\![k]\!]_{\delta} \circ \mathfrak{c})\tau &&\\
&\subseteq (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ \mathfrak{c} \circ \mathrm{Vals}^{\delta}_{|\mathbf{K}|} \circ [\![\mathfrak{c}]\!]_{\delta} \circ [\![k]\!]_{\delta} \circ \mathfrak{c})\tau && \text{from (G)}\\
&= (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ \mathfrak{c} \circ \mathrm{Vals}^{\delta}_{|\mathbf{K}|} \circ [\![k]\!]_{\delta} \circ \mathfrak{c})\tau && \text{from (J21)}\\
&= (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ \mathfrak{c} \circ \mathrm{Vals}^{\delta}_{|\mathbf{K}|} \circ \mathfrak{c} \circ [\![k]\!]_{\delta})\tau && \text{from (P2)}\\
&= (\mathrm{Val}^{\delta}_{\mathbf{K}} \circ \mathfrak{c} \circ \mathrm{Vals}^{\delta}_{|\mathbf{K}|} \circ [\![k]\!]_{\delta})\tau && \text{from (J20)}\\
&= ((\mathrm{Val}^{\delta}_{\mathbf{K}} \circ \mathfrak{c}) \circ (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^{\delta}_{|\mathbf{K}|}) \circ [\![k]\!]_{\delta})\tau && \text{from (M6)}\\
&= (\mathrm{Val}^{\varepsilon}_{\mathbf{K}} \circ \mathrm{Vals}^{\varepsilon}_{|\mathbf{K}|} \circ [\![k]\!]_{\varepsilon})\tau && \text{from (R), (Q), (O) and (25M)}
\end{aligned}
$$

(16) According to (12H2), it is enough to see that for any epsilon-termoid τ,

$$
\begin{aligned}
(\llbracket f \rrbracket_\varepsilon \circ \mathrm{Vals}^\varepsilon_X)\tau &= (\llbracket f \rrbracket_\delta \circ \mathrm{Vals}^\varepsilon_X)\tau && \text{from (O) and (25M)}\\
&= (\llbracket f \rrbracket_\delta \circ \mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_X)\tau && \text{from (Q)}\\
&= (\mathfrak{c}^{-1} \circ \llbracket f \rrbracket_\delta \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_X)\tau && \text{from (P3)}\\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \llbracket f \rrbracket_\delta \circ \mathrm{Vals}^\delta_X)\tau && \text{from (P2)}\\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_Y \circ \llbracket \llbracket f \rrbracket_\delta \rrbracket_\delta)\tau \\
& && \text{from (I16), applied for the delta-terminator}\\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_Y \circ \llbracket \llbracket f \rrbracket_\varepsilon \rrbracket_\varepsilon)\tau && \text{from (O) and (25M)}\\
&= (\mathrm{Vals}^\varepsilon_Y \circ \llbracket \llbracket f \rrbracket_\varepsilon \rrbracket_\varepsilon)\tau && \text{from (Q)}
\end{aligned}
$$

$$
\begin{array}{ccc}
\llbracket | \llbracket X \rrbracket_\varepsilon | \rrbracket_\varepsilon & \xrightarrow{\mathrm{Vals}^\varepsilon_X} & \llbracket X \rrbracket_\varepsilon \\
\Big\downarrow {\scriptstyle \llbracket \llbracket f \rrbracket_\varepsilon \rrbracket_\varepsilon} & & \Big\downarrow {\scriptstyle \llbracket f \rrbracket_\varepsilon} \\
\llbracket | \llbracket Y \rrbracket_\varepsilon | \rrbracket_\varepsilon & \xrightarrow{\mathrm{Vals}^\varepsilon_Y} & \llbracket Y \rrbracket_\varepsilon
\end{array}
$$

(17) According to (12H2), it is enough to see that for any epsilon-termoid τ,

$$
\begin{aligned}
&(\mathrm{Vals}^\varepsilon_X \circ \llbracket \mathrm{Nam}^\varepsilon_X \rrbracket_\varepsilon)\tau = \\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_X \circ \llbracket \mathrm{Nam}^\varepsilon_X \rrbracket_\varepsilon)\tau && \text{from (Q)}\\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_X \circ \llbracket \mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Nam}^\delta_X \rrbracket_\varepsilon)\tau && \text{from (S) and (25V)}\\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_X \circ \llbracket \mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Nam}^\delta_X \rrbracket_\delta)\tau && \text{from (O) and (25M)}\\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_X \circ \llbracket \mathfrak{c} \rrbracket_\delta \circ \llbracket \mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Nam}^\delta_X \rrbracket_\delta)\tau && \text{from (J21)}\\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_X \circ \llbracket \mathfrak{c} \circ \mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Nam}^\delta_X \rrbracket_\delta)\tau \\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_X \circ \llbracket \mathfrak{c} \circ \mathrm{Nam}^\delta_X \rrbracket_\delta)\tau && \text{from (M6)}\\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_X \circ \llbracket \mathfrak{c} \rrbracket_\delta \circ \llbracket \mathrm{Nam}^\delta_X \rrbracket_\delta)\tau \\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Vals}^\delta_X \circ \llbracket \mathrm{Nam}^\delta_X \rrbracket_\delta)\tau && \text{from (J21)}\\
&= (\mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{id}_{\llbracket X \rrbracket_\delta})\tau && \text{from (I17), applied for the delta-terminator}\\
&= \mathfrak{c}^{-1}(\mathfrak{c}(\tau)) \\
&= \tau && \text{from (M7)}\\
&= (\mathrm{id}_{\llbracket X \rrbracket_\varepsilon})\tau
\end{aligned}
$$

$$
\llbracket X \rrbracket_\varepsilon = \llbracket X^\circ \rrbracket_\varepsilon \xrightarrow{\llbracket \mathrm{Nam}^\varepsilon_X \rrbracket_\varepsilon} \llbracket | \llbracket X \rrbracket_\varepsilon | \rrbracket_\varepsilon \xrightarrow{\mathrm{Vals}^\varepsilon_X} \llbracket X \rrbracket_\varepsilon
$$

(18) According to (12H2), it is enough to see that for any μ belonging to an algebraic carrier of $\mathbf{M}$,

$$\begin{aligned}(\mathrm{Val}^{\varepsilon}_{\mathbf{M}} \circ \mathrm{Nam}^{\varepsilon}_{|\mathbf{M}|})\mu &= (\mathrm{Val}^{\delta}_{\mathbf{M}} \circ \mathfrak{c} \circ \mathrm{Nam}^{\varepsilon}_{|\mathbf{M}|})\mu && \text{from (R)}\\ &= (\mathrm{Val}^{\delta}_{\mathbf{M}} \circ \mathfrak{c} \circ \mathfrak{c}^{-1} \circ \mathfrak{c} \circ \mathrm{Nam}^{\delta}_{|\mathbf{M}|})\mu && \text{from (S) and (25V)}\\ &= (\mathrm{Val}^{\delta}_{\mathbf{M}} \circ \mathfrak{c} \circ \mathrm{Nam}^{\delta}_{|\mathbf{M}|})\mu && \text{from (M6)}\\ &= \mathrm{Val}^{\delta}_{\mathbf{M}}(\mathfrak{c}(\ulcorner \mu \urcorner)) &&\\ &= \mathrm{Val}^{\delta}_{\mathbf{M}}(\ulcorner \mu \urcorner) && \text{from (H)}\\ &= \{\mu\} && \text{from (I18), applied for the delta-terminator}\end{aligned}$$

$$|\mathbf{M}|^{\circ} \xrightarrow{\mathrm{Nam}^{\varepsilon}_{|\mathbf{M}|}} |[\![|\mathbf{M}|]\!]_{\varepsilon}| \xrightarrow{\mathrm{Val}^{\varepsilon}_{\mathbf{M}}} |\mathcal{P}\mathbf{M}|$$

■

U) **Proposition.** *For any epsilon-termoid* $\ulcorner n \urcorner + \tau$ *over* X, $(\ulcorner n \urcorner + \tau)[\![\mathrm{nam}_X]\!]^{[X]}_{\varepsilon} = \tau$.

Proof.

$$\begin{aligned}\{(\ulcorner n \urcorner + \tau)[\![\mathrm{nam}_X]\!]^{[X]}_{\varepsilon}\} &= \mathrm{Val}^{\varepsilon}_{[X]}((\ulcorner n \urcorner + \tau)[\![\mathrm{nam}_X]\!]_{\varepsilon}) && \text{from (14N)}\\ &= \mathrm{Val}^{\varepsilon}_{[X]}((\ulcorner n \urcorner + \tau)[\![\mathrm{nam}_X]\!]_{\delta}) && \text{from (O) and (25M)}\\ &= \mathrm{Val}^{\delta}_{[X]}\,\mathfrak{c}((\ulcorner n \urcorner + \tau)[\![\mathrm{nam}_X]\!]_{\delta}) && \text{from (R)}\\ &= \mathrm{Val}^{\delta}_{[X]}((\mathfrak{c}(\ulcorner n \urcorner + \tau))[\![\mathrm{nam}_X]\!]_{\delta}) && \text{from (P2)}\\ &= \{(\mathfrak{c}(\ulcorner n \urcorner + \tau))[\![\mathrm{nam}_X]\!]^{[X]}_{\delta}\} && \text{from (14N)}\\ &= \{\tau\} && \text{from (25X)}\end{aligned}$$

■

V) **Lemma.** *For any n-ary functional symbol* $\mathtt{f}$ *and epsilon-termoids* $\ulcorner 0 \urcorner + \tau_1, \ulcorner 0 \urcorner + \tau_2, \dots, \ulcorner 0 \urcorner + \tau_n$ *over* X *of suitable sorts,*

$$\mathtt{f}^{[\![X]\!]_{\varepsilon}}\langle \ulcorner 0 \urcorner + \tau_1, \dots, \ulcorner 0 \urcorner + \tau_n \rangle = \ulcorner 0 \urcorner + \mathtt{f}(\tau_1, \dots, \tau_n)$$

Proof.

$$\begin{aligned}&\mathfrak{c}(\mathtt{f}^{[\![X]\!]_{\varepsilon}}\langle \ulcorner 0 \urcorner + \tau_1, \dots, \ulcorner 0 \urcorner + \tau_n \rangle) = &&\\ &\quad = \mathfrak{c}(\mathfrak{c}^{-1}(\mathfrak{c}(\mathtt{f}(\ulcorner 0 \urcorner + \tau_1, \dots, \ulcorner 0 \urcorner + \tau_n)))) && \text{from (N2)}\\ &\quad = \mathfrak{c}(\mathtt{f}(\ulcorner 0 \urcorner + \tau_1, \dots, \ulcorner 0 \urcorner + \tau_n)) && \text{from (M6)}\\ &\quad = \mathfrak{c}(\mathtt{f}(\mathfrak{c}(\ulcorner 0 \urcorner + \tau_1), \dots, \mathfrak{c}(\ulcorner 0 \urcorner + \tau_n))) && \text{from (J5)}\\ &\quad = \mathfrak{c}(\mathtt{f}(\mathfrak{c}(\tau_1), \dots, \mathfrak{c}(\tau_n))) && \text{from (M9)}\\ &\quad = \mathfrak{c}(\mathtt{f}(\tau_1, \dots, \tau_n)) && \text{from (J5)}\\ &\quad = \mathfrak{c}(\ulcorner 0 \urcorner + \mathtt{f}(\tau_1, \dots, \tau_n)) && \text{from (M9)}\end{aligned}$$

Since both $\mathtt{f}^{[\![X]\!]_\varepsilon}\langle\ulcorner 0\urcorner+\tau_1,\ldots,\ulcorner 0\urcorner+\tau_n\rangle$ and $\ulcorner 0\urcorner + \mathtt{f}(\tau_1,\ldots,\tau_n)$ are epsilon-termoids, we obtain the required from (M4). ∎

W) **Proposition.** *For any term τ over X, $\tau[\mathrm{Nam}^\varepsilon_X]^{[\![X]\!]_\varepsilon} = \ulcorner 0\urcorner + \tau$.*

<u>Proof.</u> Consider the Sort-indexed function $h : |[X]| \to |[\![X]\!]_\varepsilon|$, such that $h\tau = \ulcorner 0\urcorner + \tau$ for any term τ and for any termal expression φ over X of logical sort, $h\varphi$ can be obtained from φ by replacing in φ all subexpressions of the form $\mathtt{p}(\tau_1,\ldots,\tau_n)$ where $\mathtt{p}$ is a predicate symbol with $\mathtt{p}(\ulcorner\ulcorner 0\urcorner+\tau_1\urcorner,\ldots,\ulcorner\ulcorner 0\urcorner+\tau_n\urcorner)$.

This function is a homomorphism from $[X]$ to $[\![X]\!]_\varepsilon$. Indeed, according to (V) and the definition of h, for any functional symbol $\mathtt{f}$, $h(\mathtt{f}^{[X]}\langle\tau_1,\ldots,\tau_n\rangle) = h(\mathtt{f}(\tau_1,\ldots,\tau_n)) = \ulcorner 0\urcorner + \mathtt{f}(\tau_1,\ldots,\tau_n) = \mathtt{f}^{[\![X]\!]_\varepsilon}\langle\ulcorner 0\urcorner+\tau_1,\ldots,\ulcorner 0\urcorner+\tau_n\rangle = \mathtt{f}^{[\![X]\!]_\varepsilon}\langle h\tau_1,\ldots,h\tau_n\rangle$. Moreover, for any predicate symbol $\mathtt{p}$, $h(\mathtt{p}^{[X]}\langle\tau_1,\ldots,\tau_n\rangle) = h(\mathtt{p}(\tau_1,\ldots,\tau_n)) = \mathtt{p}(\ulcorner\ulcorner 0\urcorner+\tau_1\urcorner,\ldots,\ulcorner\ulcorner 0\urcorner+\tau_n\urcorner) = \mathtt{p}(\ulcorner h\tau_1\urcorner,\ldots,\ulcorner h\tau_n\urcorner)$. Since $[\![X]\!]_\varepsilon$ is algebra, this is equal to $\mathtt{p}^{[\![X]\!]_\varepsilon}\langle h\tau_1,\ldots,h\tau_n\rangle$. Analogously, for any logical symbol $\mathtt{d}$, $h(\mathtt{d}^{[X]}\langle\tau_1,\ldots,\tau_n\rangle) = h(\mathtt{d}(\tau_1,\ldots,\tau_n)) = \mathtt{d}(h\tau_1,\ldots,h\tau_n)$. Since $[\![X]\!]_\varepsilon$ is algebra, this is equal to $\mathtt{d}^{[\![X]\!]_\varepsilon}\langle h\tau_1,\ldots,h\tau_n\rangle$.

According to definition (S), for any $\xi \in X$ we have $\ulcorner\xi\urcorner[\mathrm{Nam}^\varepsilon_X]^{[\![X]\!]_\varepsilon} = (\ulcorner\ulcorner 0\urcorner+\ulcorner\xi\urcorner\urcorner)^{[\![X]\!]_\varepsilon}$. Because of (11V2), this is equal to $\ulcorner 0\urcorner + \ulcorner\xi\urcorner = h\ulcorner\xi\urcorner$. Consequently, from (11P) we obtain that $[\mathrm{Nam}^\varepsilon_X]^{[\![X]\!]_\varepsilon} = h$ which gives us the required. ∎

X) **Proposition.** (1) *For any epsilon-formuloid φ over X, $\varphi[\![\mathrm{nam}_X]\!]^{[X]}_\varepsilon$ is obtained from φ by replacing each epsilon-termoid $\ulcorner n\urcorner + \tau$ of φ by τ.*

(2) *For any formula φ over X, $\varphi[\mathrm{Nam}^\varepsilon_X]^{[\![X]\!]_\varepsilon}$ is obtained from φ by replacing each term τ of φ by $\ulcorner 0\urcorner + \tau$.*

<u>Proof.</u> Since $[\![\mathrm{nam}_X]\!]^{[X]}_\varepsilon$ is a quasimorphism, for any predicate or logical symbol $\mathtt{d}$,

$$(\mathtt{d}^{[\![X]\!]}\langle\alpha_1,\ldots,\alpha_n\rangle)[\![\mathrm{nam}_X]\!]^{[X]}_\varepsilon = \mathtt{d}^{[X]}\langle\alpha_1[\![\mathrm{nam}_X]\!]^{[X]}_\varepsilon,\ldots,\alpha_n[\![\mathrm{nam}_X]\!]^{[X]}_\varepsilon\rangle$$

Therefore, **(1)** follows from (U) by induction on φ.

Since $[\mathrm{Nam}^\varepsilon_X]^{[\![X]\!]_\varepsilon}$ is a homomorphism, **(2)** follows analogously from (W). ∎

§27. AN ALTERNATIVE SEMANTICS

A) Throughout this section we are going to fix a Sort-indexed set X and a structure $\mathbf{M}$, such that there exist at least one assignment function

$f : X \to |\mathbf{M}|$.

B) **Definition.** A functional symbol $\mathtt{f}$ of type $\langle\langle\kappa_1,\dots,\kappa_n\rangle,\lambda\rangle$ is *accessible*, if for any $i \in \{1,\dots,n\}$ there exist at least one term over X of sort κ_i. Equivalently, $\mathtt{f}$ is accessible if there exists at least one term over X containing $\mathtt{f}$.

C) **Definition.** For any sort κ and natural number n we are going to define recursively on n two binary relations κ-$\mathfrak{sim}^\circ_n$ and κ-$\mathfrak{sim}_n$ over $\mathbf{M}_\kappa$:

(1) κ-$\mathfrak{sim}^\circ_0(\mu',\mu'')$ is true, if $\mu' = \mu''$.

(2) κ-$\mathfrak{sim}^\circ_{n+1}(\mu',\mu'')$ is true, if κ-$\mathfrak{sim}_n(\mu',\mu'')$ is true or there exists an accessible functional symbol $\mathtt{f}$ with type $\langle\langle\kappa_1,\dots,\kappa_m\rangle,\lambda\rangle$ and a natural number $i \in \{1,\dots,m\}$, such that $\kappa = \kappa_i$ and for some $\nu'_1,\dots,\nu'_m,\nu''_1,\dots,\nu''_m$ belonging to suitable carriers of $\mathbf{M}$ λ-$\mathfrak{sim}_n(\mathtt{f}^{\mathbf{M}}\langle\nu'_1,\dots,\nu'_{i-1},\mu',\nu'_{i+1},\dots,\nu'_m\rangle,\mathtt{f}^{\mathbf{M}}\langle\nu''_1,\dots,\nu''_{i-1},\mu'',\nu''_{i+1},\dots,\nu''_m\rangle)$ is true.

(3) κ-$\mathfrak{sim}_n(\mu',\mu'')$ is the transitive closure of κ-$\mathfrak{sim}^\circ_n(\mu',\mu'')$ for any natural number n.

Since there will be no danger of ambiguity, we are going to write simply $\mathfrak{sim}^\circ_n$ and $\mathfrak{sim}_n$ instead of κ-$\mathfrak{sim}^\circ_n$ and κ-$\mathfrak{sim}_n$.

D) **Proposition.** (1) *If* $\mathtt{f}$ *be an accessible functional symbol of type* $\langle\langle\kappa_1,\dots,\kappa_n\rangle,\lambda\rangle$*, then the carriers* $\mathbf{M}_{\kappa_1},\dots,\mathbf{M}_{\kappa_n}$ *and* $\mathbf{M}_\lambda$ *are nonempty sets.*

(2) *For any natural number n, the relation $\mathfrak{sim}^\circ_n$ is reflexive and symmetric, and the relation $\mathfrak{sim}_n$ is reflexive, symmetric and transitive.*

Proof. **(1)** Let $g : X \to |\mathbf{M}|$ be an arbitrary assignment function. Let $\tau_1,\dots,\tau_n$ be terms over X of respective sorts $\kappa_1,\dots,\kappa_n$. Then $\tau_1[g],\dots,\tau_n[g]$ will be terms over $|\mathbf{M}|$, so their values in $\mathbf{M}$ belong to the carriers $\mathbf{M}_{\kappa_1},\dots,\mathbf{M}_{\kappa_n}$ and $\mathtt{f}^{\mathbf{M}}\langle\tau_1^{\mathbf{M}},\dots,\tau_n^{\mathbf{M}}\rangle$ belongs to $\mathbf{M}_\lambda$.

(2) By induction on n. When $n = 0$, there is nothing to prove.

Suppose that $\mathfrak{sim}_n$ is reflexive, symmetric and transitive.

Since $\mathfrak{sim}_n(\mu',\mu'')$ implies $\mathfrak{sim}^\circ_{n+1}(\mu',\mu'')$, the relation $\mathfrak{sim}^\circ_{n+1}$ is reflexive. Since $\mathfrak{sim}^\circ_{n+1}(\mu',\mu'')$ implies $\mathfrak{sim}_{n+1}(\mu',\mu'')$, the relation $\mathfrak{sim}_{n+1}$ is reflexive as well.

Because of the way the relation $\mathfrak{sim}_{n+1}$ is defined, from the symmetry of $\mathfrak{sim}_n(\mu',\mu'')$ the symmetry of $\mathfrak{sim}^\circ_{n+1}(\mu',\mu'')$ follows. A transitive closure of a symmetric relation also is a symmetric relation, so the relation relation $\mathfrak{sim}_{n+1}$ is symmetric as well.

$\mathfrak{sim}_{n+}$ is transitive by definition. ∎

E) **Proposition.** *If* $\mathbf{M}$ *is a structure of terms, then* $\mathfrak{sim}_n(\mu,\nu)$ *is true* *if and only if* $\mu = \nu$.[77]

Proof. The "if" part follows from the reflexivity of the relation $\mathfrak{sim}_n$. Since $\mathfrak{sim}_n$ is a transitive closure of $\mathfrak{sim}_n^\circ$, it will be enough to prove that $\mathfrak{sim}_n^\circ(\mu, \nu)$ implies $\mu = \nu$. We can do this by induction on n.

When $n = 0$, this is so by definition (27C1). On the other hand, according to definition (27C2), if $\mathfrak{sim}_{n+1}(\mu', \nu')$ is true, then either $\mathfrak{sim}_n^\circ(\mu', \nu')$ is true, or $\mathfrak{sim}_n(\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_m\rangle, \mathtt{f}^{\mathbf{M}}\langle\nu_1, \ldots, \nu_m\rangle)$ is true for some functional symbol $\mathtt{f}$, natural number i and some $\mu_1, \ldots, \mu_m$ and $\nu_1, \ldots, \nu_m$ belonging to suitable carriers of $\mathbf{M}$, such that $\mu' = \mu_i$ and $\nu' = \nu_i$. In the first case (when $\mathfrak{sim}_n^\circ(\mu', \nu')$ is true), from the induction hypothesis we immediately obtain that $\mu' = \nu'$. In the other case, by induction hypothesis, from $\mathfrak{sim}_n(\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_m\rangle, \mathtt{f}^{\mathbf{M}}\langle\nu_1, \ldots, \nu_m\rangle)$ we obtain that $\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_m\rangle = \mathtt{f}^{\mathbf{M}}\langle\nu_1, \ldots, \nu_m\rangle$. But $\mathbf{M}$ is a structure of terms, so $\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle = \mathtt{f}(\mu_1, \ldots, \mu_n)$ and $\mathtt{f}^{\mathbf{M}}\langle\nu_1, \ldots, \nu_n\rangle = \mathtt{f}(\nu_1, \ldots, \nu_n)$, hence $\mathtt{f}(\mu_1, \ldots, \mu_n) = \mathtt{f}(\nu_1, \ldots, \nu_n)$, so $\mu' = \mu_i = \nu_i = \nu'$. ■

F) **Definition.** For any delta-semitermoid τ over $|\mathbf{M}|$ we are going to define a set $\mathfrak{v}_\delta(\tau)$, recursively on τ.

(1) For any $\mu \in |\mathbf{M}|$, $\mathfrak{v}_\delta(\ulcorner\mu\urcorner) = \{\mu\}$.

(2) $\mathfrak{v}_\delta(\triangle_\kappa) = \mathbf{M}_\kappa$.

(3) If $\mathtt{f}$ is a functional symbol of type $\langle\langle\kappa_1, \ldots, \kappa_n\rangle, \lambda\rangle$ and $\tau_1, \ldots, \tau_n$ are delta-termoids over $|\mathbf{M}|$ of respective sorts $\kappa_1, \ldots, \kappa_n$, then $\mathfrak{v}_\delta(\mathtt{f}(\tau_1, \ldots, \tau_n)) = \{\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle : \mu_1 \in \mathfrak{v}_\delta(\tau_1), \ldots, \mu_n \in \mathfrak{v}_\delta(\tau_n)\}$.

(4) If $\mathtt{f}$ is a functional symbol of type $\langle\langle\kappa_1, \ldots, \kappa_n\rangle, \lambda\rangle$ and τ is a delta-semitermoid over $|\mathbf{M}|$ of sort λ, then $\mathfrak{v}_\delta(\mathtt{f}_i^{-1}(\tau))$ is the set of all $\mu_i \in \mathbf{M}_{\kappa_i}$, for which there exist $\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n$ belonging to suitable carriers of $\mathbf{M}$, such that $\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle \in \mathfrak{v}_\delta(\tau)$.

(5) If n is a natural number and τ is a delta-termoid over $|\mathbf{M}|$, then $\mathfrak{v}_\delta(\ulcorner n\urcorner + \tau) = \{\mu : \text{there exists } \mu' \in \mathfrak{v}_\delta(\tau) \text{ such that } \mathfrak{sim}_n(\mu, \mu')\}$.

Our aim is to prove that $\mathfrak{v}_\delta(\tau) = \tau^{\mathcal{P}\mathbf{M}}$ for any delta-termoid τ.

G) **Lemma.** $\mathfrak{v}_\delta(\ulcorner n\urcorner + \tau) \subseteq \mathfrak{v}_\delta(\ulcorner n+1\urcorner + \tau)$ *for any natural number* n *and delta-semitermoid* τ *over* $|\mathbf{M}|$.

Proof. According to definition (C), $\mathfrak{sim}_n(\mu', \mu'')$ implies $\mathfrak{sim}_{n+1}^\circ(\mu', \mu'')$. But $\mathfrak{sim}_{n+1}$ is a transitive closure of $\mathfrak{sim}_{n+1}^\circ$, so $\mathfrak{sim}_n(\mu', \mu'')$ implies $\mathfrak{sim}_{n+1}(\mu', \mu'')$. After this observation, a quick inspection of definition (F5) reveals that $\mathfrak{v}_\delta(\ulcorner n\urcorner + \tau) \subseteq \mathfrak{v}_\delta(\ulcorner n+1\urcorner + \tau)$. ■

H) **Lemma.** $\mathrm{Val}_{\mathbf{M}}^\delta\, \tau = \mathfrak{v}_\delta(\tau)$ *for any gamma-semitermoid* τ *over* $|\mathbf{M}|$.

[77] See (14L) for the definition of the notion "structure of terms".

Proof. According to definition (25Q), $\mathrm{Val}^{\delta}_{\mathbf{M}}\,\tau$ is equal to the union of the sets of the values of all embraced by τ gamma-termoids. According to (25F), τ is the only embraced by τ gamma-semitermoid. Consequently, we have to prove that the set of the values of the gamma-semitermoid τ (according to definition 24H) is equal to $\mathfrak{v}_\delta(\tau)$. A simple comparison of definition (24H) and definition (F) is enough to tell us that this is so because the only significant difference between both definition is that (F) defines interpretation of the "plus" sign and (24H) does not, but the "plus" sign does not occur in gamma-semitermoids. ■

I) **Lemma.** $\mathrm{Val}^{\delta}_{\mathbf{M}}\,\tau \subseteq \mathfrak{v}_\delta(\tau)$ *for any delta-termoid* τ *over* $|\mathbf{M}|$.

Proof. According to definition (25Q), $\mathrm{Val}^{\delta}_{\mathbf{M}}\,\tau$ is equal to the union of the sets of the values of all embraced by τ gamma-termoids. Consequently, it will be enough to prove that if a gamma-semitermoid σ is embraced by a delta-semitermoid τ, then all elements of the set of the values of σ belong to $\mathfrak{v}_\delta(\tau)$. We are going to do this by induction on definition (25E).

If τ is a gamma-semitermoid and $\sigma = \tau$, then (H) tells us what we need.

If σ is embraced by $\ulcorner 0 \urcorner + \tau$ because σ is embraced by τ, then by induction hypothesis all elements of the set of the values of σ in $\mathbf{M}$ belong to $\mathfrak{v}_\delta(\tau)$. On the other hand, according to (C1) and (C3), $\mathfrak{sim}_0(\mu', \mu'')$ is equivalent to $\mu' = \mu''$, so according to (F5), $\mathfrak{v}_\delta(\ulcorner 0 \urcorner + \tau) = \mathfrak{v}_\delta(\tau)$.

If σ is embraced by $\ulcorner n+1 \urcorner + \tau$ because σ is embraced by $\ulcorner n \urcorner + \tau$, then by induction hypothesis, all elements of the set of the values of σ in $\mathbf{M}$ belong to $\mathfrak{v}_\delta(\ulcorner n \urcorner + \tau)$. On the other hand, according to (I), $\mathfrak{v}_\delta(\ulcorner n \urcorner + \tau) \subseteq \mathfrak{v}_\delta(\ulcorner n+1 \urcorner + \tau)$.

Let $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ be embraced by $\mathtt{f}(\tau_1, \ldots, \tau_n)$ because σ_i is embraced by τ_i for any $i \in \{1, \ldots, n\}$. Let A_i be the set of the values of σ_i in $\mathbf{M}$. By induction hypothesis, $A_i \subseteq \mathfrak{v}_\delta(\tau_i)$. According to definition (24H3), the set of the values of $\mathtt{f}(\sigma_1, \ldots, \sigma_n)$ in $\mathbf{M}$ is equal to $\{\mathtt{f}^{\mathbf{M}}\langle \mu_1, \ldots, \mu_n \rangle : \mu_1 \in A_1, \ldots, \mu_n \in A_n\}$. On the other hand, according to definition (F3), $\mathfrak{v}_\delta(\mathtt{f}(\tau_1, \ldots, \tau_n))$ is equal to $\{\mathtt{f}^{\mathbf{M}}\langle \mu_1, \ldots, \mu_n \rangle : \mu_1 \in \mathfrak{v}_\delta(\tau_1), \ldots, \mu_n \mathfrak{v}_\delta(\tau_n)\}$.

Let $\mathtt{f}_i^{-1}(\sigma')$ be embraced by $\mathtt{f}_i^{-1}(\tau')$ because σ' is embraced by τ'. Let A be the set of the values of σ' in $\mathbf{M}$. By induction hypothesis, $A \subseteq \mathfrak{v}_\delta(\tau')$. According to definition (24H4), the set of the values of $\mathtt{f}_i^{-1}(\sigma')$ in $\mathbf{M}$ is equal to $\{\mu_i :$ there exist $\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n$, such that $\mathtt{f}^{\mathbf{M}}\langle \mu_1, \ldots, \mu_n \rangle \in A\}$. On the other hand, according to definition (F3), $\mathfrak{v}_\delta(\mathtt{f}(\tau_1, \ldots, \tau_n))$ is equal to the set of all μ_i for which there exist $\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n$, such that $\mathtt{f}^{\mathbf{M}}\langle \mu_1, \ldots, \mu_n \rangle \in \mathfrak{v}_\delta(\tau')$.

Let $\mathtt{f}$ be a functional symbol of type $\langle\langle \kappa_1, \ldots, \kappa_n \rangle, \lambda\rangle$ and let $\mathtt{f}_i^{-1}(\sigma')$ be embraced by $\ulcorner n+1 \urcorner + \tau$ because σ is embraced by $\ulcorner n+1 \urcorner + \tau$ and σ' is embraced by $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_n})$.

Suppose that μ belongs to the set of the values of $\mathtt{f}_i^{-1}(\sigma')$ in $\mathbf{M}$. According to definition (24H4), there exist $\mu_1, \ldots, \mu_n$ belonging to suitable carriers of $\mathbf{M}$, such that $\mu = \mu_i$ and $\mathtt{f}^{\mathbf{M}}\langle \mu_1, \ldots, \mu_n \rangle$ belongs to the set of the values of σ' in $\mathbf{M}$. By induction hypothesis,[78] $\mathtt{f}^{\mathbf{M}}\langle \mu_1, \ldots, \mu_n \rangle$ belongs to $\mathfrak{v}_\delta(\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_n}))$. According to definition (F), there exist $\nu_1, \ldots, \nu_n$ belonging to suitable carriers of $\mathbf{M}$, such that $\nu_i \in \mathfrak{v}_\delta(\sigma)$ and $\mathfrak{sim}_n(\mathtt{f}^{\mathbf{M}}\langle \mu_1, \ldots, \mu_n \rangle, \mathtt{f}^{\mathbf{M}}\langle \nu_1, \ldots, \nu_n \rangle)$ is true. Now, from definition (C2) we obtain that $\mathfrak{sim}_{n+1}(\mu_i, \nu_i)$ is true, i.e. that $\mathfrak{sim}_{n+1}(\mu, \nu_i)$ is true.

On the other hand, since σ is a gamma-semitermoid, from (H) we obtain that ν_i belongs to the set of the values of σ in $\mathbf{M}$. By induction hypothesis,[79] ν_i belongs to $\mathfrak{v}_\delta(\ulcorner n+1 \urcorner + \tau)$. According to definition (F5), there exists $\nu \in \mathfrak{v}_\delta(\tau)$, such that $\mathfrak{sim}_{n+1}(\nu_i, \nu)$ is true. From $\mathfrak{sim}_{n+1}(\mu, \nu_i)$, $\mathfrak{sim}_{n+1}(\nu_i, \nu)$ and the transitivity of the relation $\mathfrak{sim}_{n+1}$ we obtain that $\mathfrak{sim}_{n+1}(\mu, \nu)$ is true. According to definition (F5), $\mu \in \mathfrak{v}_\delta(\ulcorner n+1 \urcorner + \tau)$. ■

J) **Lemma.** *Given a sort κ and $\mu', \mu'' \in \mathbf{M}_\kappa$:*

(1) *If $\mathfrak{sim}^\circ_n(\mu', \mu'')$, then there exist $\mu_1, \ldots, \mu_k$ and natural numbers $n_1, \ldots, n_k \in \{1, \ldots, n\}$, such that $\mu' = \mu'_1$, $\mu'' = \mu''_l$ and for any $i \in \{1, \ldots, k-1\}$, $\mathfrak{sim}^\circ_{n_i}(\mu_i, \mu_{i+1})$ is true and $\mathfrak{sim}_{n_i-1}(\mu_i, \mu_{i+1})$ is not true.*

(2) *If $\mathfrak{sim}_n(\mu', \mu'')$, then there exist $\mu_1, \ldots, \mu_k$ and natural numbers $n_1, \ldots, n_k \in \{1, \ldots, n\}$, such that $\mu' = \mu'_1$, $\mu'' = \mu''_l$ and for any $i \in \{1, \ldots, k-1\}$, $\mathfrak{sim}^\circ_{n_i}(\mu_i, \mu_{i+1})$ is true and $\mathfrak{sim}_{n_i-1}(\mu_i, \mu_{i+1})$ is not true.*

Proof. We are going to prove both conditions of the Lemma by induction on n. It will be enough to prove only (1), because $\mathfrak{sim}_n$ is a transitive closure of $\mathfrak{sim}^\circ_n$, so (2) follows from (1) for any particular n.

When $n = 0$, $\mathfrak{sim}^\circ_n(\mu', \mu'')$ implies that $\mu' = \mu''$, so we can use $k = 1$ and $\mu_1 = \mu' = \mu''$.

Suppose the condition of the Lemma is true for n. We have to prove it for $n+1$. If $\mathfrak{sim}^\circ_n(\mu', \mu'')$ is true, we obtain the required from the induction hypothesis. Otherwise, we can use $k = 2$, $\mu_1 = \mu'$, $\mu_2 = \mu''$ and $n_1 = n+1$. ■

K) **Lemma.** *Given a natural number n and gamma-semitermoid τ over $|\mathbf{M}|$, if $\mu' \in \mathfrak{v}_\delta(\tau)$ and $\mathfrak{sim}_n(\mu, \mu')$ is true, then there exists an embraced by $\ulcorner n \urcorner + \tau$ gamma-semitermoid σ, such that μ belongs to the set of the values of σ in $\mathbf{M}$.*

[78] About σ' being embraced by $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \ldots, \triangle_{\kappa_{i-1}}, \sigma, \triangle_{\kappa_{i+1}}, \ldots, \triangle_{\kappa_n})$.

[79] About σ being embraced by $\ulcorner n+1 \urcorner + \tau$.

Proof. According to (J2), whenever $\mathfrak{sim}_n(\nu', \nu'')$ is true, there exist some $\nu_1, \dots, \nu_m$, such that $\nu' = \nu_1$, $\nu'' = \nu_m$ and for any $i \in \{1, \dots, m-1\}$ there exists some $k \in \{1, \dots, n\}$, such that $\mathfrak{sim}^\circ_k(\nu_i, \nu_{i+1})$ is true and $\mathfrak{sim}_{k-1}(\nu_i, \nu_{i+1})$ is not true. Let us call the smallest possible number m *n-distance* from ν' to ν''. Notice that when $m \geq 1$, $\mathfrak{sim}_n(\nu_2, \nu_m)$ is true and the n distance from ν_2 to $\nu_m = \nu''$ is smaller than the n-distance from ν' to ν''.[80]

We are going to prove the Lemma by induction on $\omega.n + m$, where m is the n-distance from μ to μ' and ω is the smallest infinite ordinal.

When $n = 0$, notice that $\mathfrak{sim}_n(\mu, \mu')$ implies that $\mu = \mu'$. On the other hand, when the n-distance from μ to μ' is 0, then, again, $\mu = \mu'$. In both cases, from $\mu = \mu'$ it follows that $\mu \in \mathfrak{v}_\delta(\tau)$, so (H) implies that $\mu \in \mathrm{Val}^\delta_{\mathbf{M}}\, \tau$, hence from definition (25Q) we obtain that there exists an embraced by τ gamma-semitermoid σ, such that μ belongs to the set of the values of σ in $\mathbf{M}$. According to definition (25E2), σ is embraced by $\ulcorner 0 \urcorner + \tau$ as well.

This completes the proof for the case when $n = 0$ or the n-distance from μ to μ' is 0. In the following we will assume that both n and the n-distance from μ to μ' are greater than or equal to 1.

The definition of n-distance implies that there exists some μ'' and a natural number $k \in \{1, \dots, n\}$, such that $\mathfrak{sim}^\circ_k(\mu, \mu'')$ is true, $\mathfrak{sim}_{k-1}(\mu, \mu'')$ is not true, $\mathfrak{sim}_n(\mu'', \mu)$ is true and the n-distance from μ'' to μ' is strictly smaller than the n-distance from μ to μ'.

Since $\mathfrak{sim}_n(\mu'', \mu)$ is true and the n-distance from μ'' to μ' is strictly smaller than the n-distance from μ to μ', by induction hypothesis we obtain an embraced by $\ulcorner n \urcorner + \tau$ gamma-semitermoid σ', such that μ'' belongs to the set of the values of σ' in $\mathbf{M}$. According to (25F), σ' is embraced by σ', so $\mu'' \in \mathrm{Val}^\delta_{\mathbf{M}}\, \sigma'$, hence from (H) it follows that $\mu'' \in \mathfrak{v}_\delta(\sigma')$.

Since $\mathfrak{sim}^\circ_k(\mu, \mu'')$ is true but $\mathfrak{sim}_{k-1}(\mu, \mu'')$ is not true, definition (C2) implies that there exists an accessible functional symbol $\mathtt{f}$ of type $\langle\langle \kappa_1, \dots, \kappa_m \rangle, \lambda\rangle$, a natural number j and some $\mu_1, \dots, \mu_m$ and $\mu''_1, \dots, \mu''_m$ belonging to suitable carriers of $\mathbf{M}$, such that $\mu = \mu_j$, $\mu'' = \mu''_j$ and $\mathfrak{sim}_{n-1}(\mathtt{f}^{\mathbf{M}}\langle \mu_1, \dots, \mu_m \rangle, \mathtt{f}^{\mathbf{M}}\langle \mu''_1, \dots, \mu''_m \rangle)$ is true. Since $\mu'' \in \mathfrak{v}_\delta(\sigma')$ and $\mu'' = \mu''_j$, $\mathtt{f}^{\mathbf{M}}\langle \mu''_1, \dots, \mu''_m \rangle$ belongs to $\mathfrak{v}_\delta(\mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{j-1}}, \sigma', \triangle_{\kappa_{j+1}}, \dots, \triangle_{\kappa_m}))$.[81] By induction hypothesis, there exists an embraced by $\ulcorner n-1 \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{j-1}}, \sigma', \triangle_{\kappa_{j+1}}, \dots, \triangle_{\kappa_m})$ gamma-semitermoid σ'', such that $\mathtt{f}^{\mathbf{M}}\langle \mu_1, \dots, \mu_m \rangle$ belongs to the set of the values of σ'' in $\mathbf{M}$.

[80] Indeed, $\mathfrak{sim}^\circ_k(\nu_i, \nu_{i+1})$ implies $\mathfrak{sim}_k(\nu_i, \nu_{i+1})$, which implies $\mathfrak{sim}^\circ_{k+1}(\nu_i, \nu_{i+1})$, so also $\mathfrak{sim}_{k+1}(\nu_i, \nu_{i+1})$, and so on, hence $\mathfrak{sim}^\circ_n(\nu_i, \nu_{i+1})$.

[81] Since $\mathtt{f}$ is accessible, there exist terms of sorts $\kappa_1, \dots, \kappa_m$, so we are permitted to use the symbols $\triangle_{\kappa_1}, \dots, \triangle_{\kappa_m}$.

Since σ' is embraced by $\ulcorner n \urcorner + \tau$ and σ'' is embraced by $\ulcorner n-1 \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{j-1}}, \sigma', \triangle_{\kappa_{j+1}}, \dots, \triangle_{\kappa_m})$, definition (25E6) implies that $\mathtt{f}_j^{-1}(\sigma'')$ is embraced by $\ulcorner n \urcorner + \tau$. It only remains to notice that μ belongs to the set of the values of $\mathtt{f}_j^{-1}(\sigma'')$. This is so because $\mathtt{f}^{\mathbf{M}}\langle \mu_1, \dots, \mu_m \rangle$ belongs to the set of the values of σ'' in $\mathbf{M}$ and $\mu = \mu_j$. ■

L) **Lemma.** *If the gamma-semitermoid σ is embraced by τ and the gamma-semitermoid ρ is embraced by $\ulcorner n \urcorner + \sigma$, then ρ is embraced by $\ulcorner n \urcorner + \tau$.*

Proof. Suppose that σ is embraced by τ and ρ is embraced by $\ulcorner n \urcorner + \sigma$. We are going to prove by induction on definition (25E) the following statement:

> If ρ' is embraced by τ', then if $\tau' = \ulcorner m \urcorner + \sigma$ for some natural number m, then ρ' is embraced by $\ulcorner m \urcorner + \tau$.

From this we obtain the required taking $\rho' = \rho$ and $\tau' = \ulcorner n \urcorner + \sigma$.

Now we are going to consider cases regarding which item of definition (25E) has been used to prove that ρ' is embraced by τ'.

(1) In this case τ' is a gamma-semitermoid, so it can not be equal to $\ulcorner m \urcorner + \sigma$.

(2) In this case $\tau' = \ulcorner 0 \urcorner + \sigma$ and ρ' is embraced by τ' because ρ' is embraced by σ. But σ is a gamma-semitermoid, so according to (25F), $\rho' = \sigma$, hence ρ' is embraced by τ, so by (25E2), ρ' is embraced by $\ulcorner 0 \urcorner + \tau$.

(3) In this case $\tau' = \ulcorner n+1 \urcorner + \sigma$ and ρ' is embraced by τ' because ρ' is embraced by $\ulcorner n \urcorner + \sigma$. By induction hypothesis, ρ' is embraced by $\ulcorner n \urcorner + \tau$, so by (25E3), ρ' is embraced by $\ulcorner n+1 \urcorner + \tau$.

(4) and **(5)** In these cases τ' can not be equal to $\ulcorner m \urcorner + \sigma$.

(6) In this case $\tau' = \ulcorner n+1 \urcorner + \sigma$ and ρ' is embraced by τ' because there exist gamma-semitermoids σ' and σ'', an accessible functional symbol $\mathtt{f}$ and sorts $\kappa_1, \dots, \kappa_l$, such that $\rho' = \mathtt{f}_j^{-1}(\sigma'')$, σ' is embraced by $\ulcorner n+1 \urcorner + \sigma$ and σ'' is embraced by $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{j-1}}, \sigma', \triangle_{\kappa_{j+1}}, \dots, \triangle_{\kappa_l})$. In this case, by induction hypothesis,[82] σ' is embraced by $\ulcorner n+1 \urcorner + \tau$. Since σ'' is embraced by $\ulcorner n \urcorner + \mathtt{f}(\triangle_{\kappa_1}, \dots, \triangle_{\kappa_{j-1}}, \sigma', \triangle_{\kappa_{j+1}}, \dots, \triangle_{\kappa_l})$, from (25E6) we obtain that $\rho' = \mathtt{f}_j^{-1}(\sigma'')$ is embraced by $\ulcorner n+1 \urcorner + \tau$. ■

M) **Lemma.** $\mathfrak{v}_\delta(\tau) \subseteq \mathrm{Val}^\delta_{\mathbf{M}}\, \tau$ *for any delta-termoid τ over $|\mathbf{M}|$.*

Proof. According to definition (25Q), it will be enough to prove that for any delta-semitermoid τ and for any $\mu \in \mathfrak{v}_\delta(\tau)$, there exists an embraced by τ gamma-semitermoid σ, such that μ belongs to the set of the values of σ in $\mathbf{M}$. We are going to prove this by induction on τ.

[82] About σ' being embraced by $\ulcorner n+1 \urcorner + \sigma$.

We are going to consider cases with respect to the form of τ according to definition (25C).

If $\tau = \ulcorner\mu\urcorner$ for some $\mu \in |\mathbf{M}|$, then τ is not just a delta-semitermoid but also a gamma-semitermoid, so according to (25F), τ is embraced by τ. According to (24H), the set of the values of the gamma-semitermoid $\ulcorner\mu\urcorner$ is $\{\mu\}$. This set, however, is equal to $\mathfrak{v}_\delta(\ulcorner\mu\urcorner) = \mathfrak{v}_\delta(\tau)$.

If $\tau = \triangle_\kappa$ for some sort κ, then τ is not just a delta-semitermoid but also a gamma-semitermoid, so according to (25F), τ is embraced by τ. According to (24H), the set of the values of the gamma-semitermoid $\triangle_\kappa$ is $\mathbf{M}_\kappa$. This set, however, is equal to $\mathfrak{v}_\delta(\triangle_\kappa) = \mathfrak{v}_\delta(\tau)$.

If $\tau = \mathtt{f}(\tau_1, \ldots, \tau_n)$ and $\mu \in \mathfrak{v}_\delta(\tau)$, then $\mu = \mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle$ for some $\mu_1, \ldots, \mu_n$, such that μ_i belongs to the set of the values of τ_i in $\mathbf{M}$ for any $i \in \{1, \ldots, n\}$. By induction hypothesis, there exist gamma-semitermoids $\sigma_1, \ldots, \sigma_n$, embraced, respectively, by $\tau_1, \ldots, \tau_n$, such that μ_i belongs to the set of the values of σ_i in $\mathbf{M}$ for any $i \in \{1, \ldots, n\}$. Let $\sigma = \mathtt{f}(\sigma_1, \ldots, \sigma_n)$. According to (25E4), σ is embraced by τ and according to (24H), $\mu = \mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle$ belongs to the set of the values of σ.

If $\tau = \mathtt{f}_i^{-1}(\tau')$ for some n-ary functional symbol $\mathtt{f}$ and $\mu \in \mathfrak{v}_\delta(\tau)$, then there exist $\mu_1, \ldots, \mu_n$ belonging to suitable carriers of $\mathbf{M}$, such that $\mu = \mu_i$ and $\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle \in \mathfrak{v}_\delta(\tau')$. By induction hypothesis, there exists a gamma-semitermoid σ', embraced by τ', such that $\mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_n\rangle$ belongs to the set of the values of σ' in $\mathbf{M}$. Let $\sigma = \mathtt{f}_i^{-1}(\sigma')$. According to (25E5), σ is embraced by τ and according to (24H), $\mu = \mu_i$ belongs to the set of the values of σ.

Let $\tau = \ulcorner n\urcorner + \tau'$. Then for any $\mu \in \mathfrak{v}_\delta(\tau)$ there exists $\mu' \in \mathfrak{v}_\delta(\tau')$, such that $\mathfrak{sim}_n(\mu, \mu')$ is true. By induction hypothesis, there is an embraced by τ' gamma-semitermoid σ', such that μ' belongs to the set of the values of σ' in $\mathbf{M}$. According to definition (25Q), $\mu' \in \mathrm{Val}^\delta_{\mathbf{M}}\,\sigma'$, so (25F) implies that $\mu' \in \mathfrak{v}_\delta(\sigma')$, hence (K) implies that there is an embraced by $\ulcorner n\urcorner + \sigma'$ gamma-semitermoid, such that μ belongs to the set of its values in $\mathbf{M}$, so (L) implies that there is an embraced by $\ulcorner n\urcorner + \tau$ gamma-semitermoid, such that μ belongs to the set of its values in $\mathbf{M}$. ∎

N) **Proposition.** $\mathfrak{v}_\delta(\tau) = \mathrm{Val}^\delta_{\mathbf{M}}\,\tau$ *for any delta-termoid* τ *over* $|\mathbf{M}|$.

Proof. See (I) and (M). ∎

O) **Definition.** For any epsilon-termoid $\ulcorner n\urcorner + \tau$ over $|\mathbf{M}|$ we are going to define a set $\mathfrak{v}_\varepsilon(\ulcorner n\urcorner + \tau)$, recursively on τ.

(1) For any $\mu \in |\mathbf{M}|$, sort κ and $\mu \in \mathbf{M}_\kappa$, $\mathfrak{v}_\varepsilon(\ulcorner n\urcorner + \ulcorner\mu\urcorner)$ is the set of all $\nu \in \mathbf{M}_\kappa$, such that $\mathfrak{sim}_n(\nu, \mu)$ is true.

(2) For any natural number n, functional symbol $\mathtt{f}$ of type

$\langle\langle\kappa_1,\ldots,\kappa_m\rangle,\lambda\rangle$ and terms $\tau_1,\ldots,\tau_m$ over $|\mathbf{M}|$ of respective sorts $\kappa_1,\ldots,\kappa_m$, $\mathfrak{v}_\varepsilon(\ulcorner n\urcorner+\mathtt{f}(\tau_1,\ldots,\tau_m)$ is the set of all $\nu\in\mathbf{M}_\lambda$, such that for some $\mu_1\in\mathfrak{v}_\varepsilon(\ulcorner n+1\urcorner+\tau_1),\mu_2\in\mathfrak{v}_\varepsilon(\ulcorner n+1\urcorner+\tau_2),\ldots,\mu_m\in\mathfrak{v}_\varepsilon(\ulcorner n+1\urcorner+\tau_m)$, $\mathfrak{sim}_n(\nu,\mathtt{f}^{\mathbf{M}}\langle\mu_1,\ldots,\mu_m\rangle)$ is true.

P) **Proposition.** $\mathfrak{v}_\varepsilon(\ulcorner n\urcorner+\tau)=\mathrm{Val}^\varepsilon_{\mathbf{M}}(\ulcorner n\urcorner+\tau)$ *for any epsilon-termoid* $\ulcorner n\urcorner+\tau$ *over* $|\mathbf{M}|$.

Proof. According to (N) and definition (26R), we have to prove that $\mathfrak{v}_\varepsilon(\ulcorner n\urcorner+\tau)=\mathfrak{v}_\delta(\mathfrak{c}(\ulcorner n\urcorner+\tau))$. We are going to do this by induction on τ.

According to definitions (O1), (F1) and (26D1), when $\tau=\ulcorner\mu\urcorner$ for some $\mu\in|\mathbf{M}|$, then $\mathfrak{v}_\varepsilon(\ulcorner n\urcorner+\tau)=\mathfrak{v}_\varepsilon(\ulcorner n\urcorner+\ulcorner\mu\urcorner)=\{\nu:\mathfrak{sim}_n(\nu,\mu)\}=\{\nu:\mathfrak{sim}_n(\nu,\mathfrak{v}_\delta(\ulcorner\mu\urcorner))\}=\mathfrak{v}_\delta(\ulcorner n\urcorner+\ulcorner\mu\urcorner)=\mathfrak{v}_\delta(\mathfrak{c}(\ulcorner n\urcorner+\ulcorner\mu\urcorner))=\mathfrak{v}_\delta(\mathfrak{c}(\ulcorner n\urcorner+\tau))$.

Now, consider the case when $\tau=\mathtt{f}(\tau_1,\ldots,\tau_m)$. According to (O2), the induction hypothesis and (F),

$$\begin{aligned}
&\mathfrak{v}_\varepsilon(\ulcorner n\urcorner+\tau)=\\
&\quad=\mathfrak{v}_\varepsilon(\ulcorner n\urcorner+\mathtt{f}(\tau_1,\ldots,\tau_m))\\
&\quad=\{\nu:\mathfrak{sim}_n(\nu,\mathtt{f}^{\mathbf{M}}\langle\mu_1,\ldots,\mu_m\rangle)\text{ for some}\\
&\qquad\qquad\mu_1\in\mathfrak{v}_\varepsilon(\ulcorner n+1\urcorner+\tau_1),\ldots,\mu_m\in\mathfrak{v}_\varepsilon(\ulcorner n+1\urcorner+\tau_m)\}\\
&\quad=\{\nu:\mathfrak{sim}_n(\nu,\mathtt{f}^{\mathbf{M}}\langle\mu_1,\ldots,\mu_m\rangle)\text{ for some}\\
&\qquad\qquad\mu_1\in\mathfrak{v}_\delta(\mathfrak{c}(\ulcorner n+1\urcorner+\tau_1)),\ldots,\mu_m\in\mathfrak{v}_\delta(\mathfrak{c}(\ulcorner n+1\urcorner+\tau_m))\}\\
&\quad=\mathfrak{v}_\delta(\ulcorner n\urcorner+\mathtt{f}(\mathfrak{c}(\ulcorner n+1\urcorner+\tau_1),\ldots,\mathfrak{c}(\ulcorner n+1\urcorner+\tau_m)))
\end{aligned}$$

Since $\tau_1,\ldots,\tau_m$ are terms, according to (26M2), $\mathfrak{c}(\tau_i)$ has the form $\ulcorner 0\urcorner+\sigma_i$ for any $i\in\{1,\ldots,m\}$. Consequently, according to (26D2), $\mathfrak{c}(\mathtt{f}(\tau_1,\ldots,\tau_m))=\ulcorner 0\urcorner+\mathtt{f}(1\oplus\mathfrak{c}(\tau_1),\ldots,1\oplus\mathfrak{c}(\tau_m))$, so according to (26D3), $\mathfrak{c}(\ulcorner n\urcorner+\tau)=\mathfrak{c}(\ulcorner n\urcorner+\mathtt{f}(\tau_1,\ldots,\tau_m))=\ulcorner n\urcorner+\mathtt{f}((n+1)\oplus\mathfrak{c}(\tau_1),\ldots,(n+1)\oplus\mathfrak{c}(\tau_m))$. According to (26M8), this is equal to $\ulcorner n\urcorner+\mathtt{f}(\mathfrak{c}(\ulcorner n+1\urcorner+\tau_1),\ldots,\mathfrak{c}(\ulcorner n+1\urcorner+\tau_m))$. Therefore, the last expression in the above displayed sequence of equalities is equal to $\mathfrak{v}_\delta(\mathfrak{c}(\ulcorner n\urcorner+\tau))$. ∎

Q) **Corollary.** *Given a term* τ*, if* $\mu\in\mathrm{Val}^\varepsilon_{\mathbf{M}}(\ulcorner n\urcorner+\tau)$ *and* $\mathfrak{sim}_n(\nu,\mu)$ *is true, then* $\nu\in\mathrm{Val}^\varepsilon_{\mathbf{M}}(\ulcorner n\urcorner+\tau)$.

Proof. If $\tau=\ulcorner\mu'\urcorner$ is a name, then according to (O1), $\mathfrak{sim}_n(\mu,\mu')$ is true. But $\mathfrak{sim}_n$ is a transitive relation, so $\mathfrak{sim}_n(\nu,\mu)$ is also true, hence again by (O1), $\nu\in\mathrm{Val}^\varepsilon_{\mathbf{M}}(\ulcorner n\urcorner+\tau)$.

If $\tau=\mathtt{f}(\tau_1,\ldots,\tau_m)$, then according to (O2), there exist some $\mu_1\in\mathrm{Val}^\varepsilon_{\mathbf{M}}(\ulcorner n+1\urcorner+\tau_1),\ldots,\mu_m\in\mathrm{Val}^\varepsilon_{\mathbf{M}}(\ulcorner n+1\urcorner+\tau_m)$, such that $\mathfrak{sim}_n(\mu,\mathtt{f}^{\mathbf{M}}\langle\mu_1,\ldots,\mu_m\rangle)$ is true. But $\mathfrak{sim}_n$ is a transitive relation, so $\mathfrak{sim}_n(\nu, \mathtt{f}^{\mathbf{M}}\langle\mu_1, \ldots, \mu_m\rangle)$ is also true, hence again by (O2), $\nu \in \mathrm{Val}^{\varepsilon}_{\mathbf{M}}(\ulcorner n \urcorner + \tau)$. ■

§28. STRONG REDUCTORS FOR DELTA- AND EPSILON-TERMOIDS

A) **Lemma.** (1) *For any structure of terms* $\mathbf{M}$*, delta-termoid* τ *over* $|\mathbf{M}|$ *and a natural number* n*,* $(\ulcorner n \urcorner + \tau)^{\mathbf{M}} = \tau^{\mathbf{M}}$*.*

(2) *For any structure of terms* $\mathbf{M}$ *and an epsilon-termoid* $\ulcorner n \urcorner + \tau$ *over* $|\mathbf{M}|$*,* $(\ulcorner n \urcorner + \tau)^{\mathbf{M}} = \tau$*.*

Proof. **(1)** According to the alternative semantics (27F), $(\ulcorner n \urcorner + \tau)^{\mathcal{P}\mathbf{M}}$ is the set of all μ, such that $\mathfrak{sim}_n(\mu, \nu)$ is true for some $\nu \in \tau^{\mathcal{P}\mathbf{M}}$. According to (14N), $\tau^{\mathcal{P}\mathbf{M}} = \{\tau^{\mathbf{M}}\}$. Therefore, we only have to prove that $\mathfrak{sim}_n(\mu, \nu)$ is true if and only if $\mu = \nu$. This follows from (27E).

(2) By induction on τ. If $\tau = \ulcorner \sigma \urcorner$ for some term $\sigma \in |\mathbf{M}|$, then according to the alternative semantics (27O1), $(\ulcorner n \urcorner + \tau)^{\mathcal{P}\mathbf{M}}$ is the set of all $\rho \in |\mathbf{M}|$, such that $\mathfrak{sim}_n(\rho, \sigma)$ is true. According to (27E), $\mathfrak{sim}_n(\rho, \sigma)$ is equivalent to $\rho = \sigma$.

If $\tau = \mathtt{f}(\tau_1, \ldots, \tau_m)$, then according to the alternative semantics (27O2), there exist $\sigma_1, \ldots, \sigma_m$ belonging to suitable algebraic carriers of $\mathbf{M}$, such that $\sigma_1 \in \mathrm{Val}^{\varepsilon}_{\mathbf{M}}(\ulcorner n+1 \urcorner + \tau_1), \ldots, \sigma_m \in \mathrm{Val}^{\varepsilon}_{\mathbf{M}}(\ulcorner n+1 \urcorner + \tau_m)$ and $\mathfrak{sim}_n(\nu, \mathtt{f}^{\mathbf{M}}\langle\sigma_1, \ldots, \sigma_m\rangle)$ is true. By induction hypothesis, $\sigma_1 = \tau_1, \ldots, \sigma_m = \tau_m$ and according to (27E), $\mathfrak{sim}_n(\nu, \mathtt{f}^{\mathbf{M}}\langle\sigma_1, \ldots, \sigma_m\rangle)$ is equivalent to $\nu = \mathtt{f}^{\mathbf{M}}\langle\sigma_1, \ldots, \sigma_m\rangle$. Therefore, $(\ulcorner n \urcorner + \tau)^{\mathbf{M}} = \mathtt{f}^{\mathbf{M}}\langle\tau_1, \ldots, \tau_m\rangle = \tau$. ■

B) **Lemma.** *Provided we interpret the following identities as identities between delta-termoidal expressions over a* `Sort`*-indexed set* X*:*

(1) $\{\mathtt{d}(\tau_1, \ldots, \tau_n) \sim \mathtt{d}(\sigma_1, \ldots, \sigma_n)\}$ *is reducible to* $\{\tau_1 \sim \sigma_1, \ldots, \tau_n \sim \sigma_n\}$ *for any logical symbol* $\mathtt{d}$ *and delta-formuloids* $\varphi_1, \ldots, \varphi_n$ *and* $\psi_1, \ldots, \psi_n$*.*

(2) *The system* $\{\mathtt{p}(\ulcorner \tau_1 \urcorner, \ldots, \ulcorner \tau_n \urcorner) \sim \mathtt{p}(\ulcorner \sigma_1 \urcorner, \ldots, \ulcorner \sigma_n \urcorner)\}$ *is reducible to* $\{\tau_1 \sim \sigma_1, \ldots, \tau_n \sim \sigma_n\}$ *for any predicate symbol* $\mathtt{p}$ *and delta-termoids* $\tau_1, \ldots, \tau_n$ *and* $\sigma_1, \ldots, \sigma_n$ *of suitable sorts.*

(3) $\{\tau \sim \sigma\}$ *is reducible to* $\{\tau \sim \ulcorner 0 \urcorner + \sigma\}$*.*

(4) $\{\ulcorner n \urcorner + \tau \sim \ulcorner k \urcorner + \sigma\}$ *is reducible to* $\{\tau \sim \ulcorner \max\{n, k\} \urcorner + \sigma\}$*.*

(5) *The system* $\{\mathtt{f}(\tau_1, \ldots, \tau_n) \sim \ulcorner k \urcorner + \mathtt{f}(\sigma_1, \ldots, \sigma_n)\}$ *is reducible to* $\{\tau_1 \sim \ulcorner k+1 \urcorner + \sigma_1, \ldots, \tau_n \sim \ulcorner k+1 \urcorner + \sigma_n\}$ *for any functional symbol* $\mathtt{f}$*, natural number* k *and delta-termoids* $\tau_1, \ldots, \tau_n$ *and* $\sigma_1, \ldots, \sigma_n$ *of suitable sorts.*

(6) $\{\tau \sim \ulcorner n \urcorner + \ulcorner \xi \urcorner\}$ *is reducible to* $\{\ulcorner \xi \urcorner \sim \ulcorner n \urcorner + \tau\}$*.*

(7) $\{\ulcorner\xi\urcorner \sim \ulcorner n\urcorner + \ulcorner\xi\urcorner\}$ *is reducible to* $\varnothing$.

(8) *An identity of the form* $\mathtt{d}'(\tau_1, \dots, \tau_n) \sim \mathtt{d}''(\sigma_1, \dots, \sigma_m)$ *where* $\mathtt{d}'$ *and* $\mathtt{d}''$ *are different predicate or logical symbols has no solutions in any algebra.*

(9) *An identity of the form* $\mathtt{f}(\tau_1, \dots, \tau_n) \sim \ulcorner k\urcorner + \mathtt{g}(\sigma_1, \dots, \sigma_m)$ *where* $\mathtt{f}$ *and* $\mathtt{g}$ *are different functional symbols has no solutions in any structure of terms.*

(10) *An identity of the form* $\ulcorner\xi\urcorner \sim \tau$ *where* $\ulcorner\xi\urcorner$ *occurs in* τ *and* τ *contains at least one functional symbol has no solutions in any structure of terms.*

Proof. **(1)** Let $\mathbf{A}$ and $v : X \to |\mathbf{A}|$ be some arbitrary algebra and an assignment function. Then $(\mathtt{d}(\varphi_1, \dots, \varphi_n))[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ is the set of all $\mathtt{d}(\alpha_1, \dots, \alpha_n)$, such that $\alpha_1 \in \varphi_1[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta, \dots, \alpha_n \in \varphi_n[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ and $(\mathtt{d}(\psi_1, \dots, \psi_n))[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ is the set of all $\mathtt{d}(\beta_1, \dots, \beta_n)$, such that $\beta_1 \in \psi_1[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta, \dots, \beta_n \in \psi_n[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$. Since $\mathtt{d}(\alpha_1, \dots, \alpha_n) = \mathtt{d}(\beta_1, \dots, \beta_n)$ if and only if $\alpha_1, \dots, \alpha_n$ are respectively equal to $\beta_1, \dots, \beta_n$, v is a solution in $\mathbf{A}$ of the identity $\mathtt{d}(\varphi_1, \dots, \varphi_n) \sim \mathtt{d}(\psi_1, \dots, \psi_n)$ if and only if v is a solution of all of the identities $\varphi_1 \sim \psi_1, \dots, \varphi_n \sim \psi_n$.

(2) Let $\mathbf{A}$ and $v : X \to |\mathbf{A}|$ be some arbitrary algebra and an assignment function. Then $(\mathtt{d}(\ulcorner\tau_1\urcorner, \dots, \ulcorner\tau_n\urcorner))[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ is the set of all $\mathtt{d}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner)$, such that $\alpha_1 \in \tau_1[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta, \dots, \alpha_n \in \tau_n[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ and $(\mathtt{d}(\ulcorner\sigma_1\urcorner, \dots, \ulcorner\sigma_n\urcorner))[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ is the set of all $\mathtt{d}(\ulcorner\beta_1\urcorner, \dots, \ulcorner\beta_n\urcorner)$, such that $\beta_1 \in \sigma_1[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta, \dots, \beta_n \in \sigma_n[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$. Since $\mathtt{d}(\ulcorner\alpha_1\urcorner, \dots, \ulcorner\alpha_n\urcorner) = \mathtt{d}(\ulcorner\beta_1\urcorner, \dots, \ulcorner\beta_n\urcorner)$ if and only if $\alpha_1, \dots, \alpha_n$ are respectively equal to $\beta_1, \dots, \beta_n$, v is a solution in $\mathbf{A}$ of the identity $\mathtt{d}(\ulcorner\tau_1\urcorner, \dots, \ulcorner\tau_n\urcorner) \sim \mathtt{d}(\ulcorner\sigma_1\urcorner, \dots, \ulcorner\sigma_n\urcorner)$ if and only if v is a solution of all of the identities $\tau_1 \sim \sigma_1, \dots, \tau_n \sim \sigma_n$.

(3) Let $\mathbf{M}$ be an arbitrary structure. According to the alternative semantics (27F), $\mu \in \mathrm{Val}^\delta_{\mathbf{M}}(\ulcorner 0\urcorner + \sigma)$ if and only if there exists $\nu \in \mathrm{Val}^\delta_{\mathbf{M}}\,\sigma$, such that $\mathfrak{sim}_0(\mu, \nu)$ is true. But $\mathfrak{sim}_0(\mu, \nu)$ is equivalent to $\mu = \nu$, so $\mathrm{Val}^\delta_{\mathbf{M}}(\ulcorner 0\urcorner + \sigma) = \mathrm{Val}^\delta_{\mathbf{M}}\,\sigma$.

(4) Let $\mathbf{A}$ and $v : X \to |\mathbf{A}|$ be some arbitrary algebra and an assignment function. According to the alternative semantics of (27F), $(\ulcorner n\urcorner + \tau)[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ is the set of all α, such that $\mathfrak{sim}_n(\alpha, \alpha')$ is true for some $\alpha' \in \tau[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$. Also, $(\ulcorner k\urcorner + \sigma)[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ is the set of all β, such that $\mathfrak{sim}_n(\beta, \beta')$ is true for some $\beta' \in \sigma[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$. Considering that $\mathfrak{sim}_n$ implies $\mathfrak{sim}_{\max\{n,k\}}$, $\mathfrak{sim}_k$ implies $\mathfrak{sim}_{\max\{n,k\}}$ and $\mathfrak{sim}_{\max\{n,k\}}$ is a transitive relation, we can conclude that if v is a solution of the identity $\ulcorner n\urcorner + \tau \sim \ulcorner k\urcorner + \sigma$, then there exist $\alpha \in \tau[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ and $\beta \in \sigma[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$, such that $\mathfrak{sim}_{\max\{n,k\}}(\beta, \alpha)$ is true. Consequently, v is a solution of $\tau \sim \ulcorner\max\{n, k\}\urcorner + \sigma$.

(5) Let $\mathbf{A}$ and $v : X \to |\mathbf{A}|$ be some arbitrary algebra of terms and an assignment function. According to (A1), v is a solution in $\mathbf{A}$ of $\mathtt{f}(\tau_1,\dots,\tau_n) \sim \ulcorner k \urcorner + \mathtt{f}(\sigma_1,\dots,\sigma_n)$ if and only if it is a solution of $\mathtt{f}(\tau_1,\dots,\tau_n) \sim \mathtt{f}(\sigma_1,\dots,\sigma_n)$. But $\mathbf{A}$ is a structure of terms, so $(\mathtt{f}(\tau_1,\dots,\tau_n))[\![v]\!]^{\mathbf{A}}_\delta = \mathtt{f}(\tau_1[\![v]\!]^{\mathbf{A}}_\delta,\dots,\tau_n[\![v]\!]^{\mathbf{A}}_\delta)$ and $(\mathtt{f}(\sigma_1,\dots,\sigma_n))[\![v]\!]^{\mathbf{A}}_\delta = \mathtt{f}(\sigma_1[\![v]\!]^{\mathbf{A}}_\delta,\dots,\sigma_n[\![v]\!]^{\mathbf{A}}_\delta)$, hence this is so if and only if $\tau_1[\![v]\!]^{\mathbf{A}}_\delta = \sigma_1[\![v]\!]^{\mathbf{A}}_\delta,\dots,\tau_1[\![v]\!]^{\mathbf{A}}_\delta = \sigma_n[\![v]\!]^{\mathbf{A}}_\delta$. According to (A1), $\sigma_1[\![v]\!]^{\mathbf{A}}_\delta = (\ulcorner k+1 \urcorner + \sigma_1)[\![v]\!]^{\mathbf{A}}_\delta,\dots,\ \sigma_n[\![v]\!]^{\mathbf{A}}_\delta = (\ulcorner k+1 \urcorner + \sigma_n)[\![v]\!]^{\mathbf{A}}_\delta$, hence this is so if and only if v is a solution in $\mathbf{A}$ of the system $\{\tau_1 \sim \ulcorner k+1 \urcorner + \sigma_1,\dots,\tau_1 \sim \ulcorner k+1 \urcorner + \sigma_n\}$.

It only remains to see that when $\mathbf{A}$ is an arbitrary algebra (not necessarily algebra of terms), any solution of $\mathtt{f}(\tau_1,\dots,\tau_n) \sim \ulcorner k \urcorner + \mathtt{f}(\sigma_1,\dots,\sigma_n)$ is a solution also of $\{\tau_1 \sim \ulcorner k+1 \urcorner + \sigma_1,\dots,\tau_1 \sim \ulcorner k+1 \urcorner + \sigma_n\}$. Suppose that v is a solution in $\mathbf{A}$ of $\mathtt{f}(\tau_1,\dots,\tau_n) \sim \ulcorner k \urcorner + \mathtt{f}(\sigma_1,\dots,\sigma_n)$. Then there exist some μ and ν, such that $\mu \in (\mathtt{f}(\tau_1,\dots,\tau_n))[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$, $\nu \in (\mathtt{f}(\sigma_1,\dots,\sigma_n))[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ and $\mathfrak{sim}_k(\mu,\nu)$ is true. In any terminator $[\![v]\!]_\delta$ is a homomorphism and, in the delta-terminator, $\mathrm{Val}^\delta_{\mathbf{A}}$ is a homomorphism as well (see 25Q). Consequently, $[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$ is a homomorphism. In addition, according to (25L2), $\mathtt{f}(\tau_1,\dots,\tau_n) = \mathtt{f}^{[\![X]\!]_\delta}\langle\tau_1,\dots,\tau_n\rangle$ and $\mathtt{f}(\sigma_1,\dots,\sigma_n) = \mathtt{f}^{[\![X]\!]_\delta}\langle\sigma_1,\dots,\sigma_n\rangle$. Therefore, there exist $\mu_1,\dots,\mu_n$ and $\nu_1,\dots,\nu_n$ belonging to suitable carriers of $\mathbf{A}$, such that $\mu = \mathtt{f}^{\mathbf{A}}\langle\mu_1,\dots,\mu_n\rangle$, $\nu = \mathtt{f}^{\mathbf{A}}\langle\nu_1,\dots,\nu_n\rangle$, $\mu_1 \in \tau_1[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta,\dots$, $\mu_n \in \tau_n[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$, $\nu_1 \in \sigma_1[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta,\dots$, $\nu_1 \in \sigma_1[\![v]\!]^{\mathcal{P}\mathbf{A}}_\delta$.

Since $\mathfrak{sim}_k(\mu,\nu)$ is true, $\mathfrak{sim}_k(\mathtt{f}^{\mathbf{A}}\langle\mu_1,\dots,\mu_n\rangle,\mathtt{f}^{\mathbf{A}}\langle\nu_1,\dots,\nu_n\rangle)$ is true as well. According to definition (27C2), $\mathfrak{sim}_{k+1}(\mu_1,\nu_1),\dots,\mathfrak{sim}_{k+1}(\mu_n,\nu_n)$ are true, so v is a solution in $\mathbf{A}$ of $\{\tau_1 \sim \ulcorner k+1 \urcorner + \sigma_1,\dots,\tau_1 \sim \ulcorner k+1 \urcorner + \sigma_n\}$.

(6) Let $\mathbf{M}$ and $v : X \to |\mathbf{M}|$ be an arbitrary structure and an assignment function. According to the alternative semantics (27F), v is a solution of $\{\tau \sim \ulcorner n \urcorner + \ulcorner \xi \urcorner\}$ if and only if there exists μ, such that $\mu \in \mathrm{Val}^\delta_{\mathbf{M}}\,\tau$ and $\mathfrak{sim}_n(\mu, v\xi)$ is true. This is so if and only if there exists μ, such that $\mu \in \mathrm{Val}^\delta_{\mathbf{M}}\,\tau$ and $\mathfrak{sim}_n(v\xi,\mu)$ is true. According to the alternative semantics (27F), this is so if and only if v is a solution of $\{\ulcorner \xi \urcorner \sim \ulcorner n \urcorner + \tau\}$.

(7) is valid because any assignment function $v : X \to |\mathbf{A}|$ is a solution of $\{\ulcorner \xi \urcorner \sim \ulcorner n \urcorner + \ulcorner \xi \urcorner\}$.

(8) Let $\mathbf{A}$ and $v : X \to |\mathbf{A}|$ be some arbitrary algebra and an assignment function. According to (25Q) and (25M), $[\![v]\!]^{\mathcal{P}\mathbf{M}}_\delta$ is a homomorphism, so from (12C2) it follows that all elements of $(\mathtt{d}'(\tau_1,\dots,\tau_n))[\![v]\!]^{\mathcal{P}\mathbf{M}}_\delta$ have the form $\mathtt{d}'(\dots)$ and all elements of $(\mathtt{d}''(\sigma_1,\dots,\sigma_n))[\![v]\!]^{\mathcal{P}\mathbf{M}}_\delta$ have the form $\mathtt{d}''(\dots)$.

(9) Let $\mathbf{M}$ and $v : X \to |\mathbf{M}|$ be some arbitrary structure of terms and an assignment function. According to (25Q) and (25M), $[\![v]\!]^{\mathcal{P}\mathbf{M}}_\delta$ is a homomorphism, so $\mathtt{f}(\tau_1,\dots,\tau_n)[\![v]\!]^{\mathbf{M}}_\delta = (\mathtt{f}^{[\![X]\!]_\delta}\langle\tau_1,\dots,\tau_n\rangle)[\![v]\!]^{\mathbf{M}}_\delta = \mathtt{f}^{\mathbf{M}}\langle\tau_1[\![v]\!]^{\mathbf{M}}_\delta,\dots,\tau_n[\![v]\!]^{\mathbf{M}}_\delta\rangle = \mathtt{f}(\tau_1[\![v]\!]^{\mathbf{M}}_\delta,\dots,\tau_n[\![v]\!]^{\mathbf{M}}_\delta)$, hence $\mathtt{f}(\tau_1,\dots,\tau_n)[\![v]\!]^{\mathbf{M}}_\delta$

has the form $\mathtt{f}(\dots)$. Analogously, $\mathtt{g}(\sigma_1,\dots,\sigma_m)[\![v]\!]^{\mathbf{M}}_\delta$ has the form $\mathtt{g}(\dots)$, hence, according to (A1), $(\ulcorner k \urcorner + \mathtt{g}(\sigma_1,\dots,\sigma_m))[\![v]\!]^{\mathbf{M}}_\delta$ has the form $\mathtt{g}(\dots)$ as well.

(10) Let $\mathbf{M}$ and $v : X \to |\mathbf{M}|$ be an arbitrary structure of terms and an assignment function. By induction on τ we are going to prove that $\tau[\![v]\!]^{\mathbf{M}}_\delta$ contains more symbols than $\ulcorner \xi \urcorner [\![v]\!]^{\mathbf{M}}_\delta = v\xi$.

τ can not be a name because it has to contain at least one functional symbol.

If $\tau = \ulcorner k \urcorner + \sigma$, then σ has to contain at least one functional symbol and $\ulcorner \xi \urcorner$ occurs in σ. By induction hypothesis, $\sigma[\![v]\!]^{\mathbf{M}}_\delta$ contains more symbols than $v\xi$. According to (A1), $\tau[\![v]\!]^{\mathbf{M}}_\delta = \sigma[\![v]\!]^{\mathbf{M}}_\delta$.

If $\tau = \mathtt{f}(\tau_1,\dots,\tau_n)$, then ξ occurs in at least one of the termoids $\tau_1,\dots,\tau_i$. Let ξ occurs in τ_i. If τ_i contains at least one functional symbol, then, by induction hypothesis, $\tau_i[\![v]\!]^{\mathbf{M}}_\delta$ contains more symbols than $v\xi$. Otherwise, τ_i has the form $\ulcorner k_1 \urcorner + \dots + \ulcorner k_m \urcorner + \ulcorner \xi \urcorner$, so according to (A1), $\tau_i[\![v]\!]^{\mathbf{M}}_\delta = \ulcorner \xi \urcorner [\![v]\!]^{\mathbf{M}}_\delta = v\xi$. In both cases $\tau_i[\![v]\!]^{\mathbf{M}}_\delta$ contains at least as many symbols, as $v\xi$. According to (25M), $\tau[\![v]\!]^{\mathbf{M}}_\delta$ is equal to $\mathtt{f}(\tau_1[\![v]\!]^{\mathbf{M}}_\delta,\dots,\tau_n[\![v]\!]^{\mathbf{M}}_\delta)$, so $\tau[\![v]\!]^{\mathbf{M}}_\delta$ contains more symbols than $v\xi$. ∎

C) Let the partial function $\mathfrak{f}_\delta$ be defined according to the following rules:

1. If τ and σ are delta-termoidal expressions of logical sort, then τ has the form $\mathtt{d}'(\dots)$ and σ has the form $\mathtt{d}''(\dots)$ for some predicate or logical symbols $\mathtt{d}'$ and $\mathtt{d}''$. In this case:
   - if $\mathtt{d}' \neq \mathtt{d}''$, let $\mathfrak{f}_\delta(\tau \sim \sigma)$ be undefined;
   - if $\mathtt{d}' = \mathtt{d}''$, both are logical symbols, $\tau = \mathtt{d}'(\tau_1,\dots,\tau_n)$ and $\sigma = \mathtt{d}''(\sigma_1,\dots,\sigma_m)$, let $\mathfrak{f}_\delta(\tau \sim \sigma) = \{\tau_1 \sim \sigma_1,\dots,\tau_n \sim \sigma_n\}$.[83]
   - if $\mathtt{d}' = \mathtt{d}''$, both are predicate symbols, $\tau = \mathtt{d}'(\ulcorner \tau_1 \urcorner,\dots,\ulcorner \tau_n \urcorner)$ and $\sigma = \mathtt{d}''(\ulcorner \sigma_1 \urcorner,\dots,\ulcorner \sigma_m \urcorner)$, let $\mathfrak{f}_\delta(\tau \sim \sigma) = \{\tau_1 \sim \sigma_1,\dots,\tau_n \sim \sigma_n\}$.
2. If τ and σ are delta-termoids and σ does not have the form $\ulcorner k \urcorner + \rho$, let $\mathfrak{f}_\delta(\tau \sim \sigma) = \{\tau \sim \ulcorner 0 \urcorner + \sigma\}$.
3. Let $\mathfrak{f}_\delta(\mathtt{f}(\tau_1,\dots,\tau_n) \sim \ulcorner k \urcorner + \mathtt{f}(\sigma_1,\dots,\sigma_n))$ be equal to $\{\tau_1 \sim \ulcorner k+1 \urcorner + \sigma_1,\dots,\tau_n \sim \ulcorner k+1 \urcorner + \sigma_n\}$.
4. $\mathfrak{f}_\delta(\mathtt{f}(\tau_1,\dots,\tau_n) \sim \ulcorner k \urcorner + \mathtt{g}(\sigma_1,\dots,\sigma_m))$ is undefined, if $\mathtt{f}$ and $\mathtt{g}$ are different functional symbols.
5. $\mathfrak{f}_\delta(\ulcorner n \urcorner + \tau \sim \ulcorner k \urcorner + \sigma) = \{\tau \sim \ulcorner \max\{n,k\} \urcorner + \sigma\}$
6. $\mathfrak{f}_\delta(\ulcorner \xi \urcorner \sim \ulcorner n \urcorner + \ulcorner \xi \urcorner) = \varnothing$.
7. $\mathfrak{f}_\delta(\tau \sim \ulcorner n \urcorner + \ulcorner \xi \urcorner) = \{\ulcorner \xi \urcorner \sim \ulcorner n \urcorner + \tau\}$, when τ is not a name.

[83] Notice that $n = m$.

8. $\mathfrak{f}_\delta(\ulcorner\xi\urcorner \sim \ulcorner n\urcorner + \sigma)$ is undefined, when $\sigma \neq \ulcorner\xi\urcorner$.

Notice that whenever $\mathfrak{f}_\delta(\tau \sim \sigma)$ is defined and $\tau' \sim \sigma' \in \mathfrak{f}_\delta(\tau \sim \sigma)$, τ' contains less symbols than τ except in the case when τ and σ are termoids and σ does not have the form $\ulcorner k\urcorner + \rho$. In addition, if τ and σ are termoids, then σ' has the form $\ulcorner k\urcorner + \rho$. Consequently, no infinite sequence $\tau_1 \sim \sigma_1, \tau_2 \sim \sigma_2, \tau_3 \sim \sigma_3, \dots$, such that $\tau_{i+1} \sim \sigma_{i+1} \in \mathfrak{f}_\delta(\tau_i \sim \sigma_i)$ for any i, is possible. Also, notice that any identity of the form $\ulcorner\xi\urcorner \sim \tau$, where $\ulcorner\xi\urcorner$ does not occur in τ, is solving. From this and from lemma (B) we can conclude that $\mathfrak{f}_\delta$ is a strong reductor.

D) **Definition.** Let $\mathfrak{e}_\delta$ be the equaliser corresponding to the strong reductor, defined in (C). According to (18Q), $\mathfrak{e}_\delta$ is termally sound, termally complete and near-complete.

E) **Lemma.** *Provided we interpret the following identities as identities between epsilon-termoidal expressions over a* `Sort`*-indexed set X:*

(1) $\{\mathtt{d}(\tau_1, \dots, \tau_n) \sim \mathtt{d}(\sigma_1, \dots, \sigma_n)\}$ *is reducible to* $\{\tau_1 \sim \sigma_1, \dots, \tau_n \sim \sigma_n\}$ *for any logical symbol* $\mathtt{d}$ *and epsilon-formuloids* $\varphi_1, \dots, \varphi_n$ *and* $\psi_1, \dots, \psi_n$.

(2) *The system* $\{\mathtt{p}(\ulcorner\tau_1\urcorner, \dots, \ulcorner\tau_n\urcorner) \sim \mathtt{p}(\ulcorner\sigma_1\urcorner, \dots, \ulcorner\sigma_n\urcorner)\}$ *is reducible to* $\{\tau_1 \sim \sigma_1, \dots, \tau_n \sim \sigma_n\}$ *for any predicate symbol* $\mathtt{p}$ *and epsilon-termoids* $\tau_1, \dots, \tau_n$ *and* $\sigma_1, \dots, \sigma_n$ *of suitable sorts.*

(3) *The system* $\{\ulcorner n\urcorner + \mathtt{f}(\tau_1, \dots, \tau_m) \sim \ulcorner k\urcorner + \mathtt{f}(\sigma_1, \dots, \sigma_m)\}$ *is reducible to* $\{\ulcorner l+1\urcorner + \tau_1 \sim \ulcorner l+1\urcorner + \sigma_1, \dots, \ulcorner l+1\urcorner + \tau_m \sim \ulcorner l+1\urcorner + \sigma_m\}$, *where* $l = \max\{n, k\}$, *for any functional symbol* $\mathtt{f}$, *natural numbers* n *and* k *and terms* $\tau_1, \dots, \tau_m$ *and* $\sigma_1, \dots, \sigma_m$ *of suitable sorts.*

(4) *The system* $\{\ulcorner n\urcorner + \ulcorner\xi\urcorner \sim \ulcorner k\urcorner + \mathtt{f}(\tau_1, \dots, \tau_m)\}$ *is reducible to* $\{\ulcorner 0\urcorner + \ulcorner\xi\urcorner \sim \ulcorner\max\{n, k\}\urcorner + \mathtt{f}(\tau_1, \dots, \tau_m)\}$ *for any functional symbol* $\mathtt{f}$, *natural numbers* n *and* k, *terms* $\tau_1, \dots, \tau_m$ *of suitable sorts and* $\xi \in X$.

(5) *The system* $\{\ulcorner n\urcorner + \mathtt{f}(\tau_1, \dots, \tau_m) \sim \ulcorner k\urcorner + \ulcorner\xi\urcorner\}$ *is reducible to* $\{\ulcorner 0\urcorner + \ulcorner\xi\urcorner \sim \ulcorner\max\{n, k\}\urcorner + \mathtt{f}(\tau_1, \dots, \tau_m)\}$ *for any functional symbol* $\mathtt{f}$, *natural numbers* n *and* k, *terms* $\tau_1, \dots, \tau_m$ *of suitable sorts and* $\xi \in X$.

(6) *The system* $\{\ulcorner n\urcorner + \ulcorner\xi\urcorner \sim \ulcorner k\urcorner + \ulcorner\eta\urcorner\}$ *is reducible to* $\{\ulcorner 0\urcorner + \ulcorner\xi\urcorner \sim \ulcorner\max\{n, k\}\urcorner + \ulcorner\eta\urcorner\}$ *for any natural numbers* n *and* k *and* $\xi, \eta \in X$.

(7) *The system* $\{\ulcorner n\urcorner + \ulcorner\xi\urcorner \sim \ulcorner k\urcorner + \ulcorner\xi\urcorner\}$ *is reducible to* $\varnothing$ *for any natural numbers* n *and* k *and* $\xi \in X$.

(8) *An identity of the form* $\mathtt{d}'(\tau_1, \dots, \tau_n) \sim \mathtt{d}''(\sigma_1, \dots, \sigma_m)$ *where* $\mathtt{d}'$ *and* $\mathtt{d}''$ *are different predicate or logical symbols has no solutions in any algebra.*

(9) *An identity of the form* $\ulcorner n\urcorner + \mathtt{f}(\tau_1, \dots, \tau_m) \sim \ulcorner k\urcorner + \mathtt{g}(\sigma_1, \dots, \sigma_l)$ *where* $\mathtt{f}$ *and* $\mathtt{g}$ *are different functional symbols has no solutions in any struc-*

ture of terms.

(10) *An identity of the form* $\ulcorner n \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \tau$, *where* $\ulcorner \xi \urcorner$ *occurs in* τ *and* τ *contains at least one functional symbol, has no solutions in any structure of terms.*

(11) *An identity of the form* $\ulcorner n \urcorner + \tau \sim \ulcorner k \urcorner + \ulcorner \xi \urcorner$, *where* $\ulcorner \xi \urcorner$ *occurs in* τ *and* τ *contains at least one functional symbol, has no solutions in any structure of terms.*

Proof. **(1)** Analogously to the proof of (B1).

(2) Analogously to the proof of (B2).

(3) Let $\mathbf{A}$ and $v : X \to |\mathbf{A}|$ be some arbitrary algebra of terms and an assignment function. According to (A2), v is a solution in $\mathbf{A}$ of $\ulcorner n \urcorner + \mathtt{f}(\tau_1, \dots, \tau_m) \sim \ulcorner k \urcorner + \mathtt{f}(\sigma_1, \dots, \sigma_m)$ if and only if $\mathtt{f}(\tau_1[v], \dots, \tau_m[v]) = \mathtt{f}(\sigma_1[v], \dots, \sigma_m[v])$, that is, if and only if $\tau_1[v] = \sigma_1[v], \dots, \tau_m[v] = \sigma_m[v]$. Also, according to (A2), this is so if and only if v is a solution in $\mathbf{A}$ of the system $\{\ulcorner l+1 \urcorner + \tau_1 \sim \ulcorner l+1 \urcorner + \sigma_1, \dots, \ulcorner l+1 \urcorner + \tau_m \sim \ulcorner l+1 \urcorner + \sigma_m\}$.

This completes the proof of the termal equivalency of both systems. It only remains to see that when $\mathbf{A}$ is an arbitrary algebra (not necessarily an algebra of terms), any solution of $\ulcorner n \urcorner + \mathtt{f}(\tau_1, \dots, \tau_m) \sim \ulcorner k \urcorner + \mathtt{f}(\sigma_1, \dots, \sigma_m)$ is a solution also of the system $\{\ulcorner n+1 \urcorner + \tau_1 \sim \ulcorner k+1 \urcorner + \sigma_1, \dots, \ulcorner n+1 \urcorner + \tau_m \sim \ulcorner k+1 \urcorner + \sigma_m\}$.

Suppose that $v : X \to |\mathbf{A}|$ is a solution in $\mathbf{A}$ of the identity $\ulcorner n \urcorner + \mathtt{f}(\tau_1, \dots, \tau_m) \sim \ulcorner k \urcorner + \mathtt{f}(\sigma_1, \dots, \sigma_m)$. Then there exists some λ, such that $\lambda \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner n \urcorner + \mathtt{f}(\tau_1[v], \dots, \tau_m[v]))$ and $\lambda \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner k \urcorner + \mathtt{f}(\sigma_1[v], \dots, \sigma_m[v]))$. According to the alternative semantics (27O2), there exist some $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_m$ belonging to suitable carriers of $\mathbf{A}$, such that $\mathfrak{sim}_n(\lambda, \mathtt{f}^{\mathbf{A}}\langle \alpha_1, \dots, \alpha_m \rangle)$ is true, $\mathfrak{sim}_k(\lambda, \mathtt{f}^{\mathbf{A}}\langle \beta_1, \dots, \beta_m \rangle)$ is true, $\alpha_1 \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner n+1 \urcorner + \tau_1[v]), \dots, \alpha_m \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner n+1 \urcorner + \tau_m[v])$, $\beta_1 \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner k+1 \urcorner + \sigma_1[v]), \dots, \beta_m \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner k+1 \urcorner + \sigma_m[v])$.

Let $l = \max\{n, k\}$ and pick an arbitrary $i \in \{1, \dots, m\}$. Then $\mathfrak{sim}_l(\lambda, \mathtt{f}^{\mathbf{A}}\langle \alpha_1, \dots, \alpha_m \rangle)$ is true and $\mathfrak{sim}_l(\lambda, \mathtt{f}^{\mathbf{A}}\langle \beta_1, \dots, \beta_m \rangle)$ is true, so $\mathfrak{sim}_l(\mathtt{f}^{\mathbf{A}}\langle \alpha_1, \dots, \alpha_m \rangle, \mathtt{f}^{\mathbf{A}}\langle \beta_1, \dots, \beta_m \rangle)$ is true, hence according to (27C2), $\mathfrak{sim}_{l+1}(\alpha_i, \beta_i)$ is true. In addition to this, from $\alpha_i \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner n+1 \urcorner + \tau_i[v])$ it follows that $\alpha_i \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner l+1 \urcorner + \tau_i[v])$ and from $\beta_i \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner k+1 \urcorner + \sigma_i[v])$ it follows that $\beta_i \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner l+1 \urcorner + \sigma_i[v])$, so (27Q) and $\mathfrak{sim}_{k+1}(\alpha_i, \beta_i)$ imply that $\alpha_i \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner l+1 \urcorner + \sigma_i[v])$. Since α_i belongs both to $\mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner l+1 \urcorner + \tau_i[v])$ and $\mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner l+1 \urcorner + \sigma_i[v])$, v is a solution of the identity $\ulcorner l+1 \urcorner + \tau_i \sim \ulcorner l+1 \urcorner + \sigma_i$.

(4) Let $\mathbf{A}$ and $v : X \to |\mathbf{A}|$ be some arbitrary algebra of terms and an assignment function. According to (A2), v is a solution in $\mathbf{A}$ of

$\ulcorner n\urcorner + \ulcorner\xi\urcorner \sim \ulcorner k\urcorner + \mathtt{f}(\tau_1,\dots,\tau_m)$ if and only if $v\xi = \mathtt{f}(\tau_1[v],\dots,\tau_m[v])$. Also, according to (A2), this is so if and only if v is a solution in $\mathbf{A}$ of the system $\ulcorner 0\urcorner + \ulcorner\xi\urcorner \sim \ulcorner\max\{n,k\}\urcorner + \mathtt{f}(\tau_1,\dots,\tau_m)$.

This completes the proof of the termal equivalency of both systems. It only remains to see that when $\mathbf{A}$ is an arbitrary algebra (not necessarily an algebra which is a structure of terms), any solution of $\ulcorner n\urcorner + \ulcorner\xi\urcorner \sim \ulcorner k\urcorner + \mathtt{f}(\tau_1,\dots,\tau_m)$ is a solution also of the system $\ulcorner 0\urcorner + \ulcorner\xi\urcorner \sim \ulcorner\max\{n,k\}\urcorner + \mathtt{f}(\tau_1,\dots,\tau_m)$.

Suppose that $v : X \to |\mathbf{A}|$ is a solution in $\mathbf{A}$ of the identity $\ulcorner n\urcorner + \ulcorner\xi\urcorner \sim \ulcorner k\urcorner + \mathtt{f}(\tau_1,\dots,\tau_m)$. According to the alternative semantics (27O1), there exists some λ, such that $\mathfrak{sim}_n(\lambda, v\xi)$ is true and $\lambda \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner k\urcorner + \mathtt{f}(\tau_1[v],\dots,\tau_m[v]))$. According to (27O2), there exist some $\alpha_1,\dots,\alpha_m$ belonging to suitable carriers of $\mathbf{A}$, such that $\mathfrak{sim}_k(\lambda, \mathtt{f}^{\mathbf{A}}\langle\alpha_1,\dots,\alpha_m\rangle)$ is true, $\alpha_1 \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner k+1\urcorner + \tau_1[v]),\dots,\alpha_m \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner k+1\urcorner + \tau_m[v])$.

Let $l = \max\{n,k\}$. Then, $\mathfrak{sim}_n(\lambda, v\xi)$ implies $\mathfrak{sim}_l(\lambda, v\xi)$ and $\mathfrak{sim}_k(\lambda, \mathtt{f}^{\mathbf{A}}\langle\alpha_1,\dots,\alpha_m\rangle)$ implies $\mathfrak{sim}_l(\lambda, \mathtt{f}^{\mathbf{A}}\langle\alpha_1,\dots,\alpha_m\rangle)$, hence because of the transitivity of $\mathfrak{sim}_l$, $\mathfrak{sim}_l(v\xi, \mathtt{f}^{\mathbf{A}}\langle\alpha_1,\dots,\alpha_m\rangle)$ also is true. In addition to this, $\alpha_i \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner k+1\urcorner + \tau_i[v])$ implies that $\alpha_i \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner l+1\urcorner + \tau_i[v])$ for any $i \in \{1,\dots,m\}$. Consequently, from (27O2) we obtain that $v\xi \in \mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner l\urcorner + \mathtt{f}(\tau_1[v],\dots,\tau_m[v]))$, so v is a solution of the identity $\ulcorner 0\urcorner + \ulcorner\xi\urcorner \sim \ulcorner l\urcorner + \mathtt{f}(\tau_1,\dots,\tau_m)$.

(5) is analogous to (4).

(6) Let $\mathbf{A}$ and $v : X \to |\mathbf{A}|$ be some arbitrary algebra of terms and an assignment function. According to (A2), v is a solution in $\mathbf{A}$ of $\ulcorner n\urcorner + \ulcorner\xi\urcorner \sim \ulcorner k\urcorner + \ulcorner\eta\urcorner$ if and only if $\ulcorner v\xi\urcorner = \ulcorner v\eta\urcorner$. Also, according to (A2), this is so if and only if v is a solution in $\mathbf{A}$ of the system $\ulcorner 0\urcorner + \ulcorner\xi\urcorner \sim \ulcorner\max\{n,k\}\urcorner + \ulcorner\eta\urcorner$.

This completes the proof of the termal equivalency of both systems. It only remains to see that when $\mathbf{A}$ is an arbitrary algebra (not necessarily an algebra which is a structure terms), any solution of $\ulcorner n\urcorner + \ulcorner\xi\urcorner \sim \ulcorner k\urcorner + \ulcorner\eta\urcorner$ is a solution also of the system $\ulcorner 0\urcorner + \ulcorner\xi\urcorner \sim \ulcorner\max\{n,k\}\urcorner + \ulcorner\eta\urcorner$.

Suppose that $v : X \to |\mathbf{A}|$ is a solution in $\mathbf{A}$ of the identity $\ulcorner n\urcorner + \ulcorner\xi\urcorner \sim \ulcorner k\urcorner + \ulcorner\eta\urcorner$. According to the alternative semantics (27O1), there exists some λ, such that $\mathfrak{sim}_n(\lambda, v\xi)$ is true and $\mathfrak{sim}_k(\lambda, v\eta)$ is true. Let $l = \max\{n,k\}$. Then, $\mathfrak{sim}_n(\lambda, v\xi)$ implies $\mathfrak{sim}_l(\lambda, v\xi)$ and $\mathfrak{sim}_k(\lambda, v\eta)$ implies $\mathfrak{sim}_l(\lambda, v\eta)$,hence because of the transitivity of $\mathfrak{sim}_l$, $\mathfrak{sim}_l(v\xi, v\eta)$ also is true. Consequently, from (27O1) we obtain that $v\xi$ belongs both to $\mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner 0\urcorner + \ulcorner v\xi\urcorner)$ and $\mathrm{Val}^{\varepsilon}_{\mathbf{A}}(\ulcorner l\urcorner + \ulcorner v\eta\urcorner)$, so v is a solution of the identity $\ulcorner 0\urcorner + \ulcorner\xi\urcorner \sim \ulcorner l\urcorner + \ulcorner\eta\urcorner$.

(7) is valid because any assignment function $v : X \to |\mathbf{A}|$ is a solution of $\{\ulcorner n \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \ulcorner \xi \urcorner\}$.

(8) Let $\mathbf{A}$ and $v : X \to |\mathbf{A}|$ be some arbitrary algebra and an assignment function. According to (26R), (26O) and (14D), $[\![v]\!]_\varepsilon^{\mathcal{P}\mathbf{A}}$ is a quasimorphism, so $\mathtt{d}'(\tau_1, \ldots, \tau_n)[\![v]\!]_\varepsilon^{\mathcal{P}\mathbf{A}} = (\mathtt{d}'^{[\![X]\!]_\varepsilon}\langle\tau_1, \ldots, \tau_n\rangle)[\![v]\!]_\varepsilon^{\mathcal{P}\mathbf{A}} = \mathtt{d}'^{\mathcal{P}\mathbf{A}}\langle\tau_1[\![v]\!]_\varepsilon^{\mathcal{P}\mathbf{A}}, \ldots, \tau_n[\![v]\!]_\varepsilon^{\mathcal{P}\mathbf{A}}\rangle$, hence all elements of $\mathtt{d}'(\tau_1, \ldots, \tau_n)[\![v]\!]_\varepsilon^{\mathcal{P}\mathbf{A}}$ have the form $\mathtt{d}'(\ldots)$. Analogously, all elements of $\mathtt{d}''(\sigma_1, \ldots, \sigma_m)[\![v]\!]_\varepsilon^{\mathcal{P}\mathbf{A}}$ have the form $\mathtt{d}''(\ldots)$.

(9) Let $\mathbf{M}$ and $v : X \to |\mathbf{M}|$ be some arbitrary structure of terms and an assignment function. According to (26O), $(\ulcorner n \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m))[\![v]\!]_\varepsilon = \ulcorner n \urcorner + \mathtt{f}(\tau_1[v], \ldots, \tau_m[v])$. According to (A2), $(\ulcorner n \urcorner + \mathtt{f}(\tau_1[v], \ldots, \tau_m[v]))^{\mathbf{M}} = \mathtt{f}(\tau_1[v], \ldots, \tau_m[v])$. Consequently, $(\ulcorner n \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m))[\![v]\!]_\varepsilon^{\mathbf{M}}$ has the form $\mathtt{f}(\ldots)$. Analogously, $(\ulcorner k \urcorner + \mathtt{g}(\sigma_1, \ldots, \sigma_l))[\![v]\!]_\varepsilon^{\mathbf{M}}$ has the form $\mathtt{g}(\ldots)$.

(10) Let $\mathbf{M}$ and $v : X \to |\mathbf{M}|$ be some arbitrary structure of terms and an assignment function. According to (26O) and (A2), $(\ulcorner n \urcorner + \ulcorner \xi \urcorner)[\![v]\!]_\varepsilon^{\mathbf{M}} = (\ulcorner n \urcorner + \ulcorner v\xi \urcorner)^{\mathbf{M}} = \ulcorner v\xi \urcorner$ and $(\ulcorner k \urcorner + \tau)[\![v]\!]_\varepsilon^{\mathbf{M}} = (\ulcorner k \urcorner + \tau[v])^{\mathbf{M}} = \tau[v]$. Therefore, v is a solution of the identity $\ulcorner n \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \tau$ in $\mathbf{M}$ if and only if $\ulcorner v\xi \urcorner = \tau[v]$. This, however, is impossible because $\ulcorner v\xi \urcorner$ occurs in $\tau[v]$ and, in addition to this, $\tau[v]$ contains at least one functional symbol.

(11) is analogous to (10). ■

F) Let the partial function $\mathfrak{f}_\varepsilon$ be defined according to the following rules:

1. If τ and σ are epsilon-termoidal expressions of logical sort, then τ has the form $\mathtt{d}'(\ldots)$ and σ has the form $\mathtt{d}''(\ldots)$ for some predicate or logical symbols $\mathtt{d}'$ and $\mathtt{d}''$. In this case:
   - if $\mathtt{d}' \neq \mathtt{d}''$, let $\mathfrak{f}_\varepsilon(\tau \sim \sigma)$ be undefined;
   - if $\mathtt{d}' = \mathtt{d}''$, both are logical symbols, $\tau = \mathtt{d}'(\tau_1, \ldots, \tau_n)$ and $\sigma = \mathtt{d}''(\sigma_1, \ldots, \sigma_m)$, let $\mathfrak{f}_\varepsilon(\tau \sim \sigma) = \{\tau_1 \sim \sigma_1, \ldots, \tau_n \sim \sigma_n\}$.[84]
   - if $\mathtt{d}' = \mathtt{d}''$, both are predicate symbols, $\tau = \mathtt{d}'(\ulcorner \tau_1 \urcorner, \ldots, \ulcorner \tau_n \urcorner)$ and $\sigma = \mathtt{d}''(\ulcorner \sigma_1 \urcorner, \ldots, \ulcorner \sigma_m \urcorner)$, let $\mathfrak{f}_\varepsilon(\tau \sim \sigma) = \{\tau_1 \sim \sigma_1, \ldots, \tau_n \sim \sigma_n\}$.
2. Let $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m) \sim \ulcorner k \urcorner + \mathtt{f}(\sigma_1, \ldots, \sigma_m))$ be equal to $\{\ulcorner l+1 \urcorner + \tau_1 \sim \ulcorner l+1 \urcorner + \sigma_1, \ldots, \ulcorner l+1 \urcorner + \tau_m \sim \ulcorner l+1 \urcorner + \sigma_m\}$, where $l = \max\{n, k\}$.
3. $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m) \sim \ulcorner k \urcorner + \mathtt{g}(\sigma_1, \ldots, \sigma_l))$ is undefined if $\mathtt{f}$ and $\mathtt{g}$ are different functional symbols.
4. If $n \neq 0$, let $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m))$ be equal to $\{\ulcorner 0 \urcorner + \ulcorner \xi \urcorner \sim \ulcorner \max\{n, k\} \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m)\}$.
5. Let $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m) \sim \ulcorner k \urcorner + \ulcorner \xi \urcorner)$ be equal to $\{\ulcorner 0 \urcorner + \ulcorner \xi \urcorner \sim \ulcorner \max\{n, k\} \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m)\}$.

[84] Notice that $n = m$.

6. $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \ulcorner \eta \urcorner) = \{\ulcorner 0 \urcorner + \ulcorner \xi \urcorner \sim \ulcorner \max\{n, k\} \urcorner + \ulcorner \eta \urcorner\}$, if $n \neq 0$ and $\xi \neq \eta$.
7. $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \ulcorner \xi \urcorner) = \varnothing$
8. $\mathfrak{f}_\varepsilon(\ulcorner 0 \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \tau)$ is undefined if $\tau \neq \ulcorner \xi \urcorner$.

Notice that whenever $\mathfrak{f}_\varepsilon(\tau \sim \sigma)$ is defined and $\tau' \sim \sigma' \in \mathfrak{f}_\varepsilon(\tau \sim \sigma)$, τ' contains less symbols than τ except in the case when τ has the form $\ulcorner n \urcorner + \ulcorner \xi \urcorner$ for some $n \neq 0$, in which case, however, $\mathfrak{f}_\varepsilon(\tau' \sim \sigma')$ is undefined. Consequently, no infinite sequence $\tau_1 \sim \sigma_1, \tau_2 \sim \sigma_2, \tau_3 \sim \sigma_3, \ldots$, such that $\tau_{i+1} \sim \sigma_{i+1} \in \mathfrak{f}_\varepsilon(\tau_i \sim \sigma_i)$ for any i, is possible. Also, notice that according to (26S), $\mathrm{Nam}^\varepsilon_X\, \xi = \ulcorner 0 \urcorner + \ulcorner \xi \urcorner$, so any identity of the form $\ulcorner 0 \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \tau$, where $\ulcorner \xi \urcorner$ does not occur in τ, is a solving one. From this and from lemma (E) we can conclude that $\mathfrak{f}_\varepsilon$ is a strong reductor.

G) **Definition.** Let $\mathfrak{e}_\varepsilon$ be the equaliser corresponding to the strong reductor, defined in (F). According to (18Q), $\mathfrak{e}_\varepsilon$ is termally sound, termally complete and near-complete.

Finite Model Property of VED

§29. "ALMOST-EVERYWHERE" IMPLIES "SOME FINITE"

A) In (23Q) we proved that if the positive hyperresolution with clausoids saturates, then the initial set of clausoids is universally satisfiable in almost any normal algebra. In this section we are going to prove that if a finite set of clausoids is universally satisfiable in almost any algebra, then it is universally satisfiable in some normal algebra with finite carriers.

We start by proving that any finite termal system, unsolvable in some normal algebra, is unsolvable in a normal algebra with finite algebraic carriers. This result has been proved by Gladstone (for the usual case of algebras with only one sort) in [12].[85] According to Gladstone, this result (in a different form) is asserted for first time by Herbrand in [15]. However, Herbrand offered no proof. In a footnote on p. 161 in [17], Hilbert and Bernays remark that this result has deceptive plausibility and quote Schutte as having shown that the proof is by no means obvious.

In (C) I am going to present my version of this result with somewhat shorter proof (published in [32]). Afterwards, in (L) and (M), I am going to prove that the same is true for any finite delta- or epsilon-termoidal system.

B) **Lemma.** *Let τ be a term over $\mathbb{X}$ and $\xi \in \mathbb{X}$ be such that $\tau \neq \ulcorner\xi\urcorner$ and $\ulcorner\xi\urcorner$ occurs in τ. Then there exists a normal algebra with finite algebraic carriers, such that none of the identities $\ulcorner\xi\urcorner \sim \tau$ and $\tau \sim \ulcorner\xi\urcorner$ is solvable in it.*

[85] The result of Gladstone is not for finite termal systems, but for for one termal identity. This is not a significant limitation. Suppose the system $\{\tau_1 \sim \sigma_1, \ldots, \tau_n \sim \sigma_n\}$ has no solutions in a normal algebra $\mathbf{A}$. Let $\mathtt{f}$ be some new n-ary functional symbol. Then the identity $\mathtt{f}(\tau_1, \ldots, \tau_n) \sim \mathtt{f}(\sigma_1, \ldots, \sigma_n)$ is unsolvable in $[|\mathbf{A}|]$. Therefore, from the result of Gladstone we can conclude that this identity is unsolvable in some algebra with finite carrier. Obviously, in the same finite algebra, the system is unsolvable as well.

<u>Proof.</u> Since the term τ is not a name, it has the form $\mathtt{f}(\tau_1, \ldots, \tau_n)$ for some functional symbol $\mathtt{f}$ and terms $\tau_1, \ldots, \tau_n$. There exists some $k \in \{1, \ldots, n\}$, such that $\ulcorner\xi\urcorner$ occurs in τ_k.

We have to define an algebra $\mathbf{A}$ with finite algebraic carriers, such that the identities $\ulcorner\xi\urcorner \sim \tau$ and $\tau \sim \ulcorner\xi\urcorner$ have no solution in $\mathbf{A}$.

Let $T = \{T_\kappa\}_{\kappa\in\mathtt{Sort}}$ be the $\mathtt{Sort}$-indexed set of all subterms of τ in which the name $\ulcorner\xi\urcorner$ occurs, so that T_κ contains the terms of sort κ. If $T_\kappa \neq \varnothing$, let T_κ be the carrier of sort κ of $\mathbf{A}$. Otherwise, let the carrier of sort κ be an arbitrary finite set. For any functional symbol $\mathtt{g}$ of type $\langle\langle\kappa_1, \ldots, \kappa_m\rangle, \lambda\rangle$, define $\mathtt{g}^{\mathbf{A}}\langle\alpha_1, \ldots, \alpha_m\rangle$[86] arbitrarily if $T_\lambda = \varnothing$. Otherwise, let $\mathtt{g}^{\mathbf{A}}\langle\alpha_1, \ldots, \alpha_m\rangle$ be equal to the set

$$\{\mathtt{g}(\sigma_1, \ldots, \sigma_m) \in T_\lambda : \forall i \in \{1, \ldots, m\}(\sigma_i \in T_{\kappa_i} \Rightarrow \sigma_i \in \alpha_i)\} \cup \Delta$$

where

$$\Delta = \begin{cases} \{\ulcorner\xi\urcorner\} & \text{if } \mathtt{g} = \mathtt{f} \text{ and } \tau_k \notin \alpha_k, \\ \varnothing & \text{if } \mathtt{g} \neq \mathtt{f} \text{ or } \tau_k \in \alpha_k. \end{cases}$$

Let $v : \mathbb{X} \to |\mathbf{A}|$ be an arbitrary assignment function. By induction on the complexity of the term σ we are going to prove that if $\sigma \in T$ then

$$\sigma \in \sigma[v]^{\mathbf{A}} \iff \ulcorner\xi\urcorner \in v\xi \qquad (\sharp)$$

If σ is a name and $\sigma \in T$, then $\sigma = \ulcorner\xi\urcorner$, so in this case $(\sharp)$ is obvious.

Otherwise, σ has the form $\mathtt{g}(\sigma_1, \ldots, \sigma_m)$ for some functional symbol $\mathtt{g}$ of type $\langle\langle\kappa_1, \ldots, \kappa_m\rangle, \lambda\rangle$ and terms $\sigma_1, \ldots, \sigma_m$ of respective sorts $\kappa_1, \ldots, \kappa_m$. If $\sigma \in T$, according to the definition of the interpretation of $\mathbf{A}$,

$$\sigma \in \sigma[v]^{\mathbf{A}} \iff \forall i \in \{1, \ldots, m\}(\sigma_i \in T_{\kappa_i} \Rightarrow \sigma_i \in \sigma_i[v]^{\mathbf{A}})$$

By the induction hypothesis, if $\sigma_i \in T_{\kappa_i}$, then $\sigma_i \in \sigma_i[v]^{\mathbf{A}} \iff \ulcorner\xi\urcorner \in v\xi$ for any $i \in \{1, \ldots, m\}$. Therefore,

$$\sigma \in \sigma[v]^{\mathbf{A}} \iff \forall i \in \{1, \ldots, m\}(\sigma_i \in T_{\kappa_i} \Rightarrow \ulcorner\xi\urcorner \in v\xi)$$

If $\sigma \in T$, then there exists at least one i, such that $\sigma_i \in T_{\kappa_i}$, so this completes the proof of $(\sharp)$.

Since $\tau_k \in T$, from $(\sharp)$ it follows that $\tau_k \in \tau_k[v]^{\mathbf{A}}$ is equivalent to $\ulcorner\xi\urcorner \in v\xi$. But $v\xi = \ulcorner\xi\urcorner[v]^{\mathbf{A}}$, so $\tau_k \in \tau_k[v]^{\mathbf{A}}$ is equivalent to $\ulcorner\xi\urcorner \in \ulcorner\xi\urcorner[v]^{\mathbf{A}}$.

On the other hand, from the definition of the interpretation of $\mathbf{A}$ it follows that $\ulcorner\xi\urcorner \in \tau[v]^{\mathbf{A}}$ if and only if $\tau_k \notin \tau_k[v]^{\mathbf{A}}$.

[86] Notice that α_i is a subset of T_{κ_i}.

Consequently, $\ulcorner\xi\urcorner[v]^{\mathbf{A}} \neq \tau[v]^{\mathbf{A}}$. ∎

C) **Theorem.** *Any finite termal system over $\mathbb{X}$ unsolvable in some normal algebra is unsolvable in a normal algebra with finite algebraic carriers.*

Proof. Denote by k_4 the number of the identities in the system, by k_1 the number of the names in the system, by k_2 the depth of the most complex termal expression in the system if $k_4 \neq 0$ and $k_2 = 0$ if $k_4 = 0$. By k_3 denote the number of the identities containing a termal expression with depth k_2. We are going to prove the theorem by induction on the ordinal $k_1\omega^3 + k_2\omega^2 + k_3\omega + k_4$.

Case 0. The system contains no identities (i.e. it is the empty set). In this case the theorem is trivially true because in any algebra any assignment function is solution of the empty system.

Case 1. Among the identities containing a termal expression with complexity k_2, some identity has the form $\ulcorner\xi\urcorner \sim \ulcorner\xi\urcorner$ for some $\xi \in \mathbb{X}$. In this case, remove this identity from the system and use the induction hypothesis for the resulting system.

Case 2. Among the identities containing termal expression with complexity k_2, some identity has the form $\ulcorner\xi\urcorner \sim \tau$ or $\tau \sim \ulcorner\xi\urcorner$ where $\ulcorner\xi\urcorner$ is a name occurring in the termal expression τ and $\tau \not\equiv \ulcorner\xi\urcorner$. According to (B), there exist a normal algebra with finite carriers, where this identity is unsolvable, hence the whole system is unsolvable in this algebra.

Case 3. Among the identities containing termal expression with complexity k_2, some identity has the form $\ulcorner\xi\urcorner \sim \tau$ or $\tau \sim \ulcorner\xi\urcorner$ and the name $\ulcorner\xi\urcorner$ does not occur in the termal expression τ. In this case, make a new system containing the other identities replacing everywhere in them the name $\ulcorner\xi\urcorner$ by the termal expression τ. Obviously, if v is a solution of the new system in some algebra $\mathbf{A}$, then v' will be solution of the former system in $\mathbf{A}$, where

$$v'\eta = \begin{cases} v\eta & \text{if } \eta \neq \xi, \\ \tau[v]^{\mathbf{A}} & \text{if } \eta = \xi. \end{cases}$$

Since the original system is unsolvable in some algebra, the new system is unsolvable in the same algebra. By induction hypothesis, there exists an algebra $\mathbf{B}$ with finite carriers where the new system has no solutions.

Obviously, if v is a solution of the original system in some algebra, then v is a solution of the new system in the same algebra. Since the new system is unsolvable in $\mathbf{B}$, the original system is unsolvable in $\mathbf{B}$.

Case 4. Among the identities containing termal expression with complexity k_2, some identity has the form $\mathtt{f}(\tau_1, \ldots, \tau_n) \sim \mathtt{g}(\sigma_1, \ldots, \sigma_m)$

where $\mathtt{f}$ and $\mathtt{g}$ are different functional symbols with types $\langle\langle\kappa'_1,\ldots,\kappa'_n\rangle,\lambda\rangle$ and $\langle\langle\kappa''_1,\ldots,\kappa''_m\rangle,\lambda\rangle$, respectively.

In this case, define the algebra $\mathbf{A}$, so that $\mathbf{A}_\lambda = \{0,1\}$ and the other algebraic carriers of $\mathbf{A}$ are arbitrary non-empty finite sets. Let $\mathtt{f}^{\mathbf{A}}\langle\alpha_1,\ldots,\alpha_n\rangle = 0$ and $\mathtt{g}^{\mathbf{A}}\langle\beta_1,\ldots,\beta_m\rangle = 1$ for arbitrary $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_m$ belonging to suitable carriers of $\mathbf{A}$.

In this way, the identity $\mathtt{f}(\tau_1,\ldots,\tau_n) \sim \mathtt{g}(\sigma_1,\ldots,\sigma_m)$ will be unsolvable in $\mathbf{A}$, so the system will be unsolvable in $\mathbf{A}$ too.

Case 5. Among the identities containing termal expression with complexity k_2, some identity has the form $\mathtt{f}(\tau_1,\ldots,\tau_n) \sim \mathtt{f}(\sigma_1,\ldots,\sigma_n)$ where $\mathtt{f}$ is a functional symbol of type $\langle\langle\kappa_1,\ldots,\kappa_n\rangle,\lambda\rangle$. In this case make a new system by replacing this identity with the identities $\tau_1 \sim \sigma_1,\ldots,\tau_n \sim \sigma_n$. Obviously, any solution of the new system in some algebra is a solution also of the original system in the same algebra. Since the original system is unsolvable in some algebra, the new system is unsolvable there as well. By induction hypothesis, there exists an algebra $\mathbf{A}$ with finite algebraic carriers, such that the new system has no solution in $\mathbf{A}$. Define a new algebra $\mathbf{B}$, such that $\mathbf{B}_\lambda = \mathbf{A}_\lambda \times \mathbf{A}_{\kappa_1} \times \ldots \mathbf{A}_{\kappa_n}$ and all other algebraic carriers of $\mathbf{B}$ are the same as the corresponding carrier of $\mathbf{A}$.

Let $h : |\mathbf{B}| \to |\mathbf{A}|$ be a $\mathtt{Sort}$-indexed function defined as follows. Let h_κ be identity function for $\kappa \neq \lambda$ and let $h_\lambda : \mathbf{A}_\lambda \times \mathbf{A}_{\kappa_1} \times \ldots \mathbf{A}_{\kappa_n} \to \mathbf{A}_\lambda$ be the projection on the first element of the $n+1$-tuple.

If $\mathtt{g}$ is a functional symbol whose result sort is not λ, let

$$\mathtt{g}^{\mathbf{B}}\langle\alpha_1,\ldots,\alpha_m\rangle = \mathtt{g}^{\mathbf{A}}\langle h\alpha_1,\ldots,h\alpha_m\rangle$$

If the result sort of $\mathtt{g}$ is λ but $\mathtt{g} \neq \mathtt{f}$, then let

$$\mathtt{g}^{\mathbf{B}}\langle\alpha_1,\ldots,\alpha_m\rangle = \langle\mathtt{f}^{\mathbf{A}}\langle h\alpha_1,\ldots,h\alpha_m\rangle,\beta_1,\ldots,\beta_n\rangle$$

where $\beta_1\ldots,\beta_n$ are some arbitrary elements of $\mathbf{A}_{\kappa_1},\ldots,\mathbf{A}_{\kappa_n}$, respectively. Finally, let

$$\mathtt{f}^{\mathbf{B}}\langle\alpha_1,\ldots,\alpha_n\rangle = \langle\mathtt{f}^{\mathbf{A}}\langle h\alpha_1,\ldots,h\alpha_n\rangle,h\alpha_1,\ldots,h\alpha_n\rangle$$

A direct inspection shows that the $\mathtt{Sort}$-indexed function h is actually a homomorphism from $\mathbf{B}$ to $\mathbf{A}$. Suppose that v is a solution of the original system in $\mathbf{B}$. Then (11Q1) implies that $h \circ v$ will be a solution of the original system in $\mathbf{A}$. Moreover, from

$$\mathtt{f}(\tau_1,\ldots,\tau_n)[v]^{\mathbf{B}} = \mathtt{f}(\sigma_1,\ldots,\sigma_n)[v]^{\mathbf{B}}$$

it follows that

$$\langle \mathtt{f}^{\mathbf{A}}\langle h(\tau_1[v]^{\mathbf{B}}),\dots,h(\tau_n[v]^{\mathbf{B}})\rangle, h(\tau_1[v]^{\mathbf{B}}),\dots,h(\tau_n[v]^{\mathbf{B}})\rangle = \\ = \langle \mathtt{f}^{\mathbf{A}}\langle h(\sigma_1[v]^{\mathbf{B}}),\dots,h(\sigma_n[v]^{\mathbf{B}})\rangle, h(\sigma_1[v]^{\mathbf{B}}),\dots,h(\sigma_n[v]^{\mathbf{B}})\rangle$$

hence $h(\tau_i[v]^{\mathbf{B}}) = h(\sigma_i[v]^{\mathbf{B}})$ for any $i \in \{1,\dots,n\}$, which according to (11Q1) implies that $\tau_i[h\circ v]^{\mathbf{A}} = \sigma_i[h\circ v]^{\mathbf{A}}$. Consequently, $h\circ v$ turns out to be a solution of the new system in **A**, which is a contradiction.

Case 6. Among the identities containing termal expression with complexity k_2, some identity has the form $\mathtt{d}'(\tau_1,\dots,\tau_n) \sim \mathtt{d}''(\sigma_1,\dots,\sigma_m)$ where $\mathtt{d}'$ and $\mathtt{d}''$ are different predicate or logical symbols. In this case the theorem is trivially true since such an identity has no solutions in any algebra.

Case 7. Among the identities containing termal expression with complexity k_2, some identity has the form $\mathtt{d}(\tau_1,\dots,\tau_n) \sim \mathtt{d}(\sigma_1,\dots,\sigma_n)$ where $\mathtt{d}$ is a predicate or a logical symbol. In this case make a new system by replacing this identity with the identities $\tau_1 \sim \sigma_1,\dots,\tau_n \sim \sigma_n$. In any algebra, an assignment function is a solution of the original system if and only if it is a solution of the new system. Therefore, by applying the induction hypothesis to the new system we obtain the required. ∎

D) **Definition.** For any natural number n we define inductively the notion *n-term* over a `Sort`-indexed set Y:

(1) If $\xi \in Y_\kappa$, then $\ulcorner\xi\urcorner$ is an n-term over Y of sort κ (for any n).

(2) If `f` is a functional symbol of type $\langle\langle\kappa_1,\dots,\kappa_n\rangle,\lambda\rangle$ and $\tau_1,\dots,\tau_n$ are n-terms over Y of sorts $\kappa_1,\dots,\kappa_n$, respectively, then $\mathtt{f}(\tau_1,\dots,\tau_n)$ is an $(n+1)$-term over Y of sort λ.

Alternatively, we can say that τ is an n-term over Y of sort κ if τ is a term over Y of sort κ whose depth is less than or equal to n.

E) **Definition.** Given a structure **M**, for any term τ over $|\mathbf{M}|$ and natural number n we define the n-simplification of τ, written $\tau\restriction_{\mathbf{M}} n$, recursively.

(1) $\ulcorner\mu\urcorner\restriction_{\mathbf{M}} n = \ulcorner\mu\urcorner$ for any $\mu \in |\mathbf{M}|$ and n.

(2) $\mathtt{f}(\tau_1,\dots,\tau_m)\restriction_{\mathbf{M}} 0 = \ulcorner(\mathtt{f}(\tau_1,\dots,\tau_m))^{\mathbf{M}}\urcorner$.

(3) $\mathtt{f}(\tau_1,\dots,\tau_m)\restriction_{\mathbf{M}}(n+1) = \mathtt{f}(\tau_1\restriction_{\mathbf{M}} n,\dots,\tau_1\restriction_{\mathbf{M}} n)$.

F) **Proposition.** (1) *$\tau\restriction_{\mathbf{M}} n$ is an n-term over $|\mathbf{M}|$ for any natural number n and term τ over $|\mathbf{M}|$.*

(2) *$(\tau\restriction_{\mathbf{M}} n)^{\mathbf{M}} = \tau^{\mathbf{M}}$ for any natural number n and term τ over $|\mathbf{M}|$.*

(3) *$(\tau\restriction_{\mathbf{M}} n)\restriction_{\mathbf{M}} m = \tau\restriction_{\mathbf{M}} \min\{n,m\}$ for any term τ over $|\mathbf{M}|$ and natural numbers n and m.*

Proof. **(1)** By induction on n. If $\tau = \ulcorner\mu\urcorner$ for some $\mu \in |\mathbf{M}|$, then

$\tau\restriction_{\mathbf{M}}n = \ulcorner\mu\urcorner$ which is an n-term over $|\mathbf{M}|$. Otherwise, $\tau = \mathtt{f}(\tau_1,\ldots,\tau_m)$ for some m and $\tau_1,\ldots,\tau_m$. If $n=0$, then $\tau\restriction_{\mathbf{M}}n = \ulcorner\tau^{\mathbf{M}}\urcorner$, which is an n-term over $|\mathbf{M}|$. If $n>0$, then $\tau\restriction_{\mathbf{M}}n = \mathtt{f}(\tau_1\restriction_{\mathbf{M}}(n-1),\ldots,\tau_m\restriction_{\mathbf{M}}(n-1))$, which is an n-term because by induction hypothesis, $\tau_i\restriction_{\mathbf{M}}(n-1)$ is an $(n-1)$-term for any i.

(2) By simple induction on τ. The cases when $\tau = \ulcorner\mu\urcorner$ for some $\mu\in|\mathbf{M}|$ or $n=0$ follow immediately from definition (E). The case when $n>0$ and $\tau=\mathtt{f}(\tau_1,\ldots,\tau_k)$ follows from the definition and the induction hypothesis as well.

(3) By induction on τ. If $\tau = \ulcorner\mu\urcorner$ for some $\mu\in|\mathbf{M}|$, then $(\tau\restriction_{\mathbf{M}}n)\restriction_{\mathbf{M}}m = \ulcorner\mu\urcorner = \tau\restriction_{\mathbf{M}}\min\{n,m\}$.

If $n=0$, then $(\tau\restriction_{\mathbf{M}}n)\restriction_{\mathbf{M}}m = \ulcorner\tau^{\mathbf{M}}\urcorner\restriction_{\mathbf{M}}m = \ulcorner\tau^{\mathbf{M}}\urcorner$.

If $m=0$, then from (2) it follows that $(\tau\restriction_{\mathbf{M}}n)\restriction_{\mathbf{M}}m = \ulcorner(\tau\restriction_{\mathbf{M}}n)^{\mathbf{M}}\urcorner = \ulcorner\tau^{\mathbf{M}}\urcorner$.

If $n>0$, $m>0$ and $\tau=\mathtt{f}(\tau_1,\ldots,\tau_k)$, then $(\tau\restriction_{\mathbf{M}}n)\restriction_{\mathbf{M}}m = (\mathtt{f}(\tau_1,\ldots,\tau_k)\restriction_{\mathbf{M}}n)\restriction_{\mathbf{M}}m = (\mathtt{f}(\tau_1\restriction_{\mathbf{M}}(n-1),\ldots,\tau_k\restriction_{\mathbf{M}}(n-1)))\restriction_{\mathbf{M}}m = \mathtt{f}((\tau_1\restriction_{\mathbf{M}}(n-1))\restriction_{\mathbf{M}}(m-1),\ldots,(\tau_k\restriction_{\mathbf{M}}(n-1))\restriction_{\mathbf{M}}(m-1))$ which, by the induction hypothesis, is equal to

$$\begin{aligned}
&\mathtt{f}(\tau_1\restriction_{\mathbf{M}}\min\{n-1,m-1\},\ldots,\tau_k\restriction_{\mathbf{M}}\min\{n-1,m-1\})\\
&\qquad= \mathtt{f}(\tau_1\restriction_{\mathbf{M}}(\min\{n,m\}-1),\ldots,\tau_k\restriction_{\mathbf{M}}(\min\{n,m\}-1))\\
&\qquad= \mathtt{f}(\tau_1,\ldots,\tau_k)\restriction_{\mathbf{M}}\min\{n,m\}\\
&\qquad= \tau\restriction_{\mathbf{M}}\min\{n,m\}
\end{aligned}$$

■

G) **Definition.** Given an algebra $\mathbf{A}$ and a natural number n, let $\mathbf{A}\restriction n$ be the algebra, such that:

(1) For any algebraic sort κ, the carrier of sort κ of $\mathbf{A}\restriction n$ is the set of all n-terms over $|\mathbf{A}|$ of sort κ.

(2) For any m-ary functional symbol $\mathtt{f}$ and $\tau_1,\ldots,\tau_m$ belonging to suitable carriers of $\mathbf{A}\restriction n$, let $\mathtt{f}^{\mathbf{A}\restriction n}\langle\tau_1,\ldots,\tau_m\rangle = (\mathtt{f}(\tau_1,\ldots,\tau_m))\restriction_{\mathbf{A}}n$.

H) **Proposition.** *Let $\mathbf{A}$ be an algebra and $n\geq 1$. For any $m\in\{0,1,\ldots,n\}$, if τ and σ belong to an algebraic carrier of $\mathbf{A}\restriction n$, then $\mathfrak{sim}_m(\tau,\sigma)$ implies $\tau\restriction_{\mathbf{A}}(n-m) = \sigma\restriction_{\mathbf{A}}(n-m)$.*

(The relation $\mathfrak{sim}_m$ is defined according to (27C) for $\mathbf{A}\restriction n$.)

Proof. By induction on m. If $m=0$, then $\mathfrak{sim}_m(\tau,\sigma)$ implies $\tau=\sigma$ (see 27C1), so $\tau\restriction_{\mathbf{A}}(n-m) = \sigma\restriction_{\mathbf{A}}(n-m)$.

Suppose that $m<n$ and $\mathfrak{sim}_m(\tau,\sigma)$ implies $\tau\restriction_{\mathbf{A}}(n-m) = \sigma\restriction_{\mathbf{A}}(n-m)$. We are going to prove that $\mathfrak{sim}_{m+1}(\tau,\sigma)$ implies $\tau\restriction_{\mathbf{A}}(n-m-1) =$

$\sigma \restriction_{\mathbf{A}} (n-m-1)$.

According to (27C2), there are two cases to consider. The first case is when $\mathfrak{sim}_m(\tau, \sigma)$ is true. In this case the induction hypothesis implies $\tau \restriction_{\mathbf{A}} (n-m) = \sigma \restriction_{\mathbf{A}} (n-m)$, so from (F3) we obtain that $\tau \restriction_{\mathbf{A}} (n-m-1) = \tau \restriction_{\mathbf{A}} (n-m) \restriction_{\mathbf{A}} (n-m-1) = \sigma \restriction_{\mathbf{A}} (n-m) \restriction_{\mathbf{A}} (n-m-1) = \sigma \restriction_{\mathbf{A}} (n-m)$.

The second case is when there exists a functional symbol $\mathtt{f}$ of type $\langle\langle \kappa_1, \dots, \kappa_k\rangle, \lambda\rangle$ and $\tau_1, \dots, \tau_k$ and $\sigma_1, \dots, \sigma_k$ belonging to suitable carriers of $\mathbf{A} \restriction n$, such that:

- $\tau_j = \tau$ and $\sigma_j = \sigma$ for some j;
- $\mathfrak{sim}_m(\mathtt{f}^{\mathbf{A}\restriction n}\langle \tau_1, \dots, \tau_k\rangle, \mathtt{f}^{\mathbf{A}\restriction n}\langle \sigma_1, \dots, \sigma_k\rangle)$ is true.

By induction hypothesis,

$$(\mathtt{f}^{\mathbf{A}\restriction n}\langle \tau_1, \dots, \tau_k\rangle) \restriction_{\mathbf{A}} (n-m) = (\mathtt{f}^{\mathbf{A}\restriction n}\langle \sigma_1, \dots, \sigma_k\rangle) \restriction_{\mathbf{A}} (n-m)$$

Therefore,

$$\begin{aligned}
&\mathtt{f}(\tau_1 \restriction_{\mathbf{A}} (n-m-1), \dots, \tau_k \restriction_{\mathbf{A}} (n-m-1)) = \\
&\quad = \mathtt{f}(\tau_1, \dots, \tau_k) \restriction_{\mathbf{A}} (n-m) && \text{from (E3)}\\
&\quad = (\mathtt{f}(\tau_1, \dots, \tau_k) \restriction_{\mathbf{A}} n) \restriction_{\mathbf{A}} (n-m) && \text{from (F3)}\\
&\quad = (\mathtt{f}^{\mathbf{A}\restriction n}\langle \tau_1, \dots, \tau_k\rangle) \restriction_{\mathbf{A}} (n-m) && \text{from (G2)}\\
&\quad = (\mathtt{f}^{\mathbf{A}\restriction n}\langle \sigma_1, \dots, \sigma_k\rangle) \restriction_{\mathbf{A}} (n-m) &&\\
&\quad = (\mathtt{f}(\sigma_1, \dots, \sigma_k) \restriction_{\mathbf{A}} n) \restriction_{\mathbf{A}} (n-m) && \text{from (G2)}\\
&\quad = \mathtt{f}(\tau_1, \dots, \tau_k) \restriction_{\mathbf{A}} (n-m) && \text{from (F3)}\\
&\quad = \mathtt{f}(\sigma_1 \restriction_{\mathbf{A}} (n-m-1), \dots, \sigma_k \restriction_{\mathbf{A}} (n-m-1)) &&
\end{aligned}$$

Consequently, $\tau \restriction_{\mathbf{A}} (n-m-1) = \tau_j \restriction_{\mathbf{A}} (n-m-1) = \sigma_j \restriction_{\mathbf{A}} (n-m-1) = \sigma \restriction_{\mathbf{A}} (n-m-1)$. ∎

I) **Corollary.** *Let* $\mathbf{A}$ *be an algebra and* $n \geq 1$. *For any* $m \in \{0, 1, \dots, n\}$, *if* τ *and* σ *belong to an algebraic carrier of* $\mathbf{A} \restriction n$, *then* $\mathfrak{sim}_m(\tau, \sigma)$ *implies* $\tau^{\mathbf{A}} = \sigma^{\mathbf{A}}$.
(The relation $\mathfrak{sim}_m$ *is defined according to* (27C) *for* $\mathbf{A} \restriction n$.*)*

Proof. Follows immediately from the previous proposition and (F2). ∎

J) **Proposition.** *Let* $\mathbf{A}$ *be an algebra,* n *be a natural number and* τ *be a delta-termoid over* $|\mathbf{A} \restriction n|$ *such that* τ *contains no subexpressions of the form "*$\ulcorner m \urcorner +$*" with* $m > n$. *Let* $h : |\mathbf{A} \restriction n| \to |\mathbf{A}|$ *be the* `Sort`*-indexed function, such that for any* $\sigma \in |\mathbf{A} \restriction n|$, $h\sigma = \sigma^{\mathbf{A}}$. *If the term* ρ *is obtained from* τ *by removing all subexpressions of the form "*$\ulcorner l \urcorner +$*" and* $\sigma \in \tau^{\mathcal{P}(\mathbf{A}\restriction n)}$, *then* $\sigma^{\mathbf{A}} = \rho[h]^{\mathbf{A}}$.

Proof. By induction on τ. If $\tau = \ulcorner\sigma\urcorner$ for some $\sigma \in |\mathbf{A}\restriction n|$, then $\tau^{\mathcal{P}(\mathbf{A}\restriction n)} = \{\sigma\}$. Also, $\rho = \tau$, so $\rho[h]^{\mathbf{A}} = \ulcorner\sigma\urcorner[h]^{\mathbf{A}} = \ulcorner\sigma^{\mathbf{A}}\urcorner^{\mathbf{A}} = \sigma^{\mathbf{A}}$.

Let $\tau = \mathtt{f}(\tau_1, \ldots, \tau_k)$ and $\rho_1, \ldots, \rho_k$ be respectively obtained from $\tau_1, \ldots, \tau_k$ by removing all subexpressions of the form "$\ulcorner l\urcorner+$". Suppose that $\sigma \in \tau^{\mathcal{P}(\mathbf{A}\restriction n)} = \mathtt{f}(\tau_1, \ldots, \tau_k)^{\mathcal{P}(\mathbf{A}\restriction n)} = \mathtt{f}^{\mathcal{P}(\mathbf{A}\restriction n)}\langle\tau_1^{\mathcal{P}(\mathbf{A}\restriction n)}, \ldots, \tau_k^{\mathcal{P}(\mathbf{A}\restriction n)}\rangle$. Then there exist some $\sigma_1, \ldots, \sigma_k$, such that $\sigma = \mathtt{f}^{\mathbf{A}\restriction n}\langle\sigma_1, \ldots, \sigma_k\rangle$ and $\sigma_i \in \tau_i^{\mathcal{P}(\mathbf{A}\restriction n)}$ for $i \in \{1, \ldots, k\}$. Consequently,

$$\begin{aligned}
\sigma^{\mathbf{A}} &= (\mathtt{f}^{\mathbf{A}\restriction n}\langle\sigma_1, \ldots, \sigma_k\rangle)^{\mathbf{A}} \\
&= (\mathtt{f}(\sigma_1, \ldots, \sigma_k)\restriction_{\mathbf{A}} n)^{\mathbf{A}} && \text{from (G2)} \\
&= \mathtt{f}(\sigma_1, \ldots, \sigma_k)^{\mathbf{A}} && \text{from (F2)} \\
&= \mathtt{f}^{\mathbf{A}}\langle\sigma_1^{\mathbf{A}}, \ldots, \sigma_k^{\mathbf{A}}\rangle \\
&= \mathtt{f}^{\mathbf{A}}\langle\rho_1[h]^{\mathbf{A}}, \ldots, \rho_k[h]^{\mathbf{A}}\rangle && \text{by induction hypothesis} \\
&= \mathtt{f}(\rho_1[h], \ldots, \rho_k[h])^{\mathbf{A}} \\
&= \mathtt{f}(\rho_1, \ldots, \rho_k)[h]^{\mathbf{A}} = \rho[h]^{\mathbf{A}}
\end{aligned}$$

Let $\tau = \ulcorner m\urcorner + \tau'$. Notice that if we remove from τ' all subexpressions of the form "$\ulcorner l\urcorner+$", the result will be ρ. Suppose that $\sigma \in \tau^{\mathcal{P}(\mathbf{A}\restriction n)} = (\ulcorner m\urcorner + \tau')^{\mathcal{P}(\mathbf{A}\restriction n)}$. Then there exists some $\sigma' \in (\tau')^{\mathcal{P}(\mathbf{A}\restriction n)}$, such that $\mathfrak{sim}_m(\sigma, \sigma')$. According to (I), $\sigma^{\mathbf{A}} = (\sigma')^{\mathbf{A}}$ and by induction hypothesis $(\sigma')^{\mathbf{A}} = \rho[h]^{\mathbf{A}}$. ∎

K) **Proposition.** *Given a* Sort*-indexed set Y and a delta-termoid τ over Y, if the term ρ is obtained from τ by removing all subexpressions of the form "$\ulcorner l\urcorner+$", then $\rho = \tau[\![\mathrm{nam}_Y]\!]_\delta^{[Y]}$.*

Proof. If we remove from $\tau[\![\mathrm{nam}_Y]\!]_\delta$ all subexpressions of the form "$\ulcorner l\urcorner+$", the result will be $\rho[\mathrm{nam}_Y]$. According to (25R2), $\tau[\![\mathrm{nam}_Y]\!]_\delta^{[Y]} = \rho[\mathrm{nam}_Y]^{[Y]}$ which, according to (11V1), is equal to ρ. ∎

L) **Theorem.** *If a finite delta-termoidal system over $\mathbb{X}$ is unsolvable in some normal algebra of terms, then it is unsolvable in some normal algebra with finite algebraic carriers.*

Proof. Let Θ be a delta-termoidal system over $\mathbb{X}$ which has no solutions in the normal algebra of terms $\mathbf{A}$. Let Θ' is obtained from Θ by replacing each termoidal identity $\tau \sim \sigma$ with the termal identity $\tau[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]} \sim \sigma[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]}$.

First, we are going to prove that Θ' has no solutions in $\mathbf{A}$. Suppose that $v : \mathbb{X} \to |\mathbf{A}|$ is a solution of Θ' in $\mathbf{A}$. Then for any identity $\tau \sim \sigma$ of Θ,

$$\tau[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]}[v]^{\mathbf{A}} = \sigma[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]}[v]^{\mathbf{A}} \qquad (\sharp)$$

hence

$$\begin{aligned}
\tau[\![v]\!]^{\mathbf{A}} &= (\tau[\![v]\!][\![\mathrm{nam}_{|\mathbf{A}|}]\!]_\delta^{[|\mathbf{A}|]})^{\mathbf{A}} && \text{from (16I)}\\
&= \tau[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]}[v]^{\mathbf{A}} && \text{from (16D2)}\\
&= \sigma[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]}[v]^{\mathbf{A}} && \text{from } (\sharp)\\
&= (\sigma[\![v]\!][\![\mathrm{nam}_{|\mathbf{A}|}]\!]_\delta^{[|\mathbf{A}|]})^{\mathbf{A}} && \text{from (16D2)}\\
&= \tau[\![v]\!]^{\mathbf{A}} && \text{from (16I)}
\end{aligned}$$

so v is a solution of Θ' in $\mathbf{A}$ which is a contradiction.

According to (C), there exists a normal algebra $\mathbf{B}$ with finite algebraic carriers, such that Θ' is unsolvable in $\mathbf{B}$. Let n be such that no delta-termoid of Θ contains a subexpression of the form "$\ulcorner l \urcorner +$" with $l > n$. Since the algebra $\mathbf{B}\restriction n$ is normal and has finite algebraic carriers, it will be enough to prove that Θ has no solution in $\mathbf{B}\restriction n$.

Suppose that $w : \mathbb{X} \to |\mathbf{B}\restriction n|$ is a solution of Θ in $\mathbf{B} \restriction n$. Then for any identity $\tau \sim \sigma$ of Θ, there exists some $\rho \in |\mathbf{B}\restriction n|$, such that $\rho \in \tau[\![w]\!]_\delta^{\mathcal{P}(\mathbf{B}\restriction n)} \cap \sigma[\![w]\!]_\delta^{\mathcal{P}(\mathbf{B}\restriction n)}$.

Let $h : |\mathbf{B}\restriction n| \to |\mathbf{B}|$ be the `Sort`-indexed function, such that for any $\iota \in |\mathbf{B}\restriction n|$, $h\iota = \iota^{\mathbf{B}}$. Since $\rho \in \tau[\![w]\!]_\delta^{\mathcal{P}(\mathbf{B}\restriction n)}$, from (K) and (J) it follows that $\rho^{\mathbf{B}} = \tau[\![w]\!]_\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]}[h]^{\mathbf{B}}$. Therefore,

$$\begin{aligned}
\rho^{\mathbf{B}} &= \tau[\![w]\!]_\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]}[h]^{\mathbf{B}}\\
&= \tau[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]}[w][h]^{\mathbf{B}} && \text{from (16D2)}\\
&= \tau[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]}[h\circ w]^{\mathbf{B}} && \text{from (11H)}
\end{aligned}$$

and analogously, $\rho^{\mathbf{B}} = \sigma[\![\mathrm{nam}_{\mathbb{X}}]\!]_\delta^{[\mathbb{X}]}[h \circ w]^{\mathbf{B}}$. Consequently, $h \circ w$ is a solution of Θ' in $\mathbf{B}$, which is a contradiction. ∎

M) **Theorem.** *If a finite epsilon-termoidal system over $\mathbb{X}$ is unsolvable in some normal algebra of terms, then it is unsolvable in some normal algebra with finite algebraic carriers.*

Proof. Let Θ be an epsilon-termoidal system over $\mathbb{X}$ which has no solutions in the normal algebra of terms $\mathbf{A}$. Let Θ' is obtained from Θ by replacing each epsilon-termoidal identity $\tau \sim \sigma$ with the delta-termoidal identity $\mathfrak{c}(\tau) \sim \mathfrak{c}(\sigma)$. Then for any `Sort`-indexed function $v : \mathbb{X} \to |\mathbf{A}|$,

$$\begin{aligned} \tau[\![v]\!]_\varepsilon^{\mathcal{P}\mathbf{A}} &= (\mathrm{Val}_{\mathbf{A}}^{\varepsilon} \circ [\![v]\!]_\varepsilon)\tau \\ &= (\mathrm{Val}_{\mathbf{A}}^{\varepsilon} \circ [\![v]\!]_\delta)\tau && \text{compare (26O) with (25M)} \\ &= (\mathrm{Val}_{\mathbf{A}}^{\delta} \circ \mathfrak{c} \circ [\![v]\!]_\delta)\tau && \text{from (26R)} \\ &= (\mathrm{Val}_{\mathbf{A}}^{\delta} \circ [\![v]\!]_\delta \circ \mathfrak{c})\tau && \text{from (26P2)} \\ &= \mathfrak{c}(\tau)[\![v]\!]_\delta^{\mathcal{P}\mathbf{A}} \end{aligned}$$

Consequently, v is a solution of Θ in $\mathbf{A}$ if and only if it is a solution of Θ' in $\mathbf{A}$. In result, from (L) we obtain the required. ∎

N) An analogous theorem for gamma-termoidal systems can be proved as a corollary from (L). This is so because for any gamma-termoid τ there exists a delta-termoid σ, such that for any structure $\mathbf{M}$ and assignment function $v : \mathbb{X} \to |\mathbf{M}|$, $\tau[\![v]\!]_\gamma^{\mathcal{P}\mathbf{M}} \subseteq \sigma[\![v]\!]_\delta^{\mathcal{P}\mathbf{M}}$. Therefore, for any finite gamma-termoidal system Θ' we can obtain a finite delta-termoidal system Θ'' with the following two properties:

- any solution of Θ' in any structure is a solution of Θ'';
- any solution of Θ'' in an algebra which is a structure of terms is a solution of Θ'.

§30. THE CLASS VED

A) For convenience, in this section I am going to make use of (20Y) and occasionally write $\mathtt{x}$ instead of $\ulcorner\mathtt{x}\urcorner$ when $\mathtt{x} \in \mathbb{X}$.

B) In (23Q) we proved that if the positive hyperresolution with clausoids saturates, then the initial set of clausoids is universally satisfiable in almost any normal algebra. According to (29L) and (29M), this means that if the positive hyperresolution with clausoids saturates, then the initial set of clausoids is universally satisfiable in a normal algebra with finite carriers. This result can be used in order to prove that classes of predicate formulae are finite satisfiable. The plan is as follows:

- For any predicate formula φ from the class in the consideration, we can obtain a set Γ of clauses, such that φ is satisfiable if and only if Γ is universally satisfiable and φ has a finite model if and only if Γ is universally satisfiable in an algebra with finite carriers.
- Using the homomorphism $[\mathrm{Nam}_{\mathbb{X}}]^{[\![\mathbb{X}]\!]}$, the set of clauses can be converted to a set Γ' of clausoids.

- We prove that the resolution saturates for Γ' after generating finitely many resolvents.

Of course, only the last item in this plan is non-trivial.

If the resolution with clausoids saturates for a set of clausoids, then the resolution with clauses is going to saturate for the corresponding set of clauses.[87] Therefore, in order to have a chance to implement this plan, we have to find a class of formulae which is decided by the usual, the clausal resolution. In this section we are going to see that positive hyperresolution with clausoids can be used in order to prove that the class VED has the finite model property.

C) **Lemma.** *Given a term σ over $\mathbb{X}$ and a delta-termoidal substitution $s : \mathbb{X} \to [\![\mathbb{X}]\!]_\delta$, the radius of $\mathfrak{c}(\sigma[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta})$ is equal to the maximal element of the set*

$$\{0\} \cup \{\mathfrak{rad}(\mathfrak{c}(s\xi)) - n : n \text{ is depth of some occurrence of } \xi \in \mathbb{X} \text{ in } \sigma\}$$

where by $\mathfrak{rad}(\mathfrak{c}(s\xi))$ we have denoted the radius of $\mathfrak{c}(s\xi)$.

<u>Proof.</u> By induction on σ.

If σ is a name, then $\sigma = \ulcorner \mathtt{x} \urcorner$ for some $\mathtt{x} \in \mathbb{X}$ and we have to prove that the radius of $\mathfrak{c}(\sigma[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta})$ is equal to $\max\{0, \mathfrak{rad}(\mathfrak{c}(s\mathtt{x})) - 0\}$. This is so because

$$\begin{aligned} \mathfrak{c}(\sigma[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta}) &= \mathfrak{c}(\ulcorner \mathtt{x} \urcorner [\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta}) \\ &= \mathfrak{c}(\ulcorner s\mathtt{x} \urcorner^{[\![\mathbb{X}]\!]_\delta}) && \text{from (14I9)} \\ &= \mathfrak{c}(s\mathtt{x}) && \text{from (25S)} \end{aligned}$$

If σ is not a name, then $\sigma = \mathtt{f}(\sigma_1, \ldots, \sigma_k)$ for some terms $\sigma_1, \ldots, \sigma_k$ of suitable sorts. From (25M) and (25S) it follows that $\sigma[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta}$ is equal to $\mathtt{f}(\sigma_1[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta}, \ldots, \sigma_k[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta})$, so according to (26J13), the radius of $\mathfrak{c}(\sigma[\![s]\!]_\delta)$ is equal to $\max\{1, \mathfrak{rad}(\mathfrak{c}(\sigma_1[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta})), \ldots, \mathfrak{rad}(\mathfrak{c}(\sigma_k[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta}))\} - 1$. From this and the induction hypothesis we obtain the required. ∎

D) **Lemma.** *Given an epsilon-termoidal expression $\ulcorner r \urcorner + \sigma$ over $\mathbb{X}$ and a delta-termoidal substitution $s : \mathbb{X} \to [\![\mathbb{X}]\!]_\delta$, the radius of $\mathfrak{c}((\ulcorner r \urcorner + \sigma)[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta})$ is equal to the maximal element of the set*

$$\{r\} \cup \{\mathfrak{rad}(\mathfrak{c}(s\xi)) - n : n \text{ is depth of some occurrence of } \xi \in \mathbb{X} \text{ in } \sigma\}$$

[87] The opposite is not true. Recall the subsection "The Example by Baaz, Revisited" in the introductory chapter of this thesis.

where by $\mathfrak{rad}(\mathfrak{c}(s\xi))$ *we have denoted the radius of* $\mathfrak{c}(s\xi)$.

<u>Proof.</u> According to definitions (25M) and (25Q), $\mathfrak{c}((\ulcorner r\urcorner + \sigma)[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta}) = \mathfrak{c}(\ulcorner r\urcorner + \sigma[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta})$, so according to (26J14), the radius being in consideration is equal to $\max\{r, k\}$ where k is the radius of $\mathfrak{c}(\sigma[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta}))$. According to (C), k is equal to

$$\{0\} \cup \{\mathfrak{rad}(\mathfrak{c}(s\xi)) - n : n \text{ is depth of some occurrence of } \xi \in \mathbb{X} \text{ in } \sigma\}$$

∎

E) **Definition.** (1) An occurrence of $\mathtt{x} \in \mathbb{X}$ in an epsilon-termoid $\ulcorner n\urcorner + \tau$ over $\mathbb{X}$ is at depth k if this occurrence is at depth l in the term τ and $k = n + l$.

(2) An occurrence of $\mathtt{x} \in \mathbb{X}$ in an epsilon-formuloid over $\mathbb{X}$ is at depth k if this occurrence is at depth k in the termoid containing it.

Notice that according to this definition, the predicate and the logical symbols do not count when determining the depth of an occurrence of a name in a formuloid.

F) **Example.** Consider the clausoid[88]

$$\neg\mathtt{p}(\ulcorner\ulcorner 1\urcorner + \ulcorner\mathtt{y}\urcorner\urcorner, \ulcorner\ulcorner 2\urcorner + \mathtt{f}(\ulcorner\mathtt{x}\urcorner)\urcorner) \vee \mathtt{q}(\ulcorner\ulcorner 1\urcorner + \mathtt{f}(\mathtt{f}(\ulcorner\mathtt{x}\urcorner))\urcorner, \ulcorner\ulcorner 0\urcorner + \mathtt{f}(\ulcorner\mathtt{y}\urcorner)\urcorner)$$

All occurrences of $\mathtt{x}$ in it are at depth 3 and all occurrences of $\mathtt{y}$ are at depth 1;

G) **Definition.** Given a $\mathtt{Sort}$-indexed function $d : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$,

(1) an epsilon-termoid or epsilon-formuloid τ over $\mathbb{X}$ is *compatible* with d if for any $\xi \in \mathbb{X}$, all occurrences of ξ in τ are at depth $d\xi$;

(2) an epsilon-termoidal identity $\ulcorner 0\urcorner + \ulcorner\xi\urcorner \sim \ulcorner m\urcorner + \sigma$ is *compatible* with d, if $d\xi = m$ and $\ulcorner m\urcorner + \sigma$ is compatible with d;

(3) an epsilon-termoidal identity $\ulcorner n\urcorner + \tau \sim \ulcorner m\urcorner + \sigma$ whose left side does not have the form $\ulcorner 0\urcorner + \ulcorner\xi\urcorner$ is *compatible* with d, if $n = m$ and both $\ulcorner n\urcorner + \tau$ and $\ulcorner m\urcorner + \sigma$ are compatible with d;

(4) an epsilon-termoidal identity $\tau \sim \sigma$, where both τ and σ are of logical sort, is *compatible* with d, if all epsilon-termoids occurring in τ and σ are compatible with d and they all have radii equal to 0.

(5) an epsilon-termoidal substitution $s : \mathbb{X} \to [\![\mathbb{X}]\!]_\varepsilon$ is *compatible* with d, if for any $\xi \in \mathbb{X}$ either $s\xi = \mathrm{Nam}^\varepsilon_\mathbb{X}\,\xi$,[89] or $s\xi$ is a compatible with d epsilon-termoid whose radius is $d\xi$.

[88]This clausoid will become much more readable if we omit the symbols $\ulcorner$ and $\urcorner$: $\neg\mathtt{p}(1 + \mathtt{y}, 2 + \mathtt{f}(\mathtt{x})) \vee \mathtt{q}(1 + \mathtt{f}(\mathtt{f}(\mathtt{x})), 0 + \mathtt{f}(\mathtt{y}))$.

[89]Remember that, according to (26S), $\mathrm{Nam}^\varepsilon_\mathbb{X}\,\xi = \ulcorner 0\urcorner + \ulcorner\xi\urcorner$.

H) **Example.** (1) Let $d' : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$ be such that $d'\mathtt{x} = 3$ and $d'\mathtt{y} = 1$. Then the clausoid from example (F) will be compatible with d'.

(2) The identity $\ulcorner 0 \urcorner + \ulcorner \mathtt{x} \urcorner \sim \ulcorner 2 \urcorner + \mathtt{f}(\ulcorner \mathtt{x} \urcorner, \ulcorner \mathtt{x} \urcorner)$ is compatible with any Sort-indexed function $d'' : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$, such that $d''\mathtt{x} = 3$. This is so because the left side is an epsilon-name and all occurrences of x in the right side are at depth 3.

I) **Lemma.** *If $d : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$ is a* Sort*-indexed function, such that both the epsilon-termoid τ and the substitution $s : \mathbb{X} \to [\![\mathbb{X}]\!]_\varepsilon$ are compatible with d, then the epsilon-termoid $\tau[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ also is compatible with d and has the same radius as the radius of τ.*

Proof. Let $\tau = \ulcorner k \urcorner + \sigma$ and $s' = [\![\mathrm{nam}_{\mathbb{X}}]\!]_\varepsilon^{[\mathbb{X}]} \circ s$. According to (26U), for any ξ, m and ρ, if $s\xi = \ulcorner m \urcorner + \rho$, then $s'\xi = \rho$.

According to definitions (26O) and (26Q), $\tau[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} = \tau[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\varepsilon} = \mathfrak{c}^{-1}(\mathfrak{c}(\tau[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta}))$.

For any delta-termoid ρ, denote by $\mathfrak{g}(\rho)$ the result of the removal from ρ of all subexpressions of the form $\ulcorner n \urcorner +$. According to definitions (26D) and (26L), $\mathfrak{g}(\mathfrak{c}^{-1}(\mathfrak{c}(\tau[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta}))) = \mathfrak{g}(\mathfrak{c}(\tau[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta})) = \mathfrak{g}(\tau[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta})$. This, according to definitions (25M) and (25S), is equal to $\sigma[s']^{[\mathbb{X}]}$.

Therefore, $\tau[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ is equal to $\ulcorner l \urcorner + \sigma[s']^{[\mathbb{X}]}$, where l is the radius of $\mathfrak{c}(\tau[\![s]\!]_\delta^{[\![\mathbb{X}]\!]_\delta})$. According to (D), the radius l is equal to the maximal element of the set

$$\{k\} \cup \{\mathfrak{rad}(\mathfrak{c}(s\xi)) - n : n \text{ is depth of some occurrence of } \xi \in \mathbb{X} \text{ in } \sigma\}$$

Since $\tau = \ulcorner k \urcorner + \sigma$ is compatible with d, the depth of any occurrence of $\ulcorner \xi \urcorner$ in σ is $d\xi - k$. Since s is compatible with d as well, the radius of $\mathfrak{c}(s\xi)$ is equal to either 0, or $d\xi$, hence $\mathfrak{rad}(\mathfrak{c}(s\xi)) - n$ is equal to either $k - d\xi$, or k. This implies that $l = k$, so $\tau[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} = \ulcorner k \urcorner + \sigma[s']^{[\mathbb{X}]}$.

It only remains to see that the depth of any occurrence of $\ulcorner \eta \urcorner$ in $\sigma[s']^{[\mathbb{X}]}$ is equal to $d\eta - k$ as this will imply that the depth of any occurrence of $\ulcorner \eta \urcorner$ in $\tau[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} = \ulcorner k \urcorner + \sigma[s']^{[\mathbb{X}]}$ is equal to $d\eta$.

For any occurrence of $\ulcorner \eta \urcorner$ in $\sigma[s']^{[\mathbb{X}]}$, either it already occurs in σ, or it is part of $s'\xi$ replacing some occurrence of ξ in σ. In the first case the depth of η is the same as it is in σ, so it is equal to $d\eta - k$.

Suppose that an occurrence of $\ulcorner \eta \urcorner$ in $\sigma[s']^{[\mathbb{X}]}$ is part of $s'\xi$ replacing some occurrence of ξ in σ. Since s is compatible with d, according to (G5), we have to consider two cases. If $s\xi = \mathrm{Nam}_{\mathbb{X}}^{\varepsilon}\,\xi = \ulcorner 0 \urcorner + \ulcorner \xi \urcorner$, then $s'\xi = \ulcorner \xi \urcorner$, so $\xi = \eta$, hence in this case an occurrence of $\ulcorner \eta \urcorner$ in σ is replaced with $\ulcorner \eta \urcorner$, so its depth remains unchanged and equal to $d\eta - k$ as well.

Otherwise, $s\xi$ is a compatible with d epsilon-termoid whose radius is equal to $d\xi$, so $s\xi = \ulcorner d\xi \urcorner + s'\xi$. Since $s\xi$ is compatible with d, the depth of $\ulcorner \eta \urcorner$ in $s'\xi$ is equal to $d\eta - d\xi$. Therefore we are replacing an occurrence of $\ulcorner \xi \urcorner$ in σ (which is at depth $d\xi - k$) with a term in which $\ulcorner \eta \urcorner$ occcurs at depth $d\eta - d\xi$. Consequently, the occurrence of $\ulcorner \eta \urcorner$ in $\sigma[s']^{[\mathbb{X}]}$ is at depth $(d\xi - k) + (d\eta - d\xi) = d\eta - k$. ■

J) **Lemma.** *Given a formuloid φ and a termoidal substitution s, if we replace each each termoid σ occurring in φ with $\sigma[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$, the result will be equal to $\varphi[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$.*

Proof. By induction on φ (recall that any formuloid is a special kind of formula).

If $\varphi = \mathtt{p}(\ulcorner\sigma\urcorner_1, \dots, \ulcorner\sigma\urcorner_k)$ for some predicate symbol $\mathtt{p}$ and termoids $\sigma_1, \dots, \sigma_k$, then from ($\sharp$) of definition (14C) it follows that $\varphi[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} = \mathtt{p}(\ulcorner\sigma\urcorner_1, \dots, \ulcorner\sigma\urcorner_k)[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ is equal to $\mathtt{p}(\ulcorner\sigma\urcorner_1[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}, \dots, \ulcorner\sigma\urcorner_k[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon})$.

Analogously, if $\varphi = \mathtt{d}(\psi_1, \dots, \psi_k)$ for some operation symbol $\mathtt{d}$ and formuloids $\psi_1, \dots, \psi_k$, then again from ($\sharp$) of definition (14C) it follows that $\varphi[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} = \mathtt{d}(\ulcorner\psi\urcorner_1, \dots, \ulcorner\psi\urcorner_k)[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ is equal to $\mathtt{d}(\ulcorner\psi\urcorner_1[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}, \dots, \ulcorner\psi\urcorner_k[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon})$, so from the induction hypothesis we obtain the required. ■

K) **Lemma.** *If $d : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$ is a $\mathtt{Sort}$-indexed function, such that both the epsilon-formuloid φ and the substitution $s : \mathbb{X} \to [\![\mathbb{X}]\!]_\varepsilon$ are compatible with d, then the epsilon-formuloid $\varphi[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ also is compatible with d.*

Proof. Follows immediately from (I), (J) and the definition of "compatible". ■

L) **Lemma.** *Let $d : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$ be a $\mathtt{Sort}$-indexed function, both the epsilon-termoid $\ulcorner n \urcorner + \tau$ and the identity $\tau' \sim \tau''$ be compatible with d, $\mathtt{x} \in \mathbb{X}$ be such that $d\mathtt{x} = n$ and $s : \mathbb{X} \to [\![\mathbb{X}]\!]_\varepsilon$ be the substitution*

$$s\xi = \begin{cases} \mathrm{Nam}_{\mathbb{X}}^{\varepsilon}\, \xi & \text{if } \xi \neq \mathtt{x}, \\ \ulcorner n \urcorner + \tau & \text{if } \xi = \mathtt{x}. \end{cases}$$

Then the identity $\tau'[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} \sim \tau''[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ is compatible with d.

Proof. First, notice that the substitution s is compatible with d.

We have to consider three cases.

First case. $\tau' = \ulcorner 0 \urcorner + \ulcorner \mathtt{y} \urcorner$ and $\tau'' = \ulcorner k'' \urcorner + \sigma''$ for some $\mathtt{y}, k'', \sigma''$.

Since the identity $\ulcorner 0 \urcorner + \ulcorner \mathtt{y} \urcorner \sim \ulcorner k'' \urcorner + \sigma''$ is compatible with d, (G2) implies that $d\mathtt{y} = k''$. From (I) it follows that $(\ulcorner k'' \urcorner + \sigma'')[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ is compatible with d and its radius is k''.

If $\mathtt{y} \neq \mathtt{x}$, then $\tau'[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} = (\ulcorner 0 \urcorner + \ulcorner \mathtt{y} \urcorner)[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} = \ulcorner 0 \urcorner + \ulcorner \mathtt{y} \urcorner$, so from definition (G2) we can conclude that the identity $\tau'[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} \sim \tau''[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ is compatible with d.

If $\mathtt{x} = \mathtt{y}$, then $\tau'[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} = (\ulcorner 0 \urcorner + \ulcorner \mathtt{x} \urcorner)[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} = (\ulcorner 0 \urcorner + \ulcorner \ulcorner n \urcorner + \tau \urcorner)^{[\![\mathbb{X}]\!]_\varepsilon} = (\mathrm{Val}_{\mathbb{X}}^{\varepsilon} \circ \mathrm{Nam}_{\mathbb{X}}^{\varepsilon})(\ulcorner n \urcorner + \tau)) = \ulcorner n \urcorner + \tau$ which is a compatible with d epsilon-termoid. Since we already saw that the epsilon-termoid on the other side of the identity $(\ulcorner k' \urcorner + \sigma')[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} \sim (\ulcorner k'' \urcorner + \sigma'')[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ is compatible with d, from (G2) it follows that this identity is compatible with d when $n = 0$ and from (G3) it follows that the same is true when $n \neq 0$.

Second case. $\tau' = \ulcorner k' \urcorner + \sigma'$ and $\tau'' = \ulcorner k'' \urcorner + \sigma''$ for some $k', k'', \sigma', \sigma''$, such that $k' \neq 0$ or σ' is not a name.

In this case, according to (G3), $k' = k''$, so we can write k instead of k' or k''. Both $\ulcorner k \urcorner + \sigma'$ and $\ulcorner k \urcorner + \sigma''$ are compatible with d and from (I) it follows that both $(\ulcorner k' \urcorner + \sigma')[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ and $(\ulcorner k'' \urcorner + \sigma'')[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ are compatible with d and their radii are equal to k. Therefore, the identity $(\ulcorner k' \urcorner + \sigma')[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon} \sim (\ulcorner k'' \urcorner + \sigma'')[\![s]\!]_\varepsilon^{[\![\mathbb{X}]\!]_\varepsilon}$ is compatible with d.

Third case. τ' and τ'' are of logical sort. In this case we obtain the required from (I) and (J). ■

M) **Lemma.** *Given a* `Sort`*-indexed function* $d : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$, *if the partial function* $\mathfrak{f}_\varepsilon$ *is defined as in* (28F), $\tau \sim \sigma$ *is compatible with* d *and* $\mathfrak{f}_\varepsilon(\tau \sim \sigma)$ *is defined, then all elements of* $\mathfrak{f}_\varepsilon(\tau \sim \sigma)$ *are compatible with* d.

Proof. A simple inspection of (28F) suffices to prove this Lemma.

1. If τ and σ are epsilon-termoidal expressions of logical sort, then τ has the form $\mathtt{d}'(\dots)$ and σ has the form $\mathtt{d}''(\dots)$ for some predicate or logical symbols $\mathtt{d}'$ and $\mathtt{d}''$. In this case:
   - if $\mathtt{d}' \neq \mathtt{d}''$, then $\mathfrak{f}_\varepsilon(\tau \sim \sigma)$ is undefined, so there is nothing to prove;
   - if $\mathtt{d}' = \mathtt{d}''$, both are logical symbols, $\tau = \mathtt{d}'(\tau_1, \dots, \tau_n)$ and $\sigma = \mathtt{d}''(\sigma_1, \dots, \sigma_n)$, then $\mathfrak{f}_\varepsilon(\tau \sim \sigma) = \{\tau_1 \sim \sigma_1, \dots, \tau_n \sim \sigma_n\}$. According to definition (E2), the logical symbols do not count when measuring the depth, so the depth of all occurrences of any $\xi \in \mathbb{X}$ in $\tau_1, \dots, \tau_n, \sigma_1, \dots, \sigma_n$ is the same as the depth in τ and σ, namely $d\xi$. Moreover, the radii of the epsilon-termoids in $\tau_1, \dots, \tau_n, \sigma_1, \dots, \sigma_n$ are equal to 0.
   - if $\mathtt{d}' = \mathtt{d}''$, both are predicate symbols, $\tau = \mathtt{d}'(\mathrm{Nam}_{\mathbb{X}}^{\varepsilon}\, \tau_1, \dots, \mathrm{Nam}_{\mathbb{X}}^{\varepsilon}\, \tau_n)$ and $\sigma = \mathtt{d}''(\mathrm{Nam}_{\mathbb{X}}^{\varepsilon}\, \sigma_1, \dots, \mathrm{Nam}_{\mathbb{X}}^{\varepsilon}\, \sigma_n)$, then $\mathfrak{f}_\varepsilon(\tau \sim \sigma)$ is equal to $\{\tau_1 \sim \sigma_1, \dots, \tau_n \sim \sigma_n\}$. According to definition (E2), the predicate symbols do not count when measuring the depth, so the

depth of all occurrences of any $\xi \in \mathbb{X}$ in $\tau_1, \ldots, \tau_n, \sigma_1, \ldots, \sigma_n$ is the same as the depth in τ and σ, namely $d\xi$.

2. $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m) \sim \ulcorner k \urcorner + \mathtt{f}(\sigma_1, \ldots, \sigma_m))$ is equal to $\{\ulcorner l+1 \urcorner + \tau_1 \sim \ulcorner l+1 \urcorner + \sigma_1, \ldots, \ulcorner l+1 \urcorner + \tau_m \sim \ulcorner l+1 \urcorner + \sigma_m\}$, where $l = \max\{n, k\}$. If the argument of $\mathfrak{f}_\varepsilon$ is compatible with d, then $n = k$, so $l = n = k$, hence the elements of the value of $\mathfrak{f}_\varepsilon$ are compatible with d.
3. $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m) \sim \ulcorner k \urcorner + \mathtt{g}(\sigma_1, \ldots, \sigma_l))$ is undefined if $\mathtt{f}$ and $\mathtt{g}$ are different functional symbols, so there is nothing to prove.
4. If $n \neq 0$, then $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m))$ is equal to $\{\ulcorner 0 \urcorner + \ulcorner \xi \urcorner \sim \ulcorner \max\{n, k\} \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m)\}$. If the argument of $\mathfrak{f}_\varepsilon$ is compatible with d, then $n = k$, so $\max\{n, k\} = k$. According to (G2), the element of the value of $\mathfrak{f}_\varepsilon$ is compatible with d.
5. $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m) \sim \ulcorner k \urcorner + \ulcorner \xi \urcorner)$ is equal to $\{\ulcorner 0 \urcorner + \ulcorner \xi \urcorner \sim \ulcorner \max\{n, k\} \urcorner + \mathtt{f}(\tau_1, \ldots, \tau_m)\}$. If the argument of $\mathfrak{f}_\varepsilon$ is compatible with d, then $n = k$. According to (G2), the element of the value of $\mathfrak{f}_\varepsilon$ is compatible with d.
6. $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \ulcorner \eta \urcorner) = \{\ulcorner 0 \urcorner + \ulcorner \xi \urcorner \sim \ulcorner \max\{n, k\} \urcorner + \ulcorner \eta \urcorner\}$, if $n \neq 0$ and $\xi \neq \eta$. Since $n \neq 0$ and the argument of $\mathfrak{f}_\varepsilon$ is compatible with d, $n = k$. Therefore, $\max\{n, k\} = k$. According to (G2), the element of the value of $\mathfrak{f}_\varepsilon$ is compatible with d.
7. $\mathfrak{f}_\varepsilon(\ulcorner n \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \ulcorner \xi \urcorner) = \varnothing$ so there is nothing to prove.
8. $\mathfrak{f}_\varepsilon(\ulcorner 0 \urcorner + \ulcorner \xi \urcorner \sim \ulcorner k \urcorner + \tau)$ is undefined if $\tau \neq \ulcorner \xi \urcorner$, so there is nothing to prove.

■

N) **Lemma.** *Given a* `Sort`*-indexed function* $d : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$*, if all identities of the system* Θ *are compatible with* d *and we apply to* Θ *some special solving transformation*[90] *using* $\mathfrak{f}_\varepsilon$*, the result will be a system whose elements are compatible with* d*.*

Proof. If we apply the first special solving transformation, this follows from (M). If we apply the second special solving transformation, it follows from (L). ■

O) **Lemma.** *Given a* `Sort`*-indexed function* $d : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$ *and a system* Θ *of compatible with* d *identities, if* $\mathfrak{e}_\varepsilon$ *is defined as in* (28G)*, then all substitutions belonging to* $\mathfrak{e}_\varepsilon(\Theta)$ *are compatible with* d*.*

[90] See definition (18H).

Proof. According to (N), if we apply successively special solving transformations to Θ, the results will be systems compatible with d. Therefore, from (18O) we can conclude that if $s \in \mathfrak{e}_\varepsilon(\Theta)$, then s is obtained as described in (18D) from a compatible with d solved system. Namely, for any $\xi \in \mathbb{X}$, either $s\xi = \mathrm{Nam}^{\varepsilon}_{\mathbb{X}}\,\xi$, or $s\xi$ is a compatible with d epsilon-termoid whose radius is equal to $d\xi$. ■

P) **Definition.** (1) A clause δ over $\mathbb{X}$ belongs to the class *VED*, if δ is a Horn clause and for any $\mathtt{x} \in \mathbb{X}$, all occurrences of $\mathtt{x}$ in δ are at equal depth.

(2) An epsilon-clausoid δ over $\mathbb{X}$ belongs to the class *VED*, if δ is a Horn clausoid, for any $\mathtt{x} \in \mathbb{X}$, all occurrences of $\mathtt{x}$ in δ are at equal depth and the radii of all epsilon-termoids occurring in δ are equal to 0.

Q) **Lemma.** *Given a* $\mathtt{Sort}$*-indexed function* $d : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$*, any positive* $\mathfrak{e}_\varepsilon$*-resolvent of compatible with* d *epsilon-clausoids over* $\mathbb{X}$ *is compatible with* d *as well.*

Proof. Let δ be a compatible with d epsilon-clausoid over $\mathbb{X}$ with sequence $\langle \lambda_1, \ldots, \lambda_n \rangle$. Let ε be a compatible with d positive epsilon-clausoid over $\mathbb{X}$. Then, according to definition (23C1), any positive $\mathfrak{e}_\varepsilon$-resolvent of δ and ε can be obtained in the following way:

Let Γ be a non-empty set of literaloids occurring in ε and λ_i be the negative literaloid, such that there are no negative literaloids among $\lambda_1, \ldots, \lambda_{i-1}$. Let

$$s \in \mathfrak{e}_\varepsilon(\{\lambda_i \sim \overline{\mu} : \mu \in \Gamma\}).$$

Then the epsilon-clausoid whose sequence is obtained from the sequence of $\delta[\![s]\!]^{[\![\mathbb{X}]\!]}$ by replacing the literaloid corresponding to λ_i with the sequence of $(\varepsilon \setminus \Gamma)[\![s]\!]^{[\![\mathbb{X}]\!]}$ is positive $\mathfrak{e}$-resolvent of δ and ε.

Since δ and ε are compatible with d, their literaloids are compatible with d as well, hence all identities belonging to $\{\lambda_i \sim \overline{\mu} : \mu \in \Gamma\}$ are compatible with d, so according to (O) the substitution s is compatible with d, hence according to (K) the literaloids of $\delta[\![s]\!]^{[\![\mathbb{X}]\!]}$ and $(\varepsilon \setminus \Gamma)[\![s]\!]^{[\![\mathbb{X}]\!]}$ are compatible with d, so the positive $\mathfrak{e}_\varepsilon$-resolvent of δ and ε is compatible with d. ■

R) **Lemma.** *If* δ' *and* δ'' *are epsilon-clausoids having disjoint dependency and belonging to the class VED, then any positive* $\mathfrak{e}_\varepsilon$*-resolvent of* δ' *and* δ'' *belongs to the class VED.*

Proof. Since δ' and δ'' belong to VED, there exist $\mathtt{Sort}$-indexed functions $d', d'' : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$, such that δ' is compatible with d' and δ'' is compatible with d''. Let the $\mathtt{Sort}$-indexed function $d : \mathbb{X} \to \mathbb{N}^{\mathtt{Sort}}$ be such that $d\xi = d'\xi$ if the name $\ulcorner\xi\urcorner$ occurs in δ', $d\xi = d''\xi$ if the name $\ulcorner\xi\urcorner$ occurs in δ'', and $d\xi$ be defined arbitraryly, otherwise. Then $\delta'[\![d]\!] = \delta'[\![d']\!]$ and $\delta''[\![d]\!] = \delta''[\![d'']\!]$, so both δ' and δ'' are compatible with d. According to (Q), any positive $\mathfrak{e}_\varepsilon$-resolvent of δ' and δ'' is compatible with d, hence all such resolvents belong to VED. ∎

S) **Proposition.** *Given a condensing function $\mathfrak{f}$, if $\langle \delta, \varepsilon_0, \ldots, \varepsilon_n \rangle$ is a clash sequence of epsilon-clausoids belonging to the class VED, then all positive $\mathfrak{e}_\varepsilon\mathfrak{f}$-hyperresolvents defined by this clash sequence belong to the class VED.*

Proof. Follows immediately from (R) and definition (23C2). ∎

T) **Lemma.** *Suppose the clauses δ' and ε' have disjoint dependency, both belong to the class VED and $\mathfrak{e}$ is a termally sound and termally complete equaliser, such that $\mathfrak{e}(\Theta)$ contains no more than one element for any system Θ. If the epsilon-clausoids δ and ε are such that δ and ε have disjoint dependency, both belong to the class VED, δ' is a variant of $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$, ε' is a variant of $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and δ and ε have a positive $\mathfrak{e}_\varepsilon$-resolvent ζ, then δ' and ε' have a positive $\mathfrak{mgu}$-resolvent ζ', such that ζ' is a variant of $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$.*

Proof. Let $\langle \lambda_1, \ldots, \lambda_l \rangle$ be the sequence of δ, $\langle \lambda'_1, \ldots, \lambda'_l \rangle$ be the sequence of δ', $\langle \mu_0, \mu_1, \ldots, \mu_m \rangle$ be the sequence of ε and $\langle \mu'_0, \mu'_1, \ldots, \mu'_m \rangle$ be the sequence of ε'. Let λ_j be a positive literaloid, such that there are no positive literaloids among $\lambda_1, \ldots, \lambda_{j-1}$. According to (20L), this means that λ'_j is a positive literal, such that there are no positive literals among $\lambda'_1, \ldots, \lambda'_{j-1}$. Let s and Γ be the substitution and the set used to derive ζ from δ and ε. Namely, the sequence of ζ can be obtained from the sequence of $\delta[\![s]\!]^{[\mathbb{X}]}$ by replacing the literaloid corresponding to λ_j with the sequence of $(\varepsilon \setminus \Gamma)[\![s]\!]^{[\mathbb{X}]}$.

Let $\Delta = \{\lambda[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} : \lambda \in \Gamma\}$. Since all clausoids and literaloids belong to the class VED, from (26X) we can conclude that $(\varepsilon \setminus \Gamma)[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} = \varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} \setminus \Delta$.

Let $\overline{s} = [\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} \circ s$. According to (16M), the sequence of $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ can be obtained from the sequence of $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}[\![\overline{s}]\!]^{[\mathbb{X}]}$ by replacing the literal corresponding to λ_j with the sequence of $(\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} \setminus \Delta)[\![\overline{s}]\!]^{[\mathbb{X}]}$.

Since δ' is a variant of $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and ε' is a variant of $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$, there exists `Sort`-indexed functions $f, g : \mathbb{X} \to \mathbb{X}$, such that $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} = \delta'[f]$ and $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} = \varepsilon'[g]$. Since δ and ε have disjoint dependency, according to (17I2), $\delta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ have disjoint dependency, as well. But δ' and ε' also have disjoint dependency, so without loss of generality we may assume that f and g are bijective and $f = g$. Therefore, $\varepsilon[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]} = \varepsilon'[f]$, so the sequence of $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ can be obtained from the sequence of $\delta'[f][\![\overline{s}]\!]^{[\mathbb{X}]}$

by replacing the literal corresponding to λ_j (that is, λ'_j) with the sequence of $(\varepsilon' \setminus \Delta')[\![\overline{s}]\!]^{[\mathbb{X}]}$ where $\Delta' = \{\overline{\mu}[f^{-1}] : \mu \in \Delta\}$.

Since $s \in \mathfrak{e}_\varepsilon(\{\lambda_j \sim \overline{\mu} : \mu \in \Gamma\})$ and $\mathfrak{e}(\Theta)$ contains no more than one element for any system Θ, from (18U) we can conclude that $\overline{s}$ is a most general unifier of $\{\lambda_j[\![\mathrm{nam}_\mathbb{X}]\!]^{[\mathbb{X}]}\} \cup \{\overline{\mu} : \mu \in \Delta\}$. Therefore, $\overline{s} \circ f$ is a most general unifier of $\{\lambda'_j\} \cup \{\overline{\mu'} : \mu' \in \Delta'\}$. Consequently, there exists a substitution $s' \in \mathfrak{mgu}(\{\lambda'_j \sim \overline{\mu'} : \mu' \in \Delta'\})$, which is a most general unifier of $\{\lambda'_j\} \cup \{\overline{\mu'} : \mu' \in \Delta'\}$. Since both $\overline{s} \circ f$ and s' are most general unifiers of $\{\lambda'_j\} \cup \{\overline{\mu'} : \mu' \in \Delta'\}$, we can conclude that these substitutions are variants. Therefore, $\zeta[\![\mathrm{nam}_\mathbb{X}]\!]^{[\mathbb{X}]}$ is a variant of the clause whose sequence can be obtained from the sequence of $\delta'[s']^{[\mathbb{X}]}$ by replacing the literal λ'_j with the sequence of $(\varepsilon' \setminus \Delta')[\![\mathfrak{sim}']\!]^{[\mathbb{X}]}$. According to definition (B3), the clause having this sequence is a positive $\mathfrak{mgu}$-resolvent of δ' and ε'. ∎

U) **Proposition.** *Let the condensing functions $\mathfrak{f}$ and $\mathfrak{f}'$ be such that for any epsilon-clausoid ζ belonging to VED and clause ζ' over $\mathbb{X}$ belonging to VED, if ζ' is a variant of $\zeta[\![\mathrm{nam}_\mathbb{X}]\!]^{[\mathbb{X}]}$, then $\mathfrak{f}'(\zeta')$ is a variant of $\mathfrak{f}(\zeta)[\![\mathrm{nam}_\mathbb{X}]\!]^{[\mathbb{X}]}$.*

Let $\langle \delta_0, \delta_1, \ldots, \delta_n \rangle$ be a clash sequence of epsilon-clausoids over $\mathbb{X}$ belonging to VED and $\langle \delta'_0, \delta'_1, \ldots, \delta'_n \rangle$ be a clash sequence of clauses over $\mathbb{X}$ belonging to VED, such that δ'_i is a variant of $\delta_i[\![\mathrm{nam}_\mathbb{X}]\!]^{[\mathbb{X}]}$ for any i.

If ε is a positive $\mathfrak{e}_\varepsilon\mathfrak{f}$-hyperresolvent defined by the clash sequence $\langle \delta_0, \delta_1, \ldots, \delta_n \rangle$, then $\varepsilon[\![\mathrm{nam}_\mathbb{X}]\!]^{[\mathbb{X}]}$ is a variant of some positive $\mathfrak{mgu}, \mathfrak{f}'$-hyperresolvent defined by the clash sequence $\langle \delta'_0, \delta'_1, \ldots, \delta'_n \rangle$.

Proof. Follows immediately from (T) and the definitions. ∎

V) **Proposition.** *Let the condensing functions $\mathfrak{f}$ and $\mathfrak{f}'$ be such that for any epsilon-clausoid ζ belonging to VED and clause ζ' over $\mathbb{X}$ belonging to VED, if ζ' is a variant of $\zeta[\![\mathrm{nam}_\mathbb{X}]\!]^{[\mathbb{X}]}$, then $\mathfrak{f}'(\zeta')$ is a variant of $\mathfrak{f}(\zeta)[\![\mathrm{nam}_\mathbb{X}]\!]^{[\mathbb{X}]}$.*

If ε is a positive $\mathfrak{e}_\varepsilon\mathfrak{f}$-hyperresolvent defined by the clash sequence $\langle \delta_0, \delta_1, \ldots, \delta_n \rangle$, where $\delta_0, \ldots, \delta_n$ belong to VED, then the the maximal depth of the termoids in ε is not greater than the maximal depth of the termoids in $\delta_0, \ldots, \delta_n$.

Proof. Let $\delta'_i = \delta_i[\![\mathrm{nam}_\mathbb{X}]\!]^{[\mathbb{X}]}$ for $i \in \{0, \ldots, n\}$. From (U) it follows that there is a positive $\mathfrak{mgu}, \mathfrak{f}'$-hyperresolvent ε' defined by the clash sequence $\langle \delta'_0, \delta'_1, \ldots, \delta'_n \rangle$, such that ε' is a variant of $\varepsilon[\![\mathrm{nam}_\mathbb{X}]\!]^{[\mathbb{X}]}$. Since each termoid occurring in a VED clausoid has the form $\ulcorner 0 \urcorner + \tau$, from (26X) we can conclude that the maximal depth of the termoids in ε is not greater than the maximal depth of the terms in ε'. According to Lemma 3.15 on page 50 in [8], this depth is not greater than the maximal depth of the terms in $\delta'_0, \delta'_1, \ldots, \delta'_n$, so, again from (26X), we can conclude that it is not greater than the maximal depth of the termoids in $\delta_0, \delta_1, \ldots, \delta_n$. ∎

W) **Theorem.** *The class VED has the finite model property.*

Proof. Let $\mathfrak{f}$ and $\mathfrak{f}'$ be condensing functions, such that:

- for any epsilon-clausoid ζ over $\mathbb{X}$, $\mathfrak{f}(\zeta)$ is a condensation of ζ;
- for any clause ζ' over $\mathbb{X}$, $\mathfrak{f}'(\zeta')$ is a condensation of ζ' and
- if ζ' is a variant of $\zeta[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$ and ζ belongs to VED, then $\mathfrak{f}'(\zeta')$ is a variant of $\mathfrak{f}(\zeta)[\![\mathrm{nam}_{\mathbb{X}}]\!]^{[\mathbb{X}]}$.

Let $\mathfrak{g}$ be an arbitrary reducing function.[91]

Let Γ' be an arbitrary universally satisfiable finite set of clauses over $\mathbb{X}$ belonging to VED. Let $\Gamma = \{\delta[\mathrm{Nam}_{\mathbb{X}}]^{[\![\mathbb{X}]\!]} : \delta \in \Gamma'\}$. From (26X2) it follows that Γ is a finite set of epsilon-clausoids belonging to VED.

Since no reducing function may increase the maximal depth of the termoids, from (V) we can conclude that the maximal depth of any termoid occurring in a clausoid belonging to $\mathfrak{res}^*(\mathfrak{e}_\varepsilon, \mathfrak{f}, \mathfrak{g}; \Gamma)$ does not exceed the maximal depth of the termoids of the clausoids in Γ. Up to renaming of the names, there are only finitely many literaloids with limited depth of the termoids. But all hyperresolvents of Horn clausoids are clausoids with only one literaloid, hence the set $\mathfrak{res}^*(\mathfrak{e}_\varepsilon, \mathfrak{f}, \mathfrak{g}; \Gamma)$ is essentially finite.[92] From (23Q) it follows that the set Γ is universally satisfiable in almost any algebra, so from (29M) we can conclude that Γ (and so, Γ' as well) is universally satisfiable in an algebra with finite carriers. ∎

[91] See (4P) for the definition of "reducing function".

[92] See (23P) for the definition of "essentially finite".

§31. CONCLUSION

Method of resolutions is well-known method which is refutationally sound and complete. If a set of clauses is satisfiable, no contradiction is derivable. And, if a set of clauses is not satisfiable, it is possible to derive a contradiction by resolution.

Unification is an important component of method of resolution. The usual algorithm for unification of terms is sound and complete only for Herbrand structures. Therefore, the usual resolution is complete only with respect to satisfiability in such structures. Since Herbrand structures are almost never finite, it is difficult to use resolution in order to build finite models or to prove finite satisfiability. That's why in this work the notion "satisfiable in algebra" has been introduced and resolutive modifications which are complete with respect to satisfiability in algebras different from the Herbrand algebra have been studied.

The most important contribution of this work is the introduction of the notion "termoid" (14J). The unification of many kinds of termoids is complete with respect to very large class of algebras, including some finite algebras. Thus, it becomes possible to use resolution with termoids in order to prove the main results of this work about the validity of the finite model property.

Most of the existing results about finite model property have the following form: if the elements of some satisfiable set of formulae or clauses satisfy some syntactic conditions, then the set has a finite model. A new, algorithmic approach for study of the finite model property has been proposed in this work. If a special algorithm, applied to a set of formulae or clauses, gives a particular result, then the set has a finite model. More specifically, it has been proved that if the algorithm of the resolution with termoids stops after finite number of steps and it has not found a contradiction, then the set of clauses has a finite model.

As an application of the developed theory, we obtain the following important result: if Prolog fails to prove that a goal φ follows from a program Γ of Horn clauses and during the process does not go into an infinite loop, then there exists a finite model of $\Gamma \cup \{\neg\varphi\}$.

The strength of the algorithmic approach to the finite model property has been demonstrated for a certain syntactically defined class of formulae, namely the class VED (universally closed Horn formulae in which that all occurrences of every variable are at equal depth). We proved that the class VED has the finite model property.

An axiomatic theory of termoids is developed in §14 and §16. The axioms for termoids are incorporated in the definition of the notion "terminator". Several particular types of termoids are defined: alpha-termoids in §15, beta-termoids in §7 (informally), gamma-termoids in §24, delta-termoids in §25, epsilon-termoids in §26.

A rather general unification procedure for termoids is specified in §17 and §18.

Theory for the resolution with termoids is developed. Two versions of the resolution have been studied in details: the SLD-resolution (§21) and the positive hyperresolution (§22 and §23).

One characteristic feature of the resolution with termoids is that it preserves the satisfiability but generally it is not sound with respect to logical consequence. In other words, if we add new resolvents to a set, the resulting set is not necessarily equivalent with the initial set. Nevertheless, the termoidal resolution is sound with respect to satisfiability — if a set is satisfiable and we add resolvents to it, the resulting set is going to be satisfiable as well.

In mathematical logic we customarily consider the completeness a desirable property of a deductive system and the soundness — a necessary property. Nevertheless, the resolution with termoids is complete but not sound with respect to logical consequence. As far as I know this is the first time deductive systems with this property are proposed and used.

All proofs in this work about the existence of finite structures are constructive. Therefore, it should be possible to extract practical algorithms from the proofs. Of course many things should be topic for future research. For example we have to estimate the complexity of these algorithms and the size of the constructed models. We have to investigate the problem of building smaller models than what is suggested by the proofs in this work.

The algorithms we can extract from the proofs in this work tell us only what the universe of the model is and the interpretation of the functional symbols. Therefore, in order to build a complete model, we will need also an efficient method to find a suitable interpretation for the predicate symbols. Since there are only finitely many such interpretations, from theoretical point of view we can test all of them in order to find a working one. Obviously, it is impractical to run such algorithm on computer, but it seems very likely that we will be able to find a more efficient algorithm if we find a way to use the information contained in the clauses generated by the resolution.

Publications

The author has the following publications related to this thesis:

1. Anton Zinoviev. Negation as failure is finite controllable. In P. Peppas, C. Drossos, and C. Tsinakis, editors, *Proceedings of the 7th Panhellenic Logic Symposium*, pages 210–214. Patras University Press, 2009

A proof of (21L) is given. Let Γ be a finite set of Horn clauses. Suppose we ask Prolog if φ follows from Γ and after some finite computation Prolog answers with "no". In this case there exists a finite model of Γ in which φ is not valid.

2. Anton Zinoviev. Termal equations and finite controllability. *Annuaire de Université de Sofia, Faculté de Mathématiques et Informatique*, 93:49–54, 1999

In this work a new short proof of (29C) is given.

Declaration

The author declares that the current dissertation is an original scientific work. The use of previous results is fairly reflected with appropriate references.

Index

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