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Master Thesis

On the axiomatization of contact logics with measure

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1 Preface

One of the most established theory of space is classical Euclidean geometry. This theory considers *point* as a primitive notion. In contrast, Alfred Whitehead mentions in his book *"The Organization of Thought"*[5] that the point should be definable in terms of the relations between material things. The reason behind it is that points are too abstract and they do not have representation in the real world. So that, the point-free approach to the theory of space started. It is developed on the primitive notion *region* and two binary relations between regions, *part-of* and *contact*. Usually regions are considered to be elements from a Boolean algebra of regular closed sets in a topological space. The contact relation is defined as a non-empty set-theoretic intersection. In such contact logics Boolean terms are interpreted as regions. There are also two predicates $(a \leq b)$ and $C(a, b)$ which express the relations a is a *part-of* b and region a is in *contact* with region b respectively.

In this study we define a contact logic with the qualitative measure. In such way both types of information topological and size information could be expressed. We introduce a new atomic formula to the standard language of contact logics: $a \leq_\mu b$. The intended meaning is the size of region a is less than or equal to the size of region b. We interpreted formulae from our language in the following type of structures $\langle\langle \mathcal{B}, C\rangle, \mu\rangle$ where $\langle \mathcal{B}, C\rangle$ is contact algebra and μ is a positive measure on $\mathcal{B}$. Our intended structure will be all polytopes over $\mathbb{R}^+$ with the Lebesgue measure on $\mathbb{R}^+$. The current work has the following structure:

- Section 2 reminds the needed notions and well-known results from Propositional Logic and contact algebras. We also explore an algorithm for solving system of linear inequalities with rational coefficients. At the end of this section we give alternative definition for connected graph and show that it is equivalent to the most common one.
- Section 3 defines the language $\mathcal{L}$ that we will use in this work. We examine a couple of ways for interpreting formulae from our language. We also introduce the notion for contact algebra with measure and define some conditions that the measure function has to satisfy.
- Section 4 focuses on the axiomatic system $\mathcal{L}_{HL}$. We also define the notion for S_n-system. We explore an algorithm for solving such systems as well as describing how to construct formula from $\mathcal{L}$ that corresponds to a S_n-system.
- Section 5 shows that a given S_n-system has a solution exactly when the corresponding formula from $\mathcal{L}$ is satisfiable in finite relational HL-structure.
- In Section 6 we prove the soundness and completeness theorems with respect to the finite relational HL.
- Section 7 describes a procedure to associate polytopes to a given tree-like Kripke structure.

2 Preliminaries

In this introductory section we will recall some well-known entities and results related with them which will be used later in this work.

2.1 Facts about Propositional logic

The Formal Systems (FS) has three parts - language, axioms and rules. Every rule has the form $\frac{\varphi_1,\varphi_2,\ldots,\varphi_n}{\varphi}$ where $\varphi_1, \varphi_2, \ldots \varphi_n, \varphi$ are formulae from the language of the FS.

Definition 2.1 (Formal proof)**.** A finite sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ of formulae such that every φ_i is either an axiom or is obtained by applying a rule of inference on one or more elements with indices less than i is called a $formal\ proof$.

Definition 2.2 (Formal theorem)**.** A formula φ is called a $formal\ theorem$ if it is last formula of some formal proof.

Definition 2.3. Let $\mathscr{F}$ be a FS. We could define the notion formal theorems of $\mathscr{F}$ in the following inductive way:

(i) Every axiom is a theorem.

(ii) If $\varphi_1, \varphi_2, \ldots, \varphi_n$ are theorems of $\mathscr{F}$ and $\frac{\varphi_1,\varphi_2,\ldots,\varphi_n}{\varphi}$ is a rule of $\mathscr{F}$, then φ is also a theorem. We write $\vdash \varphi$.

We will consider the following Shoenfield-style FS for Propositional Logic:

Language

(i) Propositional variables: $p_0, p_1, p_2, \ldots$

(ii) Logical symbols: $\neg$ and $\vee$

(iii) Auxiliary symbols:) and (

We will define formula in the language inductively:

(i) Every propositional variable is a formula

(ii) If φ is a formula, then $\neg\varphi$ is a formula

(iii) If φ and ψ are formulae, then $(\varphi \vee \psi)$ is a formula

Axioms
The axioms of Propositional Logic are defined through the following scheme - for every formula φ, the formula $\neg\varphi \vee \varphi$ is an axiom.

Rules

(R1) $\dfrac{\varphi}{\psi \vee \varphi}$ (ER) for any formulae φ and ψ

(R2) $\dfrac{\varphi \vee \varphi}{\varphi}$ (CR) for any formula φ

(R3) $\dfrac{\varphi_1 \vee (\varphi_2 \vee \varphi_3)}{(\varphi_1 \vee \varphi_2) \vee \varphi_3}$ (AR) for any formulae φ_1, φ_2 and φ_3

(R4) $\dfrac{\varphi_1 \vee \varphi_2,\ \neg\varphi_1 \vee \varphi_3}{\varphi_2 \vee \varphi_3}$ (Cut) for any formulae φ_1, φ_2 and φ_3

We accept the standard rules in First-order logic for omission of brackets. Additionally, we will use the following common abbreviations for convenience:

(i) We will write $\varphi \wedge \psi$ instead of $\neg(\neg\varphi \vee \neg\psi)$

(ii) We will write $\varphi \Rightarrow \psi$ instead of $\neg\varphi \vee \psi$

(iii) We will write $\varphi \Leftrightarrow \psi$ instead of $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$

Lemma 2.4 (Commutative)**.** If $\vdash \varphi \vee \psi$, then $\vdash \psi \vee \varphi$.

Lemma 2.5 (Modus Ponnens)**.** If $\vdash \varphi$ and $\vdash \varphi \Rightarrow \psi$, then $\vdash \psi$.

Lemma 2.6. $\vdash \varphi \Rightarrow \psi \Rightarrow \varphi \wedge \psi$

Lemma 2.7. $\vdash (\varphi \Leftrightarrow \psi) \Leftrightarrow (\neg\varphi \Leftrightarrow \neg\psi)$

Definition 2.8 (Subformula)**.** Let φ and ψ be formulae. We say that φ is a subformula of ψ if ψ has the syntactical representation $\varphi_1\varphi\varphi_2$. The word φ is an infix of ψ.

Theorem 2.9. Let φ, φ' and ψ be formulae. Let by ψ' denote any formula obtained from ψ by substituting zero, one, many or all instances of φ as a subformula of ψ by φ'. Then if $\vdash \varphi \Leftrightarrow \varphi'$, then $\vdash \psi \Leftrightarrow \psi'$.

Definition 2.10 (Valuation, Assignment)**.** By *valuation* ν we mean an assignment to every boolean variable to either $\mathbb{T}$ or $\mathbb{F}$.

Remark. We could extend the valuation ν to the propositional formulae using truth functions $\mathrm{H}_\neg$ and $\mathrm{H}_\vee$.

Definition 2.11 (Tautology)**.** We say that φ is a propositional tautology if and only in for every value function ν, $\nu(\varphi) = \mathbb{T}$.

Theorem 2.12 (Validity Theorem)**.** If φ is a theorem, then φ is tautology.

Theorem 2.13 (Completeness Theorem)**.** If φ is tautology, then $\vdash \varphi$.

Definition 2.14 (Tautological Consequence)**.** Let for $n \geq 0$ $\varphi_1, \varphi_2, \ldots \varphi_n, \varphi$ are formulae. We say that φ is a tautological consequence of $\varphi_1, \varphi_2, \ldots \varphi_n$ if $\varphi_1 \Rightarrow \varphi_2 \Rightarrow \ldots \Rightarrow \varphi_n \Rightarrow \varphi$ is a tautology.

Theorem 2.15 (Tautology Theorem)**.** Let φ be a tautological consequence of $\varphi_1, \varphi_2, \ldots \varphi_n$ and $\vdash \varphi_1, \ldots \vdash \varphi_n$. Then $\vdash \varphi$.

2.2 Facts about Boolean and Contact algebras

A structure $\langle W, \leq \rangle$, where $\leq$ is a binary relation on W, is called a *partially ordered set or (poset)* if and only if for any $x, y \in W$ the following conditions hold:

(i) $x \leq x$ (reflexivity)

(ii) $x \leq y$ and $y \leq x \rightarrow x = y$ (antisymmetry)

(iii) $x \leq y$ and $y \leq z \rightarrow x \leq z$ (transitivity)

The relation $\leq$ is called a *partial order* on W. Let A be a non-empty subset of W. An element $a \in W$ is called an *upper bound* of A if $(\forall x \in A) : (x \leq a)$. The element a is called *least upper bound* of A if a is an upper bound of A and for all other upper bounds b of A we have that $a \leq b$. We could define the dual notions of *lower bound* of A and *greatest lower bound* of A. An element $a \in W$ such that $(\forall x \in W)(x \leq a)$ is called the *greatest* element of W. Similarly, an element $a \in W$ such that $(\forall x \in W) : (a \leq x)$ is called the *smallest* element of W.

Definition 2.16 (Lattice)**.** The partially ordered set $\langle W, \leq \rangle$ is called a *lattice* if every two-element subset of W has greatest lower bound and least upper bound. We will denote greatest lower bound of $\{a, b\}$ with $a \sqcap b$ and the least upper bound of $\{a, b\}$ with $a \sqcup b$. The structure $\langle W, \leq, \sqcap, \sqcup \rangle$ will also be called *lattice*.

Definition 2.17 (Bounded Lattice)**.** A lattice which has a greatest element and a smallest element will be called a *bounded lattice*. We will denote such lattices with $\langle W, \leq, 0, 1, \sqcap, \sqcup \rangle$, where 0 is the smallest and 1 is the greatest element.

Definition 2.18 (Distributive Lattice)**.** A lattice is called *distributive lattice* if it satisfies the following additional conditions (distributive laws):

(D) $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$

$(\widehat{D})$ $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$

Definition 2.19 (Boolean algebra)**.** Let $\mathcal{B} = \langle B, 0_{\mathcal{B}}, 1_{\mathcal{B}}, \sqcap_{\mathcal{B}}, \sqcup_{\mathcal{B}}, *_{\mathcal{B}} \rangle$ be a structure where $(B, \leq, 0_{\mathcal{B}}, 1_{\mathcal{B}}, \sqcap_{\mathcal{B}}, \sqcup_{\mathcal{B}})$ is bounded distributive lattice and the *complementation* operation $*_{\mathcal{B}}$ satisfies the following axioms:

(*1) $a \sqcup_{\mathcal{B}} a^{*_{\mathcal{B}}} = 1_{\mathcal{B}}$

(*2) $a \sqcap_{\mathcal{B}} a^{*_{\mathcal{B}}} = 0_{\mathcal{B}}$

Then $\mathcal{B}$ is called *Boolean algebra*.

Definition 2.20. Let $\mathcal{B} = \langle B, 0_{\mathcal{B}}, 1_{\mathcal{B}}, \sqcap_{\mathcal{B}}, \sqcup_{\mathcal{B}}, *_{\mathcal{B}} \rangle$ be a *Boolean algebra*. If $0_{\mathcal{B}} \neq 1_{\mathcal{B}}$ then $\mathcal{B}$ is called a *nondegenerate Boolean algebra*.

Definition 2.21 (Atom)**.** Let $\mathcal{B} = \langle B, 0_{\mathcal{B}}, 1_{\mathcal{B}}, \sqcap_{\mathcal{B}}, \sqcup_{\mathcal{B}}, *_{\mathcal{B}} \rangle$ be a Boolean algebra. An element $b \in B$ is called an *atom* if and only if $b \neq 0$ and for any $a \in B$ such that $a \leq b$ we have $a = 0$ or $a = b$. So that, the atoms are exactly the minimal elements among the non-zero elements of a Boolean algebra.

Definition 2.22 (Atomic Boolean Algebra)**.** Let $\mathcal{B}$ be a Boolean algebra and let A be the set of its atoms. We say that $\mathcal{B}$ is *atomic* if and only if for every non-zero element $b \in B$, there exists $a \in A$ such that $a \leq b$. Equivalently, every element $b \in B$ is the sum of the atoms a such that $a \leq b$.

Definition 2.23 (Precontact algebra)**.** Let $\langle \mathcal{B}, C \rangle$ be a structure such that $\mathcal{B} = \langle B, 0_{\mathcal{B}}, 1_{\mathcal{B}}, \sqcap_{\mathcal{B}}, \sqcup_{\mathcal{B}}, *_{\mathcal{B}} \rangle$ is a nondegenerate Boolean algebra and the relation $C \subseteq B \times B$ satisfies the following axioms:

$(C1)$ $C(a,b) \to a \neq 0$ and $b \neq 0$

$(C2)$ $C(a,b)$, $a \leq a'$ and $b \leq b' \to C(a',b')$

$(C3)$ $C(a, b \sqcup_{\mathcal{B}} c) \to C(a,b)$ or $C(a,c)$

$(C3')$ $C(a \sqcup_{\mathcal{B}} b, c) \to C(a,c)$ or $C(b,c)$

Then the relation C is called a *precontact relation* on B and the structure $\langle \mathcal{B}, C \rangle$ is called a *precontact algebra*.

Definition 2.24 (Contact algebra)**.** Let $\langle \mathcal{B}, C \rangle$ be a precontact algebra where the precontact C satisfies the additional axioms:

$(C4)$ $C(a,b) \to C(b,a)$

$(C5)$ $a \sqcap_{\mathcal{B}} b \neq 0 \to C(a,b)$

Then C is called a *contact relation* and $\langle \mathcal{B}, C \rangle$ is called a *contact algebra*.

Remark. If C satisfies $(C4)$, only one of the axioms $(C3)$ and $(C3')$ is needed. Also, $(C5)$ is equivalent to $(C5')$ $a \neq 0 \to C(a,a)$.

Definition 2.25 (Connected contact algebra)**.** A contact algebra is called connected contact algebra if it satisfies connectedness axiom:

(Con) $(a \neq 0) \wedge (a \neq 1) \to C(a, a^*)$

2.3 Algorithm for solving systems of linear inequalities with rational coefficients

In this section we will examine an algorithm for solving systems of linear inequalities with rational coefficients. We will consider systems of the following type:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \end{cases}$$

where $a_{ij}, b_i \in \mathbb{Q}$, for $i \in \{1, 2, \ldots m\}$ and $j \in \{1, 2, \ldots n\}$. We will apply Fourier-Motzkin elimination as a method for finding a solution for such system. The idea

of the algorithm is intuitive. It is a procedure which reduces the n-variable problem to an equivalent (n-1)-variable one. We repeat these steps eliminating variables one at a time. Eventually, we will end-ip with 1-variable problem which is easy to solve. We will be able to trace back using the solution for 1-varaible problem to find a solution for 2-variable, 3-variable and finally to n-variable problem.
We suppose that $x_1, x_2, \dots x_n$ are all variables from the system and we want to eliminate x_n. For each inequality $\sum_{1 \le i \le n} a_i x_i \le b$ we will get one of the following two inequalities: $x_n \le \frac{b}{a_n} - \sum_{1 \le i \le n-1} \frac{a_i}{a_n} x_i$ or $x_n \ge \frac{b}{a_n} - \sum_{1 \le i \le n-1} \frac{a_i}{a_n} x_i$ depending on whether $a_n > 0$ or $a_n < 0$. We will end up with $x_n \ge L_1$, $x_n \ge L_2$ $, \dots,$ $x_n \ge L_p$, $x_n \le U_1$, $x_n \le U_2$, $\dots$, $x_n \le U_q$ inequalities which are considered as a lower and upper bounds for x_n. Each L_i and U_j are expressions with variables among $x_1, x_2, \dots, x_{n-1}$. It is possible to choose a value for x_n if and only if $max\{L_1, L_2, \dots L_p\} \le min\{U_1, U_2, \dots U_q\}$. So we replace in the original system all inequalities which contain x_n with these $p.q$ new inequalities - $L_1 \le U_1$, $L_1 \le U_2$, $\dots$, $L_1 \le U_q$, $L_2 \le U_1$, $L_2 \le U_2$ $, \dots,$ $L_p \le U_1$, $\dots$, $L_p \le U_q$. The result system is with variables $x_1, x_2, \dots, x_{n-1}$. We could apply the same procedure to eliminate x_{n-1}.

2.4 Connected graphs

Later in this study we deal up with graphs that correspond to contact relation. So, in this section we will give two well-known definitions for connected graph and we will show that they are equivalent. Therefore, we will use later in this work more convenient one.

Definition 2.26 (Undirected Graph)**.** A graph is a pair $\langle V, E \rangle$ where V is a set of elements called vertices and E is a binary relation $E \subseteq V \times V$ which elements are called edges. A graph is said to be *undirected* if the relation E is symmetric.

Remark. We will consider only *undirected* graphs in this study. So, from now on we will call them only graphs.

Definition 2.27 (Path)**.** A *path* in a graph G between two distinct vertices v and w is a finite sequence of edges from E $\langle v_1, v_2 \rangle, \dots, \langle v_{n-1}, v_n \rangle$ such that:

(i) $v_1 = v$ and $v_n = w$

(ii) $v_1, v_2, \dots v_n \in V$

(iii) $v_1, v_2, \dots v_n$ are different

Definition 2.28 (Path)**.** Let $G = \langle V, E \rangle$ be a graph. A path between two distinct vertices v and w from G is a *k-sequence* $\{x_i\}_{i<k}$ which satisfies the following conditions:

(i) $k > 0$

(ii) for each $i < k-1$ $\langle x_i, x_{i+1} \rangle \in E$ and $x_i \neq x_{i+1}$

(iii) $v = x_0$ and $w = x_{k-1}$

Definition 2.29 (Connected Graph)**.** A graph $G = \langle V, E\rangle$ is said to be *connected* if there is a path between every pair of different vertices.

Lemma 2.30. Let $G = \langle V, E\rangle$ be a graph. Then the following two conditions are equivalent:

(i) G is connected

(ii) $\forall A(A \subseteq V \wedge A \neq \emptyset \wedge V \setminus A \neq \emptyset \Rightarrow (\exists\langle v_1, v_2\rangle \in E)(v_1 \in A \wedge v_2 \in V \setminus A))$

Proof. (i)⇒(ii) Let $G = \langle V, E\rangle$ be connected graph. We will prove this direction by *reductio ad absurdum.* So let us suppose A is a non-empty subset of V such that $V \setminus A \neq \emptyset$ and there is no edge $\langle v_1, v_2\rangle \in E$ such that $v_1 \in A$ and $v_2 \in V \setminus A$. We get a vertex v from A and a vertex w from $V \setminus A$. Since there is no edge connecting A and $V \setminus A$ all accessible vertices from v are from A. Similarly, all accessible vertices from w are from $V \setminus A$. So there is no path between v and w which is a contradiction with G is a connected graph.

(ii)⇒(i) We will also prove this direction by *reductio ad absurdum.* So let *(ii)* be true and G is not connected. Then, there are two vertices v and w such that there is no path between them. Let A be the set of all vertices accessible from v. It is a non-empty subset of V because at least $v \in A$. The set $V \setminus A$ is also non-empty because at least $w \in V \setminus A$. Then by the condition *(ii)* there is an edge $\langle v', w'\rangle$ between A and $V \setminus A$. Since v' is accessible from v then it follows that w' is also accessible from v. Therefore, $w' \in A$, which is a contradiction with $w' \in V \setminus A$. □

3 Contact logics with measure - language and semantics

3.1 Language and notions

We consider first-order language $\mathcal{L}$ without formal equality containing the following symbols:

(i) a countable set $BoolVars$ of *Boolean variables*

(ii) *boolean constants* - 0 and 1

(iii) *function symbols* - $\sqcap$, $\sqcup$ and $*$

(iv) *predicate symbols* - $\leq$, $\leq_\mu$ and C

(v) *propositional connectives* - $\neg$, $\wedge$ and $\vee$

(vi) *brackets* -) and (

We will define the notions of *term* and *formula* in standard way:

Definition 3.1 (Term)**.** The terms in our language $\mathcal{L}$ are constructed from Boolean variables, Boolean constants and function symbols in the following way:

(i) boolean constants 0 and 1 are terms
(ii) every boolean variable $p \in BoolVars$ is a term
(iii) if a and b are terms, then a^*, $a \sqcap b$ and $a \sqcup b$ are also terms

Definition 3.2 (Atomic Formula)**.** Let a and b be terms. Then $a \leq b$, $a \leq_\mu b$ and $C(a, b)$ are atomic formulae.

Definition 3.3 (Formula)**.** Formulae in our language are defined as follows:

(i) $\bot$ and $\top$ are formulae

(ii) atomic formulae are formulae

(iii) if φ and ψ are formulae, then $\neg\varphi$, $\varphi \wedge \psi$ and $\varphi \vee \psi$ are also formulae

Remark. We accept the standard rules in First-order logic for omission of brackets. Additionally, we will use the following abbreviations for convenience

(i) $a = b \stackrel{\text{def}}{=} (a \leq b) \wedge (b \leq a)$
(ii) $a =_\mu b \stackrel{\text{def}}{=} (a \leq_\mu b) \wedge (b \leq_\mu a)$
(iii) $a <_\mu b \stackrel{\text{def}}{=} \neg(b \leq_\mu a)$
(iv) $\varphi \Rightarrow \psi \stackrel{\text{def}}{=} \neg\varphi \vee \psi$
(v) $\varphi \Leftrightarrow \psi \stackrel{\text{def}}{=} (\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$

3.2 Semantics

In this section we will examine a couple of ways for interpreting the statements from our language into different semantic structures.

3.2.1 Algebraic semantics

We start with algebraic semantics for the language $\mathcal{L}$. Let $\mathcal{B} = \langle B, 0_\mathcal{B}, 1_\mathcal{B}, \sqcap_\mathcal{B}, \sqcup_\mathcal{B}, *_\mathcal{B}\rangle$ be a Boolean algebra and C be a contact relation on B. So, the pair $\langle \mathcal{B}, C\rangle$ is a contact algebra. We need a way for interpreting formulae like $(a \leq_\mu b)$. So, we will be interested in measure function μ that satisfies the following conditions:

(i) μ is a positive measure - $\mu : B \to [0, +\infty]$
(ii) $\mu(a) = 0$ if and only if $a = 0_\mathcal{B}$
(iii) $\mu(1_\mathcal{B}) = +\infty$
(iv) μ is additive - if $a \sqcap b = 0_\mathcal{B}$ then $\mu(a \sqcup_\mathcal{B} b) = \mu(a) + \mu(b)$
(v) μ is countably additive - if B_1 is at most countable family of pairwise disjoint elements of B, $supB_1$ exists and $supB_1 \in B$ then $\mu(B_1) = \sum_{b \in B_1} \mu(b)$

Remark. We extend the set $\mathbb{R}^+$ with one more symbol $+\infty$. For any $x \in \mathbb{R}$ we have that $x < +\infty$ and algebraic operations with $+\infty$ are defined in standard way [4].

We are interested in such structures $\mathcal{C} = \langle\langle\mathcal{B}, C\rangle, \mu\rangle$ where $\langle\mathcal{B}, C\rangle$ is a contact algebra and μ satisfies the above conditions. Let v be a valuation $v : BoolVars \to B$. It is extended to all terms of $\mathcal{L}$ in the following way:

$v(0) = 0_{\mathcal{B}}$
$v(1) = 1_{\mathcal{B}}$
$v(a^*) = (v(a))^{*_{\mathcal{B}}}$
$v(a \sqcap b) = v(a) \sqcap_{\mathcal{B}} v(b)$
$v(a \sqcup b) = v(a) \sqcup_{\mathcal{B}} v(b)$

We will call the pair $\mathcal{M} = \langle\mathcal{C}, v\rangle$ a model. The truth of a formula φ in $\mathcal{M}$ is denoted by $\mathcal{M} \models \varphi$. Similarly, we will use $\mathcal{M} \not\models \varphi$ to denote the falsehood of the formula φ in $\mathcal{M}$. We will define the conditions for truth of atomic formulae of $\mathcal{L}$:

$\mathcal{M} \models (a \leq b)$ if and only if $v(a) \leq_{\mathcal{B}} v(b)$
$\mathcal{M} \models C(a, b)$ if and only if $C(v(a), v(b))$
$\mathcal{M} \models (a \leq_\mu b)$ if and only if $\mu(v(a)) \leq \mu(v(b))$

For complex formulae, the definition is extended in the following way:

$\mathcal{M} \models \top$ and $\mathcal{M} \not\models \bot$
$\mathcal{M} \models \neg\varphi$ if and only if $\mathcal{M} \not\models \varphi$
$\mathcal{M} \models \varphi \wedge \psi$ if and only if $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$
$\mathcal{M} \models \varphi \vee \psi$ if and only if $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \psi$

We say that a formula φ is true in structure $\mathcal{C}$ ($\mathcal{C} \models \varphi$) if for all valuations v in $\mathcal{C}$ we have $\langle\mathcal{C}, v\rangle \models \varphi$.

3.2.2 Relational semantics

In this section we will explore one special case of Algebraic semantics given by terms of Kripke frame $\langle W, R\rangle$ where $W \neq \emptyset$, R is reflexive and symmetric relation and $R \subseteq W \times W$. We start with defining the Boolean algebra of all subsets of W : $\mathcal{B} = \langle\mathcal{P}(W), \emptyset, W, \cap, \cup, \setminus\rangle$. We will define the contact relation C_R for $a, b \subseteq W$ in the following way: $C_R(a, b)$ if and only if $(\exists x \in a)(\exists y \in b)R(x, y)$. As in the previous section, we will examine the tuple $\mathcal{C} = \langle\langle\mathcal{B}, C_R\rangle, \mu\rangle$ where $\langle\mathcal{B}, C_R\rangle$ is a contact algebra and μ satisfies the defined conditions for measure. So, $\mathcal{C}$ is a structure for $\mathcal{L}$. Let $v : BoolVars \to \mathcal{P}(W)$ be a valuation which is extended for all terms as follows:

$v(0) = \emptyset$
$v(1) = W$
$v(a^*) = W \setminus v(a)$
$v(a \sqcap b) = v(a) \cap v(b)$
$v(a \sqcup b) = v(a) \cup v(b)$

We will consider the model $\mathcal{M} = \langle \mathcal{C}, v \rangle$ and define the conditions for truth of atomic formulae of $\mathcal{L}$:

$\mathcal{M} \models (a \leq b)$ if and only if $v(a) \subseteq v(b)$
$\mathcal{M} \models C(a, b)$ if and only if $C_R(v(a), v(b))$
$\mathcal{M} \models (a \leq_\mu b)$ if and only if $\mu(v(a)) \leq \mu(v(b))$

It is extended for complex formulae as in the previous section.

We say that a formula φ is true in structure $\mathcal{C}$ ($\mathcal{C} \models \varphi$) if for all valuations v in $\mathcal{C}$ we have $\langle \mathcal{C}, v \rangle \models \varphi$.

3.2.3 Intended Model

Definition 3.4 (Basis Polytopes)**.** Intervals from the type *finite interval* $[m; n]$ where $0 \leq m < n$ and *infinite interval* $[m; +\infty)$ where $0 \leq m$ are called *basis polytopes.*

Definition 3.5 (Polytope)**.** *Polytope* is a finite union of basis polytopes. We denote the set of polytopes in $\mathbb{R}^+$ with $Pol(\mathbb{R}^+)$.

Remark. $\emptyset \in Pol(\mathbb{R}^+)$

We will consider the following tuple $\mathcal{B} = \langle Pol(\mathbb{R}^+), \emptyset, \mathbb{R}^+, \sqcap_\mathcal{B}, \sqcup_\mathcal{B}, *_\mathcal{B} \rangle$ and we will define the operations $\sqcap_\mathcal{B}$, $\sqcup_\mathcal{B}$ and $*_\mathcal{B}$ as follows:

$a \sqcap_\mathcal{B} b \stackrel{\text{def}}{=} Cl(Int(a \cap b))$ for $a, b \in Pol(\mathbb{R}^+)$
$a \sqcup_\mathcal{B} b \stackrel{\text{def}}{=} a \cup b$ for $a, b \in Pol(\mathbb{R}^+)$
$a^{*_\mathcal{B}} \stackrel{\text{def}}{=} Cl(\mathbb{R}^+ \setminus a)$ for $a \in Pol(\mathbb{R}^+)$

The contact relation C is defined for the elements in $Pol(\mathbb{R}^+)$ in the following way: $C(a, b)$ if and only if $a \cap b \neq \emptyset$. These definitions of $\mathcal{B}$ and C give us a contact algebra $\langle \mathcal{B}, C \rangle$. It is a sub-algebra of regular-closed sets in $\mathbb{R}^+$.
In our case we consider μ to be the Lebesgue measure on $\mathbb{R}^+$. Since intervals in $\mathbb{R}^+$ are Lebesgue measurable the measure μ satisfies the conditions in Section 3.2.1. So we will define $\mu : Pol(\mathbb{R}^+) \to [0; +\infty]$ as follows:

$$\mu(a) = \begin{cases} \sum_{[i;j] \in a} (j - i) \ , \ if\ a\ contains\ only\ finite\ intervals \\ +\infty \ , \ if\ a\ contains\ infinite\ interval \end{cases}$$

As we already mentioned μ is additive and countably additive. So,

- If $a \sqcap_\mathcal{B} b = \emptyset$, then $\mu(a \sqcup_\mathcal{B} b) = \mu(a) + \mu(b)$.
- Let $I \subseteq \omega$, for each $i \in I$ $a_i \in Pol(\mathbb{R}^+)$, for each $i, j \in I,\ i \neq j,\ a_i \sqcap_\mathcal{B} a_j = \emptyset$ and $\bigsqcup_{i \in I} a_i \in Pol(\mathbb{R}^+)$. Then $\mu(\bigsqcup_{i \in I} a_i) = \sum_{i \in I} \mu(a_i)$.

Remark. By a Birkhoff theorem, does not exist countably additive measure on regular closed sets.

Similarly to previous section $\mathcal{C} = \langle\langle\mathcal{B}, C\rangle, \mu\rangle$ is a structure for $\mathcal{L}$. We will remind the definition of valuation $v : BoolVars \to Pol(\mathbb{R}^+)$ and the way it is extended to all terms of $\mathcal{L}$ as follows:

$v(0) = \emptyset$
$v(1) = \mathbb{R}^+$
$v(a^*) = v(a)^{*_\mathcal{B}}$
$v(a \sqcap b) = v(a) \sqcap_\mathcal{B} v(b)$
$v(a \sqcup b) = v(a) \sqcup_\mathcal{B} v(b)$

Now we will remind the truth of atomic formulae in the model $\mathcal{M} = \langle\mathcal{C}, v\rangle$:

$\mathcal{M} \models (a \leq b)$ if and only if $v(a) \subseteq v(b)$
$\mathcal{M} \models C(a, b)$ if and only if $C(v(a), v(b))$
$\mathcal{M} \models (a \leq_\mu b)$ if and only if $\mu(v(a)) \leq \mu(v(b))$

It is extended for the complex formulae on the standard way.

We say that a formula φ is true in structure $\mathcal{C}$ ($\mathcal{C} \models \varphi$) if for all valuations v in $\mathcal{C}$ we have $\langle\mathcal{C}, v\rangle \models \varphi$. We are interested in all valid formulae in the structure of polytopes.

4 Axiomatization

We follow the idea for our axiomatic system from Section 3 from [1]. So that, our axiomatic system for $\mathcal{L}$ will contain one rule for inference - Modus Ponens (MP). We will take as axioms the complete set of formulae which are substitution instances of tautologies of classical prepositional logic, modification of axioms for Boolean algebra, axioms for contact algebra, measure axioms and axioms for systems of linear inequalities. We denote the set of these axioms with $\mathcal{L}_{HL}$.

4.1 Axiomatic System $\mathcal{L}_{HL}$

Axioms

(i) all formulae which are substitution instances of tautologies of classical propositional logic

(ii) a set of axiom schemes for Boolean algebra

(B1) $a \leq a$

(B2) $0 \leq a$

(B3) $a \leq 1$

(B4) $a^{**} \leq a$

(B5) $a \sqcap (b \sqcup c) \leq (a \sqcap b) \sqcup (a \sqcap c)$

(B6) $(a \leq b) \wedge (b \leq c) \Rightarrow (a \leq c)$

(B7) $(a \sqcup b) \leq c \Leftrightarrow (a \leq c) \wedge (b \leq c)$

(B8) $c \leq (a \sqcap b) \Leftrightarrow (c \leq a) \wedge (c \leq b)$

(B9) $(a \sqcap b^* \leq 0) \Leftrightarrow (a \leq b)$

(B10) $\neg(0 = 1)$

(iii) axiom schemes for contact C

(C1) $(a \neq 0) \Leftrightarrow C(a, a)$

(C2) $C(a, b \sqcup c) \Leftrightarrow C(a, b) \vee C(a, c)$

(C3) $C(a, b) \Rightarrow C(b, a)$

(Con) $(a \neq 0) \wedge (a \neq 1) \Rightarrow C(a, a^*)$

(iv) axioms for measure

(M1) $(a \leq_\mu b) \wedge (b \sqcap d = 0) \Rightarrow (a \sqcup d) \leq_\mu (b \sqcup d)$

(M2) $(a \sqcap d = 0) \wedge (b \sqcap d = 0) \wedge (d <_\mu 1) \Rightarrow ((a \leq_\mu b) \Leftrightarrow (a \sqcup d \leq_\mu b \sqcup d))$

(M3) $(a \sqcap d = 0) \wedge (b \sqcap d = 0) \wedge (d <_\mu 1) \Rightarrow ((a <_\mu b) \Leftrightarrow (a \sqcup d <_\mu b \sqcup d))$

(M4) $a =_\mu 1 \vee a^* =_\mu 1$

(M5) $a =_\mu 1 \wedge b =_\mu 1 \Rightarrow a \sqcap b =_\mu 1$

(M6) $a = 0 \Leftrightarrow a =_\mu 0$

(v) We add the set of axioms $M7_n$ for each natural number n as described in Section 4.3.

Rules of inference.

$$\frac{\varphi, (\varphi \Rightarrow \psi)}{\psi} \ (MP)$$

4.2 Substitution

We do not have *Uniform Substitution* in our axiomatic system. However, we could prove that rule.

Lemma 4.1 (Uniform Substitution)**.** Let φ be a formula from $\mathcal{L}$ and $p_1, p_2, \ldots, p_n$ be all propositional variables from φ. Let $t_1, t_2, \ldots, t_n$ be terms from $\mathcal{L}$. If $\vdash \varphi$, then $\vdash \varphi[p_1/t_1, p_2/t_2, \ldots, p_n/t_n]$.

Proof. Let φ be a theorem of Ax_M. Then there exists finite sequence $\varphi_1, \varphi_2, \ldots, \varphi_k$ where $\varphi_k = \varphi$. We will prove by induction on i - each formula φ_i in this sequence is either axiom or obtained by applying *(MP)* on formulae with indices smaller than i.

Case 1: φ_i is an axiom. In this case when we substitute propositional variables $p_1, p_2, \ldots, p_n$ for terms $t_1, t_2, \ldots t_n$ in an axiom then we obtain an instance of the same axiom. We will consider the following axiom $p_1 \sqcup p_2 \sqcup \ldots \sqcup p_n \sqcup p_{n+1} \sqcup \ldots \sqcup p_{n+l} \leq p_1 \sqcup p_2 \sqcup \ldots \sqcup p_n \sqcup p_{n+1} \sqcup \ldots \sqcup p_{n+l}$ and when we apply *Uniform Substitution* we obtain $t_1 \sqcup t_2 \sqcup \ldots \sqcup t_n \sqcup p_{n+1} \sqcup \ldots \sqcup p_{n+l} \leq t_1 \sqcup t_2 \sqcup \ldots \sqcup t_n \sqcup p_{n+1} \sqcup \ldots \sqcup p_{n+l}$ and it is an instance of the axiom $a \leq a$ where a is a term. So if $\vdash \varphi_i$, then $\vdash \varphi_i[p_1/t_1, p_2/t_2, \ldots, p_n/t_n]$. We could prove that substituting propositional variables with term in the axiom is actually an instance of the same axiom.

Case 2: φ_i is obtained by applying the *(MP)* on some formulae with indices smaller than i. Then there are formulae with indices $j, \ell < i$ such that $\varphi_\ell = \varphi_j \Rightarrow \varphi_i$ and for these formulae the induction hypothesis holds then $\vdash \varphi_j[p_1/t_1, p_2/t_2, \ldots, p_n/t_n]$ and $\vdash \varphi_\ell[p_1/t_1, p_2/t_2, \ldots, p_n/t_n]$. So, we have $\vdash \varphi_j[p_1/t_1, p_2/t_2, \ldots, p_n/t_n]$, $\vdash \varphi_j[p_1/t_1, p_2/t_2, \ldots, p_n/t_n] \Rightarrow \varphi_i[p_1/t_1, p_2/t_2, \ldots, p_n/t_n]$ and by applying *(MP)* we obtain $\vdash \varphi_i[p_1/t_1, p_2/t_2, \ldots, p_n/t_n]$.

□

4.3 Set of axioms $M7_n$

In this section we will describe a special type of linear inequalities systems that will be studied. We will explain how to associate a formula $\Phi_{\mathcal{S}}$ from our language $\mathcal{L}$ to an S_n-system. We also will prove that an S_n-system has a solution is equivalent to its corresponding $\Phi_{\mathcal{S}}$ formula to be satisfiable. At the end we will form set of axioms for an S_n-system $M7_n$.

4.3.1 Notion

Definition 4.2 ($(n, \leq)$-type inequality)**.** Let $x_1, x_2, ..., x_n$ be variables for real numbers and $I_\ell, I_r \subseteq \{1, 2, \ldots, n\}$. Then an expression of the type

$$\sum_{i \in I_\ell} x_i \leq \sum_{i \in I_r} x_i$$

is called $(n, \leq)$-type inequality.

Definition 4.3 ($(n, <)$-type inequality)**.** Let $x_1, x_2, ..., x_n$ be variables for real numbers and $I_\ell, I_r \subseteq \{1, 2, \ldots, n\}$. Then an expression of the type

$$\sum_{i \in I_\ell} x_i < \sum_{i \in I_r} x_i$$

is called $(n, <)$-type inequality.

Remark. We will denote such inequalities with σ.

First of all we consider the following system:

$$\begin{cases} \sum_{i\in I_\ell^1} x_i \leq \sum_{i\in I_r^1} x_i \\ \dots \\ \sum_{i\in I_\ell^p} x_i \leq \sum_{i\in I_r^p} x_i \\ \sum_{i\in I_\ell^{p+1}} x_i < \sum_{i\in I_r^{p+1}} x_i \\ \dots \\ \sum_{i\in I_\ell^q} x_i < \sum_{i\in I_r^q} x_i \end{cases}$$

We are interested whether there exists an algorithm which determines for finite number of steps if such system has a solution with exactly one variable equal to $+\infty$ if such exists. We will explore such an algorithm in the next section.

4.3.2 Algorithm for solving systems of $(n, \leq)$-type and $(n, <)$-type inequalities

Lemma 4.4. There exists an algorithm which for any system $\mathcal{S}$ containing only $(n, \leq)$-type and $(n, <)$-type inequalities finds a solution with exactly one variable equal to $+\infty$ if such exists or returns $\emptyset$ otherwise. The algorithm finishes for finite number of steps.

Proof. Let $\mathcal{S}$ be a system and $\sigma_1, \sigma_2, \dots, \sigma_m$ be all inequalities of that system. So, any of σ_i for $i \in \{1, 2, \dots, m\}$ is either $(n, \leq)$-type inequality or $(n, <)$-type one. We will describe an algorithm which reduces the system to another system $\mathcal{S}'$ such that the new system has $n - 1$ variables and less inequalities. We will prove that a non-negative solution for $\mathcal{S}'$ could be extended to a non-negative solution with exactly one component equal to $+\infty$ for $\mathcal{S}$. On each iteration we consider a new variable and give it value $+\infty$ so on i-th iteration we assign $+\infty$ to x_i, and then decide which inequalities have to be included in $\mathcal{S}'$. If we are in the case when the new system could not be constructed when $x_i = +\infty$ we continue with x_{i+1}, for $i + 1 \leq n$. Now we will describe the algorithm $solve_{\mathcal{S}_n}^{+\infty}$ in more details - we assume we are on the i-th iteration and we assign $x_i = +\infty$. We will apply the following rules on every inequality of $\mathcal{S}$ in order to construct $\mathcal{S}'$:

Case 1: Current inequality σ_j is of $(n, \leq)$-type (Def 4.2). Then σ_j has the following representation: $\sum_{i\in I_\ell} x_i \leq \sum_{i\in I_r} x_i$.

Case 1.1: If $i \in I_\ell$ and $i \notin I_r$ then $\mathcal{S}$ has no solution with $x_i = +\infty$ so we continue with x_{i+1}.

Case 1.2: If $i \in I_r$ then we skip this inequality.

Case 1.3: If $i \notin I_\ell$ and $i \notin I_r$ then we include σ_j in $\mathcal{S}'$.

Case 2: Current inequality σ_j is of $(n, <)$-type (Def 4.3). Then σ_j has the following representation: $\sum_{i\in I_\ell} x_i < \sum_{i\in I_r} x_i$.

Case 2.1: If $i \in I_\ell$ then $\mathcal{S}$ has no solution with $x_i = +\infty$ so we continue with x_{i+1}.

Case 2.2: If $i \notin I_\ell$ and $i \in I_r$ then we skip this inequality.

Case 2.3: If $i \notin I_\ell$ and $i \notin I_r$ then we include σ_j in $\mathcal{S}'$.

In case the algorithm could not construct a system $\mathcal{S}'$, it indicates with $\emptyset$ that no solution of desired type exists. So, $\mathcal{S}'$ is constructed by the above procedure without x_i for $i \in [1, n]$. If we are in the case when all inequalities from $\mathcal{S}$ contain x_i, then $\mathcal{S}'$ does not have any inequalities left. So, every list of real numbers $(r_1, \ldots, r_{i-1}, +\infty, r_{i+1}, \ldots, r_n)$ is a solution for $\mathcal{S}$ and the algorithm finishes with this result. The other case is when $\mathcal{S}'$ is not empty, then we apply the algorithm for solving system of linear inequalities described in the Section 2.3. If it does not find a solution then the $solve^{+\infty}_{\mathcal{S}_n}$ finishes with returning $\emptyset$. Let the algorithm from Section 2.3 finds $(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n)$ which is a non-negative solution for $\mathcal{S}'$. We will check what will happen with all inequalities which contain x_i when substitute variables with $(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n)$:

Case 1: If we perform the above operation on $(n, \leq)$-type:

Case 1.1: In the skipped inequality only I_r contains i. Then $\sum_{s \in I_\ell} r_s \leq \sum_{t \in I_r \setminus \{i\}} r_t + (+\infty)$. This is equivalent to $\sum_{s \in I_\ell} r_s \leq +\infty$ which is correct inequality.

Case 1.2: In the skipped inequality both I_ℓ and I_r contain i. Then $\sum_{s \in I_\ell \setminus \{i\}} r_s + (+\infty) \leq \sum_{t \in I_r \setminus \{i\}} r_t + (+\infty)$. This is equivalent to $+\infty \leq +\infty$ which is correct inequality.

Case 2: If we perform the above operation on $(n, <)$-type. Then $\sum_{s \in I_l} r_s < \sum_{t \in I_r \setminus \{i\}} r_t + (+\infty)$. It is equivalent to $\sum_{s \in I_\ell} r_s < +\infty$. So, we get correct inequality.

We have just proved that $(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n)$ where $r_i = +\infty$ is a solution for the skipped inequalities. We know that it is a solution for all inequalities from $\mathcal{S}$ which does not contain x_i. Then $(r_1, r_2, \ldots, r_n)$ is a non-negative solution such that exactly one component is equal to $+\infty$. $\square$

4.3.3 S_n-systems

Definition 4.5 (S_n-system)**.** A system $\mathcal{S}$ of the type

$$\begin{cases} \sum_{i\in I_\ell^1} x_i \le \sum_{i\in I_r^1} x_i \\ \dots \\ \sum_{i\in I_\ell^p} x_i \le \sum_{i\in I_r^p} x_i \\ \sum_{i\in I_\ell^{p+1}} x_i < \sum_{i\in I_r^{p+1}} x_i \\ \dots \\ \sum_{i\in I_\ell^q} x_i < \sum_{i\in I_r^q} x_i \\ 0 \le x_1 \\ 0 \le x_2 \\ \dots \\ 0 \le x_n \end{cases}$$

is called S_n-system.

4.3.4 Associate formula form $\mathcal{L}$ to S_n-system

First of all we will show how to associate formulae to both types of inequalities. Let $p_1, p_2, \dots, p_n \in BoolVars$. For inequality of $(n, \le)$-type (Def. 4.2):

$$\bigsqcup_{i\in I_\ell} p_i \le_\mu \bigsqcup_{i\in I_r} p_i$$

Using the same idea

$$\bigsqcup_{i\in I_\ell} p_i <_\mu \bigsqcup_{i\in I_r} p_i$$

corresponds to the inequality of $(n, <)$-type (Def. 4.3). For such formulae we will use φ_σ. So that, we associate φ_{σ_i} with the i-th inequality of a given S_n-system. Thus, for any S_n-system $\mathcal{S}$ we have

$$\varphi_{\mathcal{S}} = \bigwedge_{1\le i\le m} \varphi_{\sigma_i}$$

We need to add two more conditions

$$\Phi_{\mathcal{S}} = \bigwedge_{1\le i<j\le n} (p_i \sqcap p_j = 0) \wedge (\bigsqcup_{1\le i\le n} p_i = 1) \wedge \varphi_{\mathcal{S}}$$

Remark. To the discussed so far systems we added n new inequalities which ensure non-negative solution.

Remark. We will consider $\sum_{i\in\emptyset} x_i$ as abbreviation for 0. So, $0 \le x_i$ is a $(n, \le)$-type inequality where $I_\ell = \emptyset$ and $I_r = \{i\}$.

Remark. The algorithm described in Lemma 4.4 could be applied on S_n-systems and finds whether it has solution with exactly one component equals to $+\infty$.

4.3.5 Set of axioms for S_n-system

We proved so far:

(i) We could determine in finitely many steps whether a given system has a solution with exactly one variable equal to $+\infty$ - the result from Lemma 4.4.

(ii) For a given n, S_n-systems are finitely many. First of all we will consider $(n, \leq)$-type inequality $\sum_{i\in I_\ell} x_i \leq \sum_{i\in I_r} x_i$ and $I_\ell, I_r \subseteq \{1, 2, \ldots, n\}$. Since the sets I_ℓ, I_r are subsets of the numbers $\{1, 2, \ldots, n\}$ then all $(n, \leq)$-type inequalities for given n are 2^{2n}. We apply the same argument for $(n, <)$-type inequality and evaluate that for fixed n the number of all inequalities of both types is 2^{2n+1}. Finally, all S_n-systems for given n are $2^{2^{2n+1}}$.

So, for each $n \in \mathbb{N}$ for all S_n-systems $\mathcal{S}$ that do not have a solution with one component equal to $+\infty$ we associate the following formula:

$$\Phi^{ax}_{\mathcal{S}} = \bigwedge_{1\leq i<j\leq n} (p_i \sqcap p_j = 0) \wedge (\bigsqcup_{1\leq i\leq n} p_i = 1) \Rightarrow \neg\varphi_{\mathcal{S}}$$

We take all substitution instances of $\Phi^{ax}_{\mathcal{S}}$ as axioms. That is $\Phi^{ax}_{\mathcal{S}}[p_1/t_1, p_2/t_2, \ldots, p_n/t_n]$ where $t_1, t_2, \ldots, t_n$ are terms.

4.4 Equivalences with the classic set of axioms for Boolean algebras

In this section we will show that with the axioms *(B1)÷(B10)* we could prove the standard axioms for Boolean algebra.

Lemma 4.6. Let a,b and c be terms from $\mathcal{L}$. Then:

(1) $\vdash a \sqcap a = a$

(1′) $\vdash a \sqcup a = a$

(2) $\vdash a \leq a \sqcup b$

(2′) $\vdash b \leq a \sqcup b$

(3) $\vdash a \sqcap b \leq a$

(3′) $\vdash a \sqcap b \leq b$

(4) $\vdash a \sqcap b = b \sqcap a$

(4′) $\vdash a \sqcup b = b \sqcup a$

(5) $\vdash a \sqcup 0 = a$

(5′) $\vdash a \sqcap 0 = 0$

(6) $\vdash a \sqcup 1 = 1$

(6′) $\vdash a \sqcap 1 = a$

(7) $\vdash a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c$

(7′) $\vdash a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$

(8) $\vdash a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$

(8′) $\vdash a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$

(9) $\vdash (a \sqcup b)^* = a^* \sqcap b^*$

(9′) $\vdash (a \sqcap b)^* = a^* \sqcup b^*$

(10) $\vdash a \sqcup a^* = 1$

(10′) $\vdash a \sqcap a^* = 0$

(11) $\vdash a \le b \Leftrightarrow b^* \le a^*$

(12) $\vdash a = a^{**}$

(13) $\vdash 0^* = 1$

(14) $\vdash 1^* = 0$

Proof. We will show a proof for some of them and the others could be proved using similar arguments.

(1) We need to prove $\vdash a \sqcap a \le a$ and $\vdash a \le a \sqcap a$ since $=$ is abbreviation.

(1.1) We will prove $\vdash a \le a \sqcap a$. We start with an instance of axiom (B8): $\vdash a \le (a \sqcup a) \Leftrightarrow (a \le a) \wedge (a \le a)$. We use that "$\Leftrightarrow$" is abbreviation and the tautology $\varphi \wedge \psi \Rightarrow \varphi$ to infer $\vdash (a \le a) \wedge (a \le a) \Rightarrow a \le (a \sqcup a)$. We consider that $\vdash (a \le a) \Rightarrow (a \le a) \Rightarrow (a \le a) \wedge (a \le a)$, $a \le a$ is an instance of *(B1)* and *(MP)* so we obtain $\vdash (a \le a) \wedge (a \le a)$. Thus, $\vdash a \le (a \sqcup a)$.

(1.2) The other direction is $\vdash a \sqcap a \le a$ and we start with an instance of (B1): $\vdash a \sqcap a \le a \sqcap a$. We apply the axiom *(B8)* $\vdash a \sqcap a \le a \sqcap a \Leftrightarrow (a \sqcap a \le a) \wedge (a \sqcap a \le a)$. Similarly to (1.1) using that "$\Leftrightarrow$" is abbreviation, the axiom $\vdash a \sqcap a \le a \sqcap a$ and *(MP)* we obtain $\vdash (a \sqcap a \le a) \wedge (a \sqcap a \le a)$. From here and the tautology $\vdash (a \sqcap a \le a) \wedge (a \sqcap a \le a) \Rightarrow a \sqcap a \le a$ we infer $\vdash a \sqcap a \le a$.

(2) We will prove $a \le a \sqcup b$ but using exactly the same proof except the last step where we use the tautology $\varphi \wedge \psi \Rightarrow \varphi$. If we use $\varphi \wedge \psi \Rightarrow \psi$, then it will be a proof for $(3')$. We start with the axiom *(B7)*: $\vdash a \sqcup b \le a \sqcup b \Leftrightarrow (a \le a \sqcup b) \wedge (b \le a \sqcup b)$. From here and using that "$\Leftrightarrow$" is an abbreviation, $\vdash a \sqcup b \le a \sqcup b$ and *(MP)* we infer $\vdash (a \le a \sqcup b) \wedge (b \le a \sqcup b)$. Here is the step where we use the tautology $\varphi \wedge \psi \Rightarrow \varphi$ and obtain $\vdash a \le a \sqcup b$. If we use $\varphi \wedge \psi \Rightarrow \psi$ then we infer $\vdash b \le a \sqcup b$ which is a proof for $(3')$.

(3) Similarly to (2) the proof for (3) is the same as the proof for $3'$ except the last step. We start with an instance of the axiom *(B8)*: $\vdash a \sqcap b \le a \sqcap b \Leftrightarrow (a \sqcap b \le a) \wedge (a \sqcap b \le b)$. We infer $\vdash (a \sqcap b \le a) \wedge (a \sqcap b \le b)$ from the tautology $\varphi \wedge \psi \Rightarrow \varphi$, the axiom $a \sqcap b \le a \sqcap b$ and *(MP)*. We will use one more time that tautology and obtain $\vdash a \sqcap b \le a$. If we consider the tautology $\varphi \wedge \psi \Rightarrow \psi$ then we will infer $\vdash a \sqcap b \le b$.

$(4')$ We will prove $\vdash a \sqcup b \le b \sqcup a$ and $\vdash b \sqcup a \le a \sqcup b$. We will show a proof only for $\vdash a \sqcup b \le b \sqcup a$ because the other direction is the same. We know that $\vdash a \le b \sqcup a$ and $\vdash b \le b \sqcup a$. We use that $\vdash a \le b \sqcup a \Rightarrow b \le b \sqcup a \Rightarrow (a \le b \sqcup a) \wedge (b \le b \sqcup a)$ and *(MP)* to infer $\vdash (a \le b \sqcup a) \wedge (b \le b \sqcup a)$. From the axiom *(B7)* we obtain $\vdash (a \le b \sqcup a) \wedge (b \le b \sqcup a) \Rightarrow a \sqcup b \le b \sqcup a$. From here and *(MP)* $\vdash a \sqcup b \le b \sqcup a$. Similarly, we obtain $\vdash b \sqcup a \le a \sqcup b$ and so $\vdash a \sqcup b = b \sqcup a$.

(9) We will prove $\vdash (a \sqcup b)^* \le a^* \sqcap b^*$ and $\vdash a^* \sqcap b^* \le (a \sqcup b)^*$.

(9.1) We start with $\vdash a \le a \sqcup b \Leftrightarrow (a \sqcup b)^* \le a^*$. So, we infer $\vdash a \le a \sqcup b \Rightarrow (a \sqcup b)^* \le a^*$. We proved in (2) that $\vdash a \le a \sqcup b$ and using *(MP)* we obtain

$(a \sqcup b)^* \leq a^*$. Similarly, we infer $(a \sqcup b)^* \leq b^*$. From here and using the tautology $\varphi \Rightarrow \psi \Rightarrow \varphi \wedge \psi$ we prove $\vdash (a \sqcup b)^* \leq a^* \wedge (a \sqcup b)^* \leq b^*$. Now we consider an instance of the axiom *(B8)*: $\vdash (a \sqcup b)^* \leq a^* \sqcap b^* \Leftrightarrow (a \sqcup b)^* \leq a^* \wedge (a \sqcup b)^* \leq b^*$. So we infer $\vdash (a \sqcup b)^* \leq a^* \wedge (a \sqcup b)^* \leq b^* \Rightarrow (a \sqcup b)^* \leq a^* \sqcap b^*$. By *(MP)* $\vdash (a \sqcup b)^* \leq a^* \sqcap b^*$.

(9.2) We start with $\vdash a^* \sqcap b^* \leq a^*$. We infer $\vdash a^{**} \leq (a^* \sqcap b^*)^*$ from the last formula, *(MP)* and $\vdash a^* \sqcap b^* \leq a^* \Leftrightarrow a^{**} \leq (a^* \sqcap b^*)^*$. We consider the following tautology $a \leq a^{**} \Rightarrow a^{**} \leq (a^* \sqcap b^*)^* \Rightarrow a \leq (a^* \sqcap b^*)^*$ and using *(MP)* two times we get $\vdash a \leq (a^* \sqcap b^*)^*$. Analogously, we obtain $\vdash b \leq (a^* \sqcap b^*)^*$. From the last two formulae and the propositional tautology $\varphi \Rightarrow \psi \Rightarrow \varphi \wedge \psi$ we infer $\vdash a \leq (a^* \sqcap b^*)^* \wedge b \leq (a^* \sqcap b^*)^*$. Now we use an instance of the axiom *(B7)* $\vdash a \sqcup b \leq (a^* \sqcap b^*)^* \Leftrightarrow a \leq (a^* \sqcap b^*)^* \wedge b \leq (a^* \sqcap b^*)^*$. Then from the tautology $\varphi \wedge \psi \Rightarrow \varphi$ follows $\vdash a \leq (a^* \sqcap b^*)^* \wedge b \leq (a^* \sqcap b^*)^* \Rightarrow a \sqcup b \leq (a^* \sqcap b^*)^*$. So we infer $\vdash a \sqcup b \leq (a^* \sqcap b^*)^*$. We have proved that $\vdash a \sqcup b \leq (a^* \sqcap b^*)^* \Leftrightarrow (a^* \sqcap b^*)^{**} \leq (a \sqcup b)^*$. From the last two formulae we obtain $\vdash (a^* \sqcap b^*)^{**} \leq (a \sqcup b)^*$. The formula $a^* \sqcap b^* \leq (a \sqcup b)^*$ is a tautological consequence of the last formula and $a^* \sqcap b^* \leq (a^* \sqcap b^*)^{**}$. Thus, $\vdash a^* \sqcap b^* \leq (a \sqcup b)^*$.

(9′) We have to prove $\vdash (a \sqcap b)^* \leq a^* \sqcup b^*$ and $\vdash a^* \sqcup b^* \leq (a \sqcap b)^*$.

(9′.1) The direction $\vdash a^* \sqcup b^* \leq (a \sqcap b)^*$ is similar to (9.1) so it is briefly mentioned. We start with $\vdash a \sqcap b \leq a$ and infer $\vdash a^* \leq (a \sqcap b)^*$. From here and $\vdash b^* \leq (a \sqcap b)^*$ we obtain $\vdash a^* \leq (a \sqcap b)^* \wedge b^* \leq (a \sqcap b)^*$. From the axiom *(B7)* we get that $\vdash a^* \leq (a \sqcap b)^* \wedge b^* \leq (a \sqcap b)^* \leq (a \sqcap b)^* \Rightarrow (a^* \sqcup b^*) \leq (a \sqcap b)^*$. Thus, $\vdash a^* \sqcup b^* \leq (a \sqcap b)^*$.

(9′.2) Analogously, the direction $(a \sqcap b)^* \leq a^* \sqcup b^*$ is similar to (9.2). We start with $\vdash a^* \leq a^* \sqcup b^*$. We infer $\vdash (a^* \sqcup b^*)^* \leq a$ as a tautological consequence of $a^{**} \leq a$ and $(a^* \sqcup b^*)^* \leq a^{**}$. Similarly we obtain $\vdash (a^* \sqcup b^*)^* \leq b$ so we have $\vdash (a^* \sqcup b^*)^* \leq a \wedge (a^* \sqcup b^*)^* \leq b$. From the axiom *(B8)* and *(MP)* we derive $\vdash (a^* \sqcup b^*)^* \leq a \sqcap b$. We will use (11) and infer $\vdash (a \sqcap b)^* \leq (a^* \sqcup b^*)^{**}$. We obtain $\vdash (a \sqcap b)^* \leq a^* \sqcup b^*$ as a tautological consequence of the last formula and $(a^* \sqcup b^*)^{**} \leq a^* \sqcup b^*$.

(10′) We will prove $\vdash a \sqcap a^* \leq 0$ because we have an axiom $\vdash 0 \leq a \sqcap a^*$. We start with the axiom *(B9)*: $\vdash a \leq a \Leftrightarrow a \sqcap a^* \leq 0$. Then we obtain $\vdash a \sqcap a^* \leq 0$ from $\vdash a \leq a \Leftrightarrow a \sqcap a^* \leq 0$, the axiom $a \leq a$ and *(MP)*. So, we have $\vdash a \sqcap a^* \leq 0$ and $\vdash 0 \leq a \sqcap a^*$. Thus, $\vdash a \sqcap a^* = 0$.

(11) We have to prove $\vdash a \leq b \Rightarrow b^* \leq a^*$ and $\vdash b^* \leq a^* \Rightarrow a \leq b$.

(11.1) In order to prove $\vdash a \leq b \Rightarrow b^* \leq a^*$ we start with the axiom *(B9)*: $\vdash a \leq b \Leftrightarrow a \sqcap b^* \leq 0$ and we obtain $\vdash a \leq b \Rightarrow a \sqcap b^* \leq 0$. We also proved that $\vdash a \sqcap b^* = b^* \sqcap a$. The formula $a \leq b \Rightarrow b^* \sqcap a \leq 0$ is tautological consequence from the last two formulae so that $\vdash a \leq b \Rightarrow b^* \sqcap a \leq 0$. We infer analogously $\vdash a \leq b \Rightarrow b^* \sqcap a^{**} \leq 0$ as tautological consequence from last formula and $\vdash a =$

a^{**}. Now we use an instance of the axiom *(B9)* $\vdash b^* \sqcap a^{**} \leq 0 \Leftrightarrow b^* \leq a^*$. From here and Theorem 2.9 we obtain $\vdash (a \leq b \Rightarrow b^* \leq a^*) \Leftrightarrow (a \leq b \Rightarrow b^* \sqcap a^{**} \leq 0)$. Thus, we infer $\vdash a \leq b \Rightarrow b^* \leq a^*$ using that "$\Leftrightarrow$" is an abbreviation, *(MP)* and the propositional tautology $\vdash \varphi \wedge \psi \Rightarrow \varphi$.

(11.2) The proof for $\vdash b^* \leq a^* \Rightarrow a \leq b$ starts with the following instance of the axiom *(B9)* $\vdash b^* \leq a^* \Leftrightarrow b^* \sqcap a^{**} \leq 0$ and the other steps are similar.

(12) We have to prove $\vdash a^{**} = a$. We start with the axiom *(B9)*: $\vdash a^{**} \leq a \Leftrightarrow a^{**} \sqcap a^* \leq 0$. We obtain $\vdash a^{**} \sqcap a^* \leq 0$ using that "$\Leftrightarrow$" is an abbreviation, the axiom $a^{**} \leq a$ and *(MP)*. We also have that $\vdash 0 \leq a^{**} \sqcap a^*$. Then, $\vdash a^{**} \sqcap a^* = 0$. We consider the following tautology $a^{**} \sqcap a^* = 0 \Rightarrow a \sqcap a^* = 0 \Rightarrow a^{**} = a$. Hence, we infer $\vdash a^{**} = a$.

□

5 S_n-systems and HL-structures

In this section we will describe HL-structures. We will prove that the S_n-system $\mathcal{S}$ has a solution and $\Phi_{\mathcal{S}}$ is satisfiable in HL-structure are equivalent. We start with giving some definitions that will be used later in this study.

5.1 HL-structures

Definition 5.1 (HL-measure)**.** A measure μ on Boolean algebra $\mathcal{B} = \langle B, 0_{\mathcal{B}}, 1_{\mathcal{B}}, \sqcap_{\mathcal{B}}, \sqcup_{\mathcal{B}}, *_{\mathcal{B}} \rangle$ is an *HL-measure* if:

(i) μ is positive, i.e. $\mu(a) = 0$ if and only if $a = 0_{\mathcal{B}}$
(ii) $\mu(1_{\mathcal{B}}) = +\infty$
(iii) if $\mu(a) = \mu(b) = +\infty$ then $\mu(a \sqcap_{\mathcal{B}} b) = +\infty$

Remark. If $\mu(a) = +\infty$ and $\mu(a) \leq \mu(b)$, then $\mu(a) = \mu(b)$.

Remark. If $\mathcal{B}$ is a finite Boolean algebra then a measure μ on $\mathcal{B}$ is an HL-measure if and only if the following conditions are satisfied:

(i) for all atoms a, $\mu(a) > 0$
(ii) for exactly one atom b, $\mu(b) = +\infty$

Definition 5.2 (HL-structure)**.** A structure $\mathcal{C} = \langle \langle \mathcal{B}, C \rangle, \mu \rangle$ is an HL-structure when $\langle \mathcal{B}, C \rangle$ is a contact algebra and μ is an HL-measure.

Remark. The tuple $\langle \langle \mathcal{B}, C \rangle, \mu_L \rangle$ where $\langle \mathcal{B}, C \rangle$ is the contact algebra of polytopes in $\mathbb{R}^+$ and μ_L is the Lebesgue measure is an HL-structure.

Definition 5.3 (Finite Relational HL-structure)**.** Let $R \subseteq W \times W$ be reflexive and symmetric relation and let the graph $\langle W, R \rangle$ be connected. Then a structure $\mathcal{C} = \langle \langle \mathcal{B}, C \rangle, \mu \rangle$ is a finite relational HL-structure if:

(i) $\mathcal{B} = \langle \mathcal{P}(W), \emptyset, W, \cap, \cup, \setminus \rangle$
(ii) $C \subseteq \mathcal{P}(W) \times \mathcal{P}(W)$ and $C(a, b) \leftrightarrow (\exists x \in a)(\exists y \in b)(R(x, y))$
(iii) μ is an HL-measure

5.2 Soundness of $\Phi_{\mathcal{S}}$

Proposition 5.4. Let $\mathcal{S}$ be an S_n-system of inequalities. Then the following two conditions are equivalent:

(i) $\mathcal{S}$ has a solution $(r_1, r_2, ..., r_n)$ where $r_i = +\infty$ for exactly one $i \in [1, n]$
(ii) $\Phi_{\mathcal{S}}$ is satisfiable in finite relational HL-structure $\mathcal{C} = \langle \langle \mathcal{B}, C \rangle, \mu \rangle$.

Proof. $(i \Rightarrow ii)$ Let $(r_1, r_2, ..., r_n)$ be a solution of $\mathcal{S}$ and exactly one $r_i, 1 \leq i \leq n$ is equal to $+\infty$. Let $(r_{i_1}, r_{i_2}, \dots\ r_{i_t})$ be all numbers from the solution $(r_1, r_2, ..., r_n)$ that are different from 0 and let denote the set of their indices with $I = \{i_1, i_2, \dots, i_t\}$. Let $a_1, a_2, \dots, a_t$ be different objects and let denote this set with $A = \{a_1, a_2, \dots, a_t\}$. We will define a Boolean algebra $\mathcal{B} = \langle \mathcal{P}(A), \emptyset, A, \cap, \cup, \setminus \rangle$. The contact relation C is defined in arbitrary way through reflexive and symmetric $R \subseteq A \times A$ such that the graph $\langle A, R \rangle$ is connected. We will define a measure μ in the following way: $\mu(\{a_j\}) \stackrel{\text{def}}{=} r_{i_j}$, for $j = 1, 2, \dots, t$ and $\mu(\emptyset) \stackrel{\text{def}}{=} 0$. If $A_1 \subseteq A$, then $\mu(A_1) = \sum_{a \in A_1} \mu(a)$. We will prove that μ is an HL-measure:

(i) If $A_1, A_2 \in \mathcal{P}(A)$ and $A_1 \cap A_2 = \emptyset$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$
(ii) Exactly one element from the list $(r_1, r_2, ..., r_n)$ is equal to $+\infty$. Since it is non-zero it is mapped to a_j for some $j, 1 \leq j \leq t$. So that, μ returns $+\infty$ for exactly one atom.
(iii) $A_1, A_2 \in \mathcal{P}(A)$ and $\mu(A_1) = \mu(A_2) = +\infty$ then both sets contain a_j such that $\mu(\{a_j\}) = r_{i_j} = +\infty$. Thus, $a_j \in A_1 \cap A_2$ and $\mu(A_1 \cap A_2) = +\infty$

The measure μ satisfies the three conditions for HL-measure. So that, the tuple $\mathcal{C} = \langle \langle \mathcal{B}, C \rangle, \mu \rangle$ is an HL-structure. We will define valuation ν from $BoolVars \cup \{0, 1\}$ on the domain of our Boolean algebra in the following way:

- $\nu(p_{i_j}) = \begin{cases} \{a_j\}, \text{ if } i_j \in I \\ \emptyset, otherwise \end{cases}$
- $\nu(0) = \emptyset$
- $\nu(1) = A$

It is extended for terms in standard way:

- $\nu(a^*) = A \setminus \nu(a)$
- $\nu(a \sqcap b) = \nu(a) \cap \nu(b)$
- $\nu(a \sqcup b) = \nu(a) \cup \nu(b)$

We have to prove that $\mathcal{M} \models \Phi_{\mathcal{S}}$ where $\mathcal{M} = \langle \mathcal{C}, v \rangle$. The formula $\Phi_{\mathcal{S}}$ has the following representation:

$$\bigwedge_{1 \leq i < j \leq n} (p_i \sqcap p_j = 0) \wedge (\bigsqcup_{1 \leq i \leq n} p_i = 1) \wedge \varphi_{\mathcal{S}}$$

We start proving $\mathcal{M} \models \bigwedge_{1 \leq i < j \leq n}(p_i \sqcap p_j = 0)$. By definitions for truth in model and our valuation we get the following $\mathcal{M} \models p_i \sqcap p_j = 0 \leftrightarrow v(p_i \sqcap p_j) = v(0) \leftrightarrow v(p_i) \cap v(p_j) = \emptyset$. We chose elements of A to be different. Then in the case when $v(p_i) = \{a_i\}$ and $v(p_j) = \{a_j\}$ we have that $v(p_i) \cap v(p_j) = \{a_i\} \cap \{a_j\} = \emptyset$. In the other cases when $v(p_i) = \emptyset$ or $v(p_j) = \emptyset$ or $v(p_i) = v(p_j) = \emptyset$ it is clear that $v(p_i) \cap v(p_j) = \emptyset$. We proved that for an arbitrary conjunct $p_i \sqcap p_j = 0$ that $\mathcal{M} \models p_i \sqcap p_j = 0$ so using similar arguments we could show the same for the others. Therefore, $\mathcal{M} \models \bigwedge_{1 \leq i < j \leq n}(p_i \sqcap p_j = 0)$.

We continue with $\mathcal{M} \models \bigsqcup_{1 \leq i \leq n} p_i = 1$. We have to prove that $v(p_1) \cup v(p_2) \cup \ldots \cup v(p_n) = A$. Since $t \leq n$ it follows that $I \subseteq \{1, 2, \ldots, n\}$. So, $v(p_j) = \{a_j\}$ for all $j \in I$ and in the case when $t < n$ we will have $v(p_j) = \emptyset$ where $j \notin I$ but $j \in \{1, 2, \ldots, n\}$. Thus, $v(p_1) \cup v(p_2) \cup \ldots \cup v(p_n) = \{a_{i_1}\} \cup \{a_{i_2}\} \cup \ldots \cup \{a_{i_t}\} \cup \emptyset \cup \ldots \cup \emptyset = \{a_{i_1}, a_{i_2} \ldots a_{i_t}\} = A$.

We have to prove $\mathcal{M} \models \varphi_{\mathcal{S}}$ and we use that $\varphi_{\mathcal{S}} = \bigwedge_{1 \leq i \leq m} \varphi_{\sigma_i}$. So that, our goal is to show $\mathcal{M} \models \bigwedge_{1 \leq i \leq m} \varphi_{\sigma_i}$. We know from the previous section that each φ_{σ_i} corresponds to a formula of $(n, \leq)$-type or $(n, <)$-type inequalities. We will prove for inequality of $(n, \leq)$-type and it is analogous for the other type. Let assume $\varphi_{\sigma_j} = \bigsqcup_{i \in I_\ell^j} p_i \leq_\mu \bigsqcup_{i \in I_r^j} p_i$. Then we will apply valuation v and measure μ and get the following result: $\mathcal{M} \models \bigsqcup_{i \in I_\ell^j} p_i \leq_\mu \bigsqcup_{i \in I_r^j} p_i \leftrightarrow \mu(\bigcup_{i \in I_\ell^j} v(p_i)) \leq \mu(\bigcup_{i \in I_r^j} v(p_i)) \leftrightarrow \Sigma_{i \in I_\ell^j} \mu(v(p_i)) \leq \Sigma_{i \in I_r^j} \mu(v(p_i)) \leftrightarrow \Sigma_{i \in I_\ell^j} r_i \leq \Sigma_{i \in I_r^j} r_i$. The formula φ_{σ_j} corresponds to the σ_j inequality $\Sigma_{i \in I_\ell^j} x_i \leq \Sigma_{i \in I_l^j} x_i$ from the system $\mathcal{S}$. We know that $(r_1, r_2, \ldots, r_n)$ is a solution for the system. So, when we substitute x_i variables with the corresponding numbers from the solution in $\Sigma_{i \in I_\ell^j} r_i \leq \Sigma_{i \in I_r^j} r_i$ it has to be correct inequality. Therefore, σ_j is correct inequality. Thus, $\mathcal{M} \models \varphi_{\sigma_j}$. Therefore, we have $\mathcal{M} \models \bigwedge_{1 \leq i \leq m} \varphi_{\sigma_i}$. The formula $\Phi_{\mathcal{S}}$ is conjuction of three formulae and we have proved that every conjuction member of $\Phi_{\mathcal{S}}$ is true in $\mathcal{M} = \langle \mathcal{C}, v \rangle$ then $\mathcal{M} \models \Phi_{\mathcal{S}}$.

$(ii \Rightarrow i)$ Let $\mathcal{M} = \langle \mathcal{C}, v \rangle$ where $\mathcal{C} = \langle \langle \mathcal{B}, C \rangle, \mu \rangle$ is a finite relational HL-structure and v is valuation from $BoolVars$ on the domain of the Boolean algebra $\mathcal{B}$ and $\mathcal{M} \models \Phi_{\mathcal{S}}$. We use that $\Phi_{\mathcal{S}}$ is a conjuction of three formulae. We start with formula $\varphi_{\mathcal{S}}$ which is also a conjuction of formulae corresponding to all inequalities of $\mathcal{S}$. We consider the formula φ_{σ_j} which corresponds to the σ_j inequality of $\mathcal{S}$. We know that $\mathcal{M} \models \varphi_{\sigma_j}$ and similarly to the other direction we will apply valuation v, measure μ and truth in model: $\mathcal{M} \models \bigsqcup_{i \in I_l^j} p_i \leq_\mu \bigsqcup_{i \in I_r^j} p_i \leftrightarrow \Sigma_{i \in I_\ell^j} \mu(v(p_i)) \leq \Sigma_{i \in I_r^j} \mu(v(p_i))$. The last is correct inequality because $\mathcal{M}$ is a model for $\Phi_{\mathcal{S}}$. We could prove $(\mu(v(p_1)), \mu(v(p_2)), \ldots, \mu(v(p_n)))$ is a solution for the inequalities $\sigma_1, \sigma_2, \ldots, \sigma_m$ using mentioned argument. We need to prove that there is exactly one element $v(p_i)$ which has measure $+\infty$. From $\mathcal{M} \models \Phi_{\mathcal{S}}$ follows $\mathcal{M} \models \bigsqcup_{1 \leq i \leq n} p_i = 1$. We develop the formula and obtain $v(p_1) \sqcup_{\mathcal{B}} v(p_2) \sqcup_{\mathcal{B}} \ldots \sqcup_{\mathcal{B}}$

$\upsilon(p_n) = \upsilon(1) = 1_{\mathcal{B}}$. Now we check the measures and consider that all $\upsilon(p_i)$ are pair-wise disjoint (it comes from $\mathcal{M} \models \bigwedge_{1 \leq i < j \leq n}(p_i \sqcap p_j = 0)$). So, we have $\mu(\upsilon(p_1)) + \mu(\upsilon(p_2)) + \cdots + \mu(\upsilon(p_n)) = \mu(1_{\mathcal{B}}) = +\infty$. Then at least one of these elements has a measure $+\infty$. In fact, it is only one. Let us assume there are two such elements. Then by the third condition for HL-measure their intersection has also measure $+\infty$. On the other hand, we have that $\mathcal{M} \models \bigwedge_{1 \leq i < j \leq n}(p_i \sqcap p_j = 0)$. So, we obtain that $0 =_\mu 1$ which is a contradiction. □

6 Soundness and Completeness

In this section we will prove Soundness and Completeness theorems with respect to the finite relational HL-structures.

6.1 Soundness

Lemma 6.1. All axioms from our axiomatic system $\mathcal{L}_{HL}$ are true in the class of HL-structures $\mathcal{C}_{HL}$.

Proof. Let $\mathcal{M} = \langle \mathcal{C}, \upsilon \rangle$ be a model where $\mathcal{C} = \langle \langle \mathcal{B}, C \rangle, \mu \rangle$ is an HL-structure and υ is an arbitrary valuation from $BoolVars$ to the Boolean algebra. We will begin with the axioms for measure:

1. $((a \leq_\mu b) \wedge (b \sqcap d = 0)) \Rightarrow (a \sqcup d) \leq_\mu (b \sqcup d)$
   Let $\mathcal{M} \models ((a \leq_\mu b) \wedge (b \sqcap d = 0))$ which is equivalent to $\mathcal{M} \models (a \leq_\mu b)$ and $\mathcal{M} \models (b \sqcap d = 0)$. It follows that $\mu(\upsilon(a)) \leq \mu(\upsilon(b))$ and $\upsilon(b) \sqcap_{\mathcal{B}} \upsilon(d) = 0_{\mathcal{B}}$. We continue with the right side of the axiom: $\mu(\upsilon(a \sqcup d)) \leq \mu(\upsilon(b \sqcup d)) \leftrightarrow \mu(\upsilon(a) \sqcup_{\mathcal{B}} \upsilon(d)) \leq \mu(\upsilon(b) \sqcup_{\mathcal{B}} \upsilon(d))$ using the result $\upsilon(b) \sqcap_{\mathcal{B}} \upsilon(d) = 0_{\mathcal{B}}$ we get $\mu(\upsilon(a) \sqcup_{\mathcal{B}} \upsilon(d)) \leq \mu(\upsilon(b)) + \mu(\upsilon(d))$. We do not know whether $\upsilon(a) \sqcap_{\mathcal{B}} \upsilon(d) = 0_{\mathcal{B}}$ or $\upsilon(a) \sqcap_{\mathcal{B}} \upsilon(d) \neq 0_{\mathcal{B}}$ but in both cases $\mu(\upsilon(a) \sqcup_{\mathcal{B}} \upsilon(d)) \leq \mu(\upsilon(a)) + \mu(\upsilon(d))$. Thus, $\mu(\upsilon(a) \sqcup_{\mathcal{B}} \upsilon(d)) \leq \mu(\upsilon(a)) + \mu(\upsilon(d)) \leq \mu(\upsilon(b)) + \mu(\upsilon(d))$. So that, $\mathcal{M} \models (a \sqcup d) \leq_\mu (b \sqcup d)$.

2. $((a \sqcap d = 0) \wedge (b \sqcap d = 0) \wedge (d <_\mu 1)) \Rightarrow ((a \leq_\mu b) \Leftrightarrow (a \sqcup d \leq_\mu b \sqcup d))$
   Let $\mathcal{M} \models ((a \sqcap d = 0) \wedge (b \sqcap d = 0) \wedge (d <_\mu 1))$. It is equivalent to $\upsilon(a) \sqcap_{\mathcal{B}} \upsilon(d) = 0_{\mathcal{B}}$ and $\upsilon(b) \sqcap_{\mathcal{B}} \upsilon(d) = 0_{\mathcal{B}}$ and $\mu(\upsilon(d)) < +\infty$. Later in this proof we will use $\mu(\upsilon(d)) < +\infty$ which means that $\mu(\upsilon(d))$ is a real number.

   ($\Rightarrow$) Let $\mathcal{M} \models (a \leq_\mu b)$ which is equivalent to $\mu(\upsilon(a)) \leq \mu(\upsilon(b))$. We will add $\mu(\upsilon(d))$ to both sides $\mu(\upsilon(a)) + \mu(\upsilon(d)) \leq \mu(\upsilon(b)) + \mu(\upsilon(d)))$ and let denote this inequality with *(ineq-1)*. We know from the premise of the axiom $\upsilon(a) \sqcap_{\mathcal{B}} \upsilon(d) = 0_{\mathcal{B}}$ and $\upsilon(b) \sqcap_{\mathcal{B}} \upsilon(d) = 0_{\mathcal{B}}$ then $(a \sqcup d \leq_\mu b \sqcup d) \leftrightarrow \mu(\upsilon(a)) + \mu(\upsilon(d)) \leq \mu(\upsilon(b)) + \mu(\upsilon(d))$ which is correct inequality due to *(ineq-1)*. So that, $\mathcal{M} \models (a \sqcup d \leq_\mu b \sqcup d)$.

   ($\Leftarrow$) Let $\mathcal{M} \models (a \sqcup d \leq_\mu b \sqcup d)$ then it follows $\mu(\upsilon(a) \sqcup_{\mathcal{B}} \upsilon(d)) \leq \mu(\upsilon(b) \sqcup_{\mathcal{B}} \upsilon(d))$ and using the result from the premise of the axiom $\mu(\upsilon(a)) + \mu(\upsilon(d)) \leq$

$\mu(\nu(b)) + \mu(\nu(d))$. We use that $\mu(\nu(d))$ is a real number and remove it from both sides. So we obtain a correct inequality $\mu(\nu(a)) \leq \mu(\nu(b))$ and let denote this inequality with *ineq-2*. We will develop $a \leq_\mu b$ which is equivalent to $\mu(\nu(a)) \leq \mu(\nu(b))$. The last is correct inequality using the result *ineq-2*. Thus, $\mathcal{M} \models (a \leq_\mu b)$.

3. $((a \sqcap d = 0) \wedge (b \sqcap d = 0) \wedge (d <_\mu 1)) \Rightarrow ((a <_\mu b) \Leftrightarrow (a \sqcup d <_\mu b \sqcup d))$
The proof is almost the same as previous one.

4. $a =_\mu 1 \vee a^* =_\mu 1$
We have to prove that $\mathcal{M} \models (a =_\mu 1 \vee a^* =_\mu 1)$. It is equivalent to $\mathcal{M} \models (a =_\mu 1)$ or $\mathcal{M} \models (a^* =_\mu 1)$. Let develop the first formula $\mu(\nu(a)) = \mu(1_\mathcal{B}) \leftrightarrow \mu(\nu(a)) = +\infty$. The other formula $\mu(\nu(a^*)) = \mu(1_\mathcal{B}) \leftrightarrow \mu(1_\mathcal{B} \setminus \nu(a)) = \mu(1_\mathcal{B}) \leftrightarrow \mu(1_\mathcal{B} \setminus \nu(a)) = +\infty$. Now we use $a \sqcup a^* = 1$ which comes from the Lemma 4.6. We apply the definitions and get that $\mu(\nu(a)) + \mu(\nu(a^*)) = \mu(\nu(1)) = \mu(1_\mathcal{B}) = +\infty$. We further know that $0 \leq \mu(\nu(a)), \mu(\nu(a^*))$. So, we have that at least one of $\nu(a)$ and $\nu(a^*)$ has measure $+\infty$. In order to prove that exactly one has measure $+\infty$ we use axiom *(M5)* and a result from Lemma 4.6 $a \sqcap a^* = 0$. If we assume that both $a =_\mu 1$ and $a^* =_\mu 1$ then $a \sqcap a^* =_\mu 1$. On the other hand $a \sqcap a^* = 0$ and we obtain that $0 =_\mu 1$ which is contradiction.

5. $a =_\mu 1 \wedge b =_\mu 1 \Rightarrow a \sqcap b =_\mu 1$
Let $\mathcal{M} \models a =_\mu 1 \wedge b =_\mu 1$ then it follows $\mathcal{M} \models a =_\mu 1$ and $\mathcal{M} \models b =_\mu 1$. We apply the definitions: $\mu(\nu(a)) = \mu(\nu(1)) \leftrightarrow \mu(\nu(a)) = +\infty$. We know that μ is defined to give $+\infty$ for $1_\mathcal{B}$ and so $1_\mathcal{B} \in \nu(a)$. Using the same arguments $1_\mathcal{B} \in \nu(b)$. Then $1_\mathcal{B} \in \nu(a) \sqcap_\mathcal{B} \nu(b)$. So that, $\mu(\nu(a) \sqcap_\mathcal{B} \nu(b)) = +\infty$.

6. $a = 0 \Leftrightarrow a =_\mu 0$
We start with the implication $a = 0 \Rightarrow a =_\mu 0$ and apply the definitions $\mathcal{M} \models a = 0 \leftrightarrow \nu(a) = \nu(0) = 0_\mathcal{B}$. Now we develop the conclusion of the implication $\mu(\nu(a)) = \mu(\nu(0))$. We know that $\nu(a) = \nu(0) = 0_\mathcal{B}$. Then, $\mathcal{M} \models a =_\mu 0$. We consider the opposite direction $a =_\mu 0 \Rightarrow a = 0$ and again apply the definitions $\mathcal{M} \models a =_\mu 0 \leftrightarrow \mu(\nu(a)) = \mu(\nu(0))$. We further develop the right-hand side of the equality and obtain $\mu(\nu(0)) = \mu(0_\mathcal{B}) = 0$ and so $\mu(\nu(a)) = 0$. We use that μ is an HL-measure and get that $\nu(a) = 0_\mathcal{B}$. Therefore, $\mathcal{M} \models a = 0$.

Now we will prove that the axioms for S_n-system are true in $\mathcal{M}$. So, we have to show $\mathcal{M} \models \bigwedge_{1 \leq i < j \leq n}(t_i \sqcap t_j = 0) \wedge (\bigsqcup_{1 \leq i \leq n} p_i = 1) \Rightarrow \neg\varphi_\mathcal{S}$. Since it is an axiom then the system that corresponds to $\varphi_\mathcal{S}$ does not have a solution with exactly one component equals to $+\infty$. So from the Proposition 5.4 we get that $\mathcal{M} \not\models \varphi_\mathcal{S}$. Since the axiom is an implication then it is enough to prove that the conclusion is true so the whole implication is also true. We need to prove that $\mathcal{M} \models \neg\varphi_\mathcal{S}$ which by definition is equivalent to $\mathcal{M} \not\models \varphi_\mathcal{S}$. According to the above proposition then we have $\mathcal{M} \models \neg\varphi_\mathcal{S}$. The other axioms could be proved using similar arguments. □

Lemma 6.2. Modus Ponens (MP) preserves validity in the class of HL-structures $\mathcal{C}_{HL}$.

Proof. We need to prove that whenever the premise of the rule is true in HL-structures then the conclusion is also true. Let $\mathcal{C} = \langle\langle\mathcal{B}, C\rangle, \mu\rangle$ be an HL-structure. Assume that $\langle\mathcal{C}, v\rangle \models \varphi$ and $\langle\mathcal{C}, v\rangle \models \varphi \Rightarrow \psi$. When the implication is true and the premise of that implication is true then the conclusion is also true. So, $\langle\mathcal{C}, v\rangle \models \psi$. □

Theorem 6.3 (Soundness theorem)**.** All theorems of $\mathcal{L}_{HL}$ are true in the class of HL-structures $\mathcal{C}_{HL}$.

Proof. Let φ be a theorem of $\mathcal{L}_{HL}$. Then there exists finite sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ where $\varphi_n = \varphi$. We will prove by induction on i that φ_i is true in the class of HL-structures $\mathcal{C}_{HL}$. The first member of the proof φ_1 has to be an axiom. Then from (Lemma 6.1) it follows that φ_1 is true in $\mathcal{C}_{HL}$. Let suppose that the statement is true for $i = 1, 2, \ldots, k$ and $k < n$. We will check for $k + 1 \leq n$:

Case 1: φ_{k+1} is an axiom. Then the statement is true by (Lemma 6.1).

Case 2: φ_{k+1} is obtained by applying *(MP)* on some formulae φ_l and $\varphi_j = \varphi_l \Rightarrow \varphi_{k+1}$ where $j, l < k + 1$. By the induction hypothesis they are true in HL-structures and by (Lemma 6.2) it follows that the statement is true for φ_{k+1}.

□

6.2 Completeness

We will introduce the abbreviation for p^ϵ where p is a boolean variable and ϵ is a number such that $\epsilon \in \{0, 1\}$ in the following way: $p^\epsilon = \begin{cases} p, \ if \ \epsilon = 0 \\ p^*, \ if \ \epsilon = 1 \end{cases}$

Definition 6.4. Let $p_1, p_2, \ldots, p_k$ be Boolean variables. Then the term of the following type $p_1^{\epsilon_1} \sqcap p_2^{\epsilon_2} \sqcap \ldots \sqcap p_k^{\epsilon_k}$ is called *k-monom.*

Remark. All *k-monoms* that could be constructed with boolean variables $p_1, p_2, \ldots, p_k$ are 2^k.

Definition 6.5. We define $\bigsqcup_{i\in I} t_i$ for every finite set I and a family of terms $\{t_i\}_{i\in I}$:

Case 1: $I = \emptyset$ then $\bigsqcup_{i\in I} t_i = 0$
Case 2: $I = \{i_1\}$ for some natural number i_1 then $\bigsqcup_{i\in I} t_i = t_{i_1}$
Case 3: $I = \{i_1, i_2\}$ for some natural numbers i_1 and i_2 then $\bigsqcup_{i\in I} t_i = t_{i_1} \sqcup t_{i_2}$
Case 4: $I = I_1 \cup \{i_0\}$ for some set I_1 and natural number i_0 then $\bigsqcup_{i\in I} t_i = (\bigsqcup_{i\in I_1} t_i) \sqcup t_{i_0}$

Remark. The above definition is correct because of associativity and commutativity of $\sqcup$.

Lemma 6.6. Let $p_1, p_2, \ldots, p_k$ be propositional variables. We construct all 2^k $m_1, m_2, \ldots, m_{2^k}$ *k-monoms.* Then $\vdash (\bigsqcup_{1 \leq i \leq 2^k} m_i) = 1$.

Proof. We will prove the lemma by induction on k, the number of propositional variables.

- Let k = 1. Then we have $m_1 = p_1$ and $m_2 = p_1^*$. So $\vdash p_1 \sqcup p_1^* = 1$
- Let the lemma be true for some k. So $\vdash \bigsqcup_{1 \leq i \leq 2^k} m_i = 1$.
- We will prove the proposition for k+1. We have to add the new propositional variable p_{k+1} to the constructed so far *k-monoms.* By the induction hypothesis we have $\vdash \bigsqcup_{1 \leq i \leq 2^k} m_i = 1$ and as in the base case $\vdash p_{k+1} \sqcup p_{k+1}^* = 1$. The formula $(\bigsqcup_{1 \leq i \leq 2^k} m_i) \sqcap (p_{k+1} \sqcup p_{k+1}^*) = 1$ is tautological consequence of the above two formulae and so $\vdash (\bigsqcup_{1 \leq i \leq 2^k} m_i) \sqcap (p_{k+1} \sqcup p_{k+1}^*) = 1$. Now we apply the axiom *(B5)* and get $\vdash (\bigsqcup_{1 \leq i \leq 2^k} m_i \sqcap p_{k+1}) \sqcup (\bigsqcup_{1 \leq i \leq 2^k} m_i \sqcap p_{k+1}^*) = 1$. We use the symmetry of $\sqcap$ and axiom *(B5)* and get the following: $\vdash (m_1 \sqcap p_{k+1}) \sqcup \ldots \sqcup (m_{2^k} \sqcap p_{k+1}) \sqcup (m_1 \sqcap p_{k+1}^*) \sqcup \ldots \sqcup (m_{2^k} \sqcap p_{k+1}^*) = 1$. Actually, $(m_i \sqcap p_{k+1})$ is a term that looks like $p_1^{\epsilon_1} \sqcap \ldots \sqcap p_k^{\epsilon_k} \sqcap p_{k+1}$. It is true for all other terms. There are 2^{k+1} terms in the above sum. So they are *(k+1)-monoms* and we denote them with m'. So, $\vdash \bigsqcup_{1 \leq i \leq 2^{k+1}} m'_i = 1$.

□

Lemma 6.7. Let $p_1, p_2, \ldots, p_k$ for $k \geq 1$ be different boolean variables and let $m_1, m_2, m_3, \ldots, m_{2^k}$ be all $k - monoms$ for these variables. Then there exists an algorithm which for every boolean term a containing variables among $p_1, p_2, \ldots, p_k$ returns another term which is the representation of the original term as a sum of monoms $m_{i_1} \sqcup m_{i_2} \sqcup \ldots \sqcup m_{i_s}$.

Proof. Let $p_1, p_2, \ldots, p_k$ for $k \geq 1$ be different boolean variables and let m_1, m_2, m_3 $\ldots, m_{2^k}$ be all $k - monoms$ for these variables. Let the term a contains propositional variables among $p_1, p_2, \ldots, p_k$. We will prove the lemma by induction on the construction of a:

Case 1: The term a is a propositional variable p_i for some $1 \leq i \leq k$. Let denote with I_{p_i} the indices of all *k-monoms* which contain p_i. According to Remark 6.2, there are $2^{(k-1)}$ such *k-monoms* and the term $\bigsqcup_{j \in I_{p_i}} m_j$ is the representation of term a as a sum of *k-monoms.*

Case 2: The term $a = a_1 \sqcup a_2$ and terms $\bigsqcup_{i \in I_s} m_i$, $\bigsqcup_{i \in I_t} m_i$ where $I_s, I_t \subseteq \{1, 2, \ldots, 2^k\}$ are the representations as a sum of *k-monoms* for terms a_1 and a_2 respectively. We construct $I_u = I_s \cup (I_t \setminus (I_s \cap I_t))$. So, $\bigsqcup_{l \in I_u} m_l$ is the representation of a as a sum of *k-monoms.*

Case 3: The term $a = a_1 \sqcap a_2$ and terms $\bigsqcup_{i \in I_s} m_i$, $\bigsqcup_{i \in I_t} m_i$ where $I_s, I_t \subseteq \{1, 2, \ldots, 2^k\}$ are the representations as a sum of *k-monoms* for terms a_1 and a_2 respectively. Then $\bigsqcup_{i \in I_s, j \in I_t} m_i \sqcap m_j$ is a representation of a. We will consider that $m_i \sqcap m_j = 0$ for $i \neq j$ and $m_i \sqcap m_j = m_i$ for $i = j$. In this case $I_u = I_s \cap I_t$ and $\bigsqcup_{i \in I_u} m_i$ is the representation of a.

Case 4: The term $a = a_1^*$ and $a_1 = \bigsqcup_{i \in I} m_i$ where the set $I \subseteq \{1, 2, 3, \ldots, 2^k\}$. Let $J = \{1, 2, 3, \ldots, 2^k\} \setminus I$ and now we define $a_1^* = \bigsqcup_{j \in J} m_j$. We have that $I \cup J = \{1, 2, 3, \ldots, 2^k\}$ so according to Lemma 6.6 we get that $a_1 \sqcup a_1^* = 1$. We want to prove that $a_1 \sqcap a_1^* = 0$. We have $a_1 \sqcap a_1^* = \bigsqcup_{i \in I} m_i \sqcap \bigsqcup_{j \in J} m_j = \bigsqcup_{i \in I, j \in J} m_i \sqcap m_j$. We consider an arbitrary m_i such that $i \in I$ and $m_j, j \in J$. We know that $I \cap J = \emptyset$ so that m_i and m_j are different *k-monoms*. Then there is an index l such that p_l is in m_i and p_l^* is in m_j. We use that $p_l \sqcap p_l^* = 0$ so $m_i \sqcap m_j = 0$. Then $a_1 \sqcap a_1^* = \bigsqcup_{i \in I, j \in J} m_i \sqcap m_j = 0$.

□

Definition 6.8 (Negation Normal Form (NNF))**.** We say that formula φ is in Negation Normal Form or NNF if all connectives are $\wedge$, $\vee$, $\neg$ and $\neg$ occurs only in front of atomic formulae.

Definition 6.9 (Complexity of formula)**.** We will define complexity of formula φ and we denote it with $|\varphi|$:

$$|\varphi| = \begin{cases} 1 \text{ , } if\ \varphi\ is\ atomic \\ |\varphi_1| + 1 \text{ , } if\ \varphi = \neg\varphi_1 \\ |\varphi_1| + |\varphi_2| + 1 \text{ , } if\ \varphi = \varphi_1 \wedge \varphi_2\ or\ \varphi = \varphi_1 \vee \varphi_2 \end{cases}$$

Lemma 6.10 (Negation Normal Form Lemma)**.** Let φ be a formula from $\mathscr{L}$. There exists an algorithm which constructs a formula φ' in NNF for finite number of steps and $\vdash \varphi \Leftrightarrow \varphi'$.

Proof. We will prove the lemma by induction on n = $|\varphi|$. We start with n = 1 then φ is atomic and φ is NNF. So $\vdash \varphi \Leftrightarrow \varphi'$. Let the proposition is true for all formulae with complexity less than or equal to n and we will prove for formula $|\varphi|$ = n + 1:

Case 1: The formula $\varphi = \neg\varphi_1$. We will consider all cases for φ_1:

Case 1.1: φ_1 is atomic. Then the negation is in front of atomic formula and φ is in NNF. So, $\vdash \varphi \Leftrightarrow \varphi'$.

Case 1.2: $\varphi_1 = \neg\varphi_2$. We will check the complexities $|\varphi_2| < |\varphi_1|$ and $|\varphi_1| < |\varphi| = n+1$. So, by the induction hypothesis $\vdash \varphi_2 \Leftrightarrow \varphi_2{}'$ and we also have $\varphi = \neg\neg\varphi_2$. We will consider the propositional tautology $\vdash \neg\neg\psi \Leftrightarrow \psi$ then we could infer $\vdash \varphi \Leftrightarrow \varphi_2$. Thus, $\vdash \varphi \Leftrightarrow \varphi_2{}'$ and $\varphi_2{}'$ is in NNF.

Case 1.3: $\varphi_1 = \varphi_2 \wedge \varphi_3$. We will consider the propositional tautology $\vdash \neg(\psi_1 \wedge \psi_2) \Leftrightarrow \neg\psi_1 \vee \neg\psi_2$ and we infer $\vdash \varphi \Leftrightarrow \neg\varphi_2 \vee \neg\varphi_3$. We will check the complexities $|\neg\varphi_2| = |\varphi_2| + 1 < |\varphi_2| + |\varphi_3| + 1 = |\varphi_1| < |\varphi| = n + 1$. So, $|\neg\varphi_2| < n$ and by induction hypothesis $\varphi_2{}'$ is in NNF and $\vdash \neg\varphi_2 \Leftrightarrow \varphi_2{}'$. Similarly, we could infer $\vdash \neg\varphi_3 \Leftrightarrow \varphi_3{}'$. So, $\vdash \varphi \Leftrightarrow \varphi_2{}' \vee \varphi_3{}'$ and $\varphi_2{}' \vee \varphi_3{}'$ is in NNF.

Case 1.4: $\varphi_1 = \varphi_2 \vee \varphi_3$. We will use propositional tautology $\vdash \neg(\psi_1 \vee \psi_2) \Leftrightarrow \neg\psi_1 \wedge \neg\psi_2$ and infer that $\vdash \varphi \Leftrightarrow \varphi_2{}' \wedge \varphi_3{}'$ and the formula $\varphi_2{}' \wedge \varphi_3{}'$ is in NNF.

Case 2: $\varphi = \varphi_1 \wedge \varphi_2$. We will check the complexities $|\varphi_1| + |\varphi_2| < |\varphi| = n + 1$. Hence, $|\varphi_1| + |\varphi_2| \leq n$, $|\varphi_1|, |\varphi_2| < n$. By the induction hypothesis we have $\vdash \varphi_1 \Leftrightarrow \varphi_1{}'$ and $\vdash \varphi_2 \Leftrightarrow \varphi_2{}'$ where $\varphi_1{}'$ and $\varphi_2{}'$ are in NNF. Thus, $\vdash \varphi \Leftrightarrow \varphi_1{}' \wedge \varphi_2{}'$ and formula $\varphi_1{}' \wedge \varphi_2{}'$ is in NNF.

Case 3: $\varphi = \varphi_1 \vee \varphi_2$. We will prove that $\vdash \varphi \Leftrightarrow \varphi_1{}' \vee \varphi_2{}'$ and formula $\varphi_1{}' \vee \varphi_2{}'$ is in NNF using the same arguments as in previous case.

□

Later in this study we will need $\vdash C(\bigsqcup_{i \in I} t_i, \bigsqcup_{j \in J} s_j) \Leftrightarrow \bigvee_{i \in I, j \in J} C(t_i, s_j)$. We will start with some lemmas to help us to prove it.

Lemma 6.11. Let for $n \geq 0$ $t_1, t_2, \ldots, t_n, s$ are terms of $\mathscr{L}$ and let denote with $I = \{1, 2, \ldots, n\}$. Then, $\vdash C(s, \bigsqcup_{i \in I} t_i) \Leftrightarrow \bigvee_{i \in I} C(s, t_i)$.

Proof. We will prove the lemma by induction on $m = |I|$.

Case 1: $I = \emptyset$ then by definition $\bigsqcup_{i \in I} t_i = 0$. So we have $\vdash C(s, 0) \Leftrightarrow C(s, 0)$.

Case 2: $I = \{i_1\}$ for some natural number $1 \leq i_1 \leq n$ then by definition $\bigsqcup_{i \in I} t_i$ $= t_{i_1}$. Hence, $\vdash C(s, t_{i_1}) \Leftrightarrow C(s, t_{i_1})$.

Case 3: $I = \{i_1, i_2\}$ for some natural numbers i_1 and i_2 such that $1 \leq i_1 \leq n$ and $1 \leq i_2 \leq n$. Then, $\bigsqcup_{i \in I} t_i = t_{i_1} \sqcup t_{i_2}$. We have $\vdash C(s, t_{i_1} \sqcup t_{i_2}) \Leftrightarrow C(s, t_{i_1}) \vee C(s, t_{i_2})$ from the axiom *(C2)*.

Case 4: Let the lemma is true for all sets $I \subseteq \{1, 2, \ldots, n\}$ such that $|I| \leq m$. Now we will prove the lemma for $|I| = m+1$. We have $I = (I \setminus \{i_0\}) \cup \{i_0\}$ and $|I \setminus \{i_0\}| < m+1$. So by definition $\bigsqcup_{i \in I} t_i = (\bigsqcup_{i \in I \setminus \{i_0\}} t_i) \sqcup t_{i_0}$. Then we have $C(s, \bigsqcup_{i \in I} t_i) = C(s, (\bigsqcup_{i \in I \setminus \{i_0\}} t_i) \sqcup t_{i_0})$. This is an instance of the axiom *(C3)*, then $\vdash C(s, (\bigsqcup_{i \in I \setminus \{i_0\}} t_i) \sqcup t_{i_0}) \Leftrightarrow C(s, \bigsqcup_{i \in I \setminus \{i_0\}} t_i) \vee C(s, t_{i_0})$. Since $|I \setminus \{i_0\}| < m+1$ by the induction hypothesis $\vdash C(s, \bigsqcup_{i \in I \setminus \{i_0\}} t_i) \Leftrightarrow \bigvee_{i \in I \setminus \{i_0\}} C(s, t_i)$. So we have $\vdash C(s, \bigsqcup_{i \in I} t_i) \Leftrightarrow \bigvee_{i \in I} C(s, t_i)$.

□

Lemma 6.12. Let for $n \geq 0$ $t_0, t_1, t_2, \ldots, t_n$ be terms of $\mathscr{L}$ and let denote with $I = \{1, 2, \ldots, n\}$. Then, $\vdash C(\bigsqcup_{i \in I} t_i, t_0) \Leftrightarrow \bigvee_{i \in I} C(t_i, t_0)$.

Proof. Follows from Lemma 6.11 and $\vdash C(a, b) \Rightarrow C(b, a)$. □

Lemma 6.13. Let for $n \geq 0$ $t_1, t_2, \ldots, t_n$ and for $m \geq 0$ $s_1, s_2, \ldots, s_m$ be terms. Then, $\vdash C(\bigsqcup_{i \in I} t_i, \bigsqcup_{j \in J} s_j) \Leftrightarrow \bigvee_{i \in I, j \in J} C(t_i, s_j)$.

Proof. We apply Lemma 6.11 and Lemma 6.12. □

Lemma 6.14. Let for $n \geq 1$ $t_1, t_2, \ldots, t_n$ be terms from our language $\mathscr{L}$. Then $\vdash \bigsqcup_{i \in \{1,2,\ldots,n\}} t_i \leq 0 \Leftrightarrow \bigwedge_{i \in \{1,2,\ldots,n\}} (t_i = 0)$.

Proof. We will prove the lemma by induction on the number of terms n.

Case 1: In this case we will prove the lemma for n = 1 so $\vdash t_1 \leq 0 \Leftrightarrow t_1 = 0$. We will show $\vdash t_1 \leq 0 \Rightarrow t_1 = 0$ and $\vdash t_1 = 0 \Rightarrow t_1 \leq 0$. We have that $0 \leq t_1 \Rightarrow (t_1 \leq 0 \Rightarrow t_1 = 0)$ is a tautology, then from Theorem 2.13 $\vdash 0 \leq t_1 \Rightarrow (t_1 \leq 0 \Rightarrow t_1 = 0)$. We will use that $\vdash 0 \leq t_1$ and *(MP)* so we infer $\vdash t_1 \leq 0 \Rightarrow t_1 = 0$. We continue with the other direction and the tautology $t_1 = 0 \Rightarrow t_1 \leq 0$ because $t_1 = 0$ is an abbreviation for $(t_1 \leq 0) \wedge (0 \leq t_1)$. Hence, we derive $\vdash t_1 = 0 \Rightarrow t_1 \leq 0$ again from Theorem 2.13.

Case 2: Let $n \geq 1$ be a natural number and the lemma holds for all numbers $\leq n$.

Case 3: We will prove the lemma for $n + 1$. The term $\bigsqcup_{i\in\{1,2,\ldots,n+1\}} t_i \leq 0$ is equal by definition to $(\bigsqcup_{i\in\{1,2,\ldots,n+1\}\setminus\{i_0\}} t_i) \sqcup t_{i_0} \leq 0$ for some $i_0 \in \{1, 2, \ldots, n+1\}$. Then we use the axiom *(B7)* and we get $\vdash (\bigsqcup_{i\in\{1,2,\ldots,n+1\}\setminus\{i_0\}} t_i) \sqcup t_{i_0} \leq 0 \Leftrightarrow (\bigsqcup_{i\in\{1,2,\ldots,n+1\}\setminus\{i_0\}} t_i \leq 0) \wedge (t_{i_0} \leq 0)$. By the induction hypothesis we have $\vdash \bigsqcup_{i\in\{1,2,\ldots,n+1\}\setminus\{i_0\}} t_i \leq 0 \Leftrightarrow \bigwedge_{i\in\{1,2,\ldots,n+1\}\setminus\{i_0\}}(t_i = 0)$ and from the base case we get that $\vdash t_{i_0} \leq 0 \Leftrightarrow t_{i_0} = 0$. So from the result of Theorem 2.9 $\vdash (\bigsqcup_{i\in\{1,2,\ldots,n+1\}\setminus\{i_0\}} t_i) \sqcup t_{i_0} \leq 0 \Leftrightarrow \bigwedge_{i\in\{1,2,\ldots,n+1\}\setminus\{i_0\}}(t_i = 0) \wedge t_{i_0} = 0$. Thus, by the definitions $\vdash (\bigsqcup_{i\in\{1,2,\ldots,n+1\}} t_i \leq 0 \Leftrightarrow \bigwedge_{i\in\{1,2,\ldots,n+1\}}(t_i = 0)$.

□

We will introduce some convenient abbreviations which will be used later in this section. We start with a formula which determines whether *k-monom* is 0 or not:

$$\varphi^P = \bigwedge_{i\in I^{pos}} (m_i = 0) \wedge \bigwedge_{i\in I^{neg}} \neg(m_i = 0)$$

We know that all *k-monoms* with k boolean variables are 2^k so for each $i = 1, 2, 3, \ldots, 2^k$ i belongs to exactly one of I^{pos} or I^{neg}. The next abbreviation is for a formula that gives us contacts between monoms:

$$\varphi^C = \bigwedge_{(i,j)\in J^{pos}} C(m_i, m_j) \wedge \bigwedge_{(i,j)\in J^{neg}} \neg C(m_i, m_j)$$

Similarly, for each pair (i, j) we have either $C(m_i, m_j)$ or $\neg C(m_i, m_j)$. It determines whether m_i and m_j are in contact. The formula might contain contradictions - when for fixed i and j we have in the formula $C(m_i, m_j)$ but $\neg C(m_j, m_i)$ which breaks the symmetry of contact relation C. We will explain how to handle such situations later.

Definition 6.15 (Good elementary formula)**.** We say a formula is *good elementary formula* if it has the following type $\varphi^P \wedge \varphi^C \wedge \varphi^M$ where φ^P and φ^C are the explained above formulae and the formula φ^M is a boolean combination of formulae of the type $a \leq_\mu b$. We denote such good elementary formula with ψ^E.

We will develop an algorithm that takes as an input a formula φ from our language $\mathcal{L}$ and returns another formula which is disjunction of good elementary formulae. We will prove that this formula is equivalent to the input one using the definitions and lemmas above.

Proposition 6.16. There exists an algorithm which takes a formula φ from our language $\mathscr{L}$ and returns a formula $\Psi_{\varphi}^{E} = \psi_1^E \vee \psi_2^E \vee \ldots \vee \psi_{\ell}^E$ where for each $i = 1, 2, \ldots, \ell$ the formula φ_i^E is a good elementary formula. The input formula φ is equivalent to the output formula Ψ_{φ}^{E} in the sense $\vdash \varphi \Leftrightarrow \Psi_{\varphi}^{E}$. The algorithm finishes for finite number of steps.

Proof. We will start developing such an algorithm and prove that on each step the input formula and the result formula are equivalent. So the final formula will be obtained from the original formula only applying operations that preserve validity. Let the original formula be φ and let $p_1, p_2, \ldots, p_k$ be all boolean variables from φ.

Step 1: On this step we will push the negation connective $\neg$ to be only in front of atomic formulae. The result formula is φ_1 and it is in NNF. According to (Lemma 6.10) $\vdash \varphi \Leftrightarrow \varphi_1$.

Step 2: All sub-formulae from φ_1 which have the type $a \leq b$ are substituted with $a \sqcap b^* = 0$. In this way we obtain φ_2. We have that $\vdash a \leq b \Leftrightarrow a \sqcap b^* = 0$. So, by the Theorem 2.9 $\vdash \varphi_1 \Leftrightarrow \varphi_2$.

Step 3: We construct $m_1, m_2, \ldots, m_{2^k}$ all $k-monoms$ from $p_1, p_2, \ldots, p_k$ and substitute every term a in φ_2 with its representation of a sum of $k-monoms$ (Lemma 4.1 and Lemma 6.7). We will denote the result formula φ_3 and $\vdash \varphi_2 \Leftrightarrow \varphi_3$.

Step 4: As a result of the above substitution φ_3 might contain atomic formulae from the following type - $C(m_{i_1} \sqcup m_{i_2}, \sqcup \ldots \sqcup m_{i_s}, m_{j_1} \sqcup m_{j_2}, \sqcup \ldots \sqcup m_{j_t})$. Based on the result from (Lemma 6.13) each of these formulae could be replaced with $\bigvee_{1 \leq x \leq s, 1 \leq y \leq t} C(m_{i_x}, m_{j_y})$. So we obtain φ_4 from φ_3 applying this operation and due to the same lemma $\vdash \varphi_3 \Leftrightarrow \varphi_4$.

Step 5: Again as a result from the substitution from *Step 3* φ_4 might contain atomic formulae from the type: $(m_1 \sqcup m_2 \sqcup \ldots \sqcup m_n \leq 0)$. These formulae have to be replaced with $(m_1 = 0) \wedge (m_2 = 0) \wedge \ldots \wedge (m_n = 0)$ (Lemma 6.14). We denote this new formula with φ_5 and we have $\vdash \varphi_4 \Leftrightarrow \varphi_5$

Step 6: After the previous steps φ_5 is constructed from - $m_i = 0$, $C(m_i, m_j)$ and $m_{i_1} \sqcup m_{i_2} \sqcup \ldots \sqcup m_{i_s} \leq_{\mu} m_{j_1} \sqcup m_{j_2} \sqcup \ldots \sqcup m_{j_s}$ using the connectives $\neg$, $\wedge$ and $\vee$. We will add to φ_5 formulae $(m_i = 0 \vee \neg(m_i = 0))$, $(C(m_i, m_j) \vee \neg C(m_i, m_j))$ and $(0 \leq_{\mu} 1)$. We proved that the original formula is equivalent to the result formula - so at the end of each sub-step we will obtain a new formula equivalent to the input one. We apply the following sub-steps consequently:

Step 6.1: If the *k-monom* m_i for $i = 1, 2, 3, \ldots 2^k$ is missing we add $(m_i = 0 \vee \neg(m_i = 0))$ to φ_5 and apply distributive law. We perform this operation until in all disjunctive members all $k-monoms$ are included with either $m_i = 0$ or $\neg(m_i = 0)$. We have to prove that $\vdash \varphi_5 \Leftrightarrow \varphi_5 \wedge (m_i = 0 \vee \neg(m_i =$

0)). The formula $m_i = 0 \vee \neg(m_i = 0) \Rightarrow (\varphi_5 \Leftrightarrow \varphi_5 \wedge (m_i = 0 \vee \neg(m_i = 0)))$ is tautology. Since $m_i = 0 \vee \neg(m_i = 0)$ is an axiom and by *(MP)* we infer $\vdash \varphi_5 \Leftrightarrow \varphi_5 \wedge (m_i = 0 \vee \neg(m_i = 0))$. The result formula is $\varphi_{6.1}$ and $\vdash \varphi_5 \Leftrightarrow \varphi_{6.1}$.

Step 6.2: We add to each disjunctive member $(C(m_i, m_j) \vee \neg C(m_i, m_j))$ for each missing pair (i, j) and apply distributive law. After this operation every disjunctive member will contain all possible pairs (i, j), $for\ i = 1, 2, 3, \ldots, 2^k$ and $j = 1, 2, 3, \ldots, 2^k$ either as $C(m_i, m_j)$ or $\neg C(m_i, m_j)$. We prove that $\varphi_{6.1} \Leftrightarrow \varphi_{6.1} \wedge (C(m_i, m_j) \vee \neg C(m_i, m_j))$ similarly as in *(Step 6.1)*. We denote the result formula with $\varphi_{6.2}$ and $\vdash \varphi_{6.1} \Leftrightarrow \varphi_{6.2}$.

Step 6.3: Our goal after this step is all disjunctive members to be *good elementary formulae*. So that, if there is a member which does not contain formula from the type: $m_{i_1} \sqcup m_{i_2} \sqcup \ldots \sqcup m_{i_s} \leq_\mu m_{j_1} \sqcup m_{j_2} \sqcup \ldots \sqcup m_{j_s}$ we add to it $0 \leq_\mu 1$. Similarly to *(Step 1)*, we prove $\vdash \varphi_{6.2} \Leftrightarrow \varphi_{6.2} \wedge (0 \leq_\mu 1)$. We apply this operation until all disjunctive members could be represented as $\varphi^P \wedge \varphi^C \wedge \varphi^M$. So, the result of this step is a formula $\Psi^E_\varphi = \psi^E_1 \vee \psi^E_2 \vee \ldots \vee \psi^E_l$.

On each step we applied an operation which infer syntactically result formula from the input formula. So, $\vdash \varphi \Leftrightarrow \Psi^E_\varphi$. □

Lemma 6.17. Let ψ^E be a good elementary formula. Then there exists an algorithm which processes ψ^E syntactically and returns $\vdash \neg\psi^E$ or a model for ψ^E over finite relational HL-structure. The algorithm finishes for finite number of steps.

Proof. The formula ψ^E is a good elementary formula so $\psi^E = \varphi^P \wedge \varphi^C \wedge \varphi^M$. The algorithm checks the following conditions:

Case 1: $\varphi^P = \bigwedge_{1 \leq i \leq 2^k} (m_i = 0)$. According to Lemma 6.14 we have $\vdash m_1 \sqcup m_2 \sqcup \ldots \sqcup m_{2^k} = 0 \Leftrightarrow (m_1 = 0) \wedge (m_2 = 0) \wedge \ldots \wedge (m_{2^k} = 0)$. We also proved in Lemma 6.6 that $\vdash m_1 \sqcup m_2 \sqcup \ldots \sqcup m_{2^k} = 1$. From the above formulae we obtain $\vdash (m_1 \sqcup m_2 \sqcup \ldots \sqcup m_{2^k} = 1) \wedge (m_1 \sqcup m_2 \sqcup \ldots \sqcup m_{2^k} = 0) \Rightarrow (0 = 1)$. We have that $\vdash \neg(0 = 1)$ and $\vdash m_1 \sqcup m_2 \sqcup \ldots \sqcup m_{2^k} = 1$. Hence, $\vdash \neg(m_1 \sqcup m_2 \sqcup \ldots \sqcup m_{2^k} = 0)$. Now we will use the propositional tautology $(\varphi \Leftrightarrow \psi) \Leftrightarrow (\neg\varphi \Leftrightarrow \neg\psi)$ and infer $\vdash \neg\varphi^P$. So that, $\vdash \neg(\varphi^P \wedge \varphi^C \wedge \varphi^M)$ and the algorithm ends.

Case 2: $\varphi^P = \bigwedge_{i \in I^{pos}} (m_i = 0) \wedge \bigwedge_{i \in I^{neg}} \neg(m_i = 0)$ and $I^{pos} \cup I^{neg} = \{1, 2, 3, \ldots, 2^k\}$ and $I^{pos} \cap I^{neg} = \emptyset$ and $\varphi^C = \bigwedge_{(i,j) \in J^{pos}} C(m_i, m_j) \wedge \bigwedge_{(i,j) \in J^{neg}} \neg C(m_i, m_j)$.

Case 2.1: There is an index $i_0 \in I^{neg}$ and $(i_0, i_0) \in J^{neg}$. So, we have a subformula $\neg(m_{i_0} = 0) \wedge \neg C(m_{i_0}, m_{i_0})$. We will consider an instance of the axiom *(C1)*: $\neg(m_{i_0} = 0) \Rightarrow C(m_{i_0}, m_{i_0})$. Since the "$\Rightarrow$" symbol is an abbreviation for $(m_{i_0} = 0) \vee C(m_{i_0}, m_{i_0})$, the subformula $\neg(m_{i_0} = 0) \wedge \neg C(m_{i_0}, m_{i_0})$ is a negation of that instance of the axiom *(C1)*. Hence, we have proof of $\vdash \neg(\neg(m_{i_0} = 0) \wedge \neg C(m_{i_0}, m_{i_0}))$. From here we obtain that

$\vdash \neg(\neg(m_{i_0} = 0) \wedge \neg C(m_{i_0}, m_{i_0})) \wedge \bigwedge_{i \in I^{pos} \setminus \{i_0\}} (m_i = 0) \wedge \bigwedge_{i \in I^{neg}} \neg(m_i = 0) \wedge \bigwedge_{(i,j) \in J^{pos}} C(m_i, m_j) \wedge \bigwedge_{(i,j) \in J^{neg} \setminus \{(i_0, i_0)\}\})} \neg C(m_i, m_j) \wedge \varphi^M$. Thus, $\vdash \neg\psi^E$ and the algorithm finishes.

Case 2.2: There are two indices $i_0, j_0 \in I^{neg}$ such that $(i_0, j_0) \in J^{pos}$ and $(j_0, i_0) \in J^{neg}$. We have subformula $\neg(m_{i_0} = 0) \wedge \neg(m_{j_0} = 0) \wedge C(m_{i_0}, m_{j_0}) \wedge \neg C(m_{j_0}, m_{i_0})$. We consider the axiom *(C3):* $C(m_{i_0}, m_{j_0}) \Rightarrow C(m_{j_0}, m_{i_0})$. We will use again that the "$\Rightarrow$" symbol is an abbreviation for $\neg C(m_{i_0}, m_{j_0}) \vee C(m_{j_0}, m_{i_0})$. Hence, the subformula $C(m_{i_0}, m_{j_0}) \wedge \neg C(m_{j_0}, m_{i_0})$ is a negation of the axiom *(C3)*. Thus, $\vdash \neg(\neg C(m_{i_0}, m_{j_0}) \vee C(m_{j_0}, m_{i_0}))$. Using the same idea as in previous case - when we find a proof for negation of one conjunction member it is a proof for the negation of the whole conjunction, we infer $\vdash \neg\psi^E$. So, the algorithm stops.

If ψ^E does not satisfy any of the above conditions we will show how to construct a model for it. The formula φ^M corresponds to a system. In order to transform it to an S_n-system, we possibly need to add inequalities of the following type $0 \leq_\mu m_i$ for all *k-monoms* m_i such that m_i is a member of φ^M and there is no such inequality. We also substitute m_i with x_i and $\sqcup$ with $+$. It is how we obtain $\mathcal{S}$ from φ^M. We use $solve^{+\infty}_{\mathcal{S}_n}$ to find a solution for $\mathcal{S}$ because it is a S_n-system. If the result from the $solve^{+\infty}_{\mathcal{S}_n}$ is $\emptyset$ then φ^M corresponds to an S_n-system which does not have a solution with $+\infty$ for only one variable. Then we have an axiom of the form $\theta \Rightarrow \neg\varphi^M$. So, we get that $(\theta \Rightarrow \neg\varphi^M) \Rightarrow (\psi^E \Rightarrow \bot)$ is a tautology. From here we infer $\vdash \psi^E \Rightarrow \bot$ and we use that "$\Rightarrow$" is an abbreviation to obtain $\vdash \neg\psi^E$. The other case is when the system has a solution of desired type and let $(r_1, r_2, \ldots, r_s)$ be such solution and without loss of generality $r_1 = +\infty$. Let $M = \{M_1, M_2, \ldots M_s\}$ be a set of s different objects where $|I^{neg}| = s$. Then $\mathcal{B} = \langle \mathcal{P}(M), \emptyset, M, \cap, \cup, \setminus \rangle$ is a Boolean algebra of all subsets of the set M. We will define the contact relation in terms of Kripke semantics: $(j, k) \in J^{pos} \leftrightarrow R(M_j, M_k)$. Now we are ready to define $C_R \subseteq \mathcal{P}(M) \times \mathcal{P}(M)$ and $C(a, b) \leftrightarrow (\exists M_j \in a)(\exists M_k \in b)(R(M_j, M_k))$. Now we will prove that this definition of C_R satisfies the axioms *(C1)÷(C3)*:

(C1): $a \neq 0 \Rightarrow C(a, a)$. In this case the premise says that there exists point M_j such that $M_j \in a$, and we will use that R is reflexive (*Case 2.1*). So $R(M_j, M_j)$ and then $C_R(a, a)$.

(C2): $C_R(a, b \sqcup c) \Leftrightarrow C_R(a, b) \vee C_R(a, c)$. We start with "$\Rightarrow$" direction. Here we have $M_j \in a$, $M_k \in b \sqcup c$ such that $R(M_j, M_k)$. There are two cases - $M_k \in b$ then $C_R(a, b)$ and the second case $M_k \in c$ then $C_R(a, c)$. In both cases the conclusion is true. The opposite direction "$\Leftarrow$" could be proved using similar arguments.

(C3): $C_R(a, b) \Rightarrow C_R(b, a)$. In this case we use that the relation R is symmetric. We proved that if $(j, k) \in J^{pos}$ then also $(k, j) \in J^{pos}$ (*Case 2.2*). From the premise follows $M_j \in a$, $M_k \in b$ such that $R(M_j, M_k)$ and by the symmetry of R $M_k \in b$, $M_j \in a$ such that $R(M_k, M_j)$ which is the definition for $C_R(b, a)$.

Then $\langle \mathcal{B}, C_R \rangle$ is a contact algebra. Next step is to define the measure $\mu : \mathcal{P}(M) \to [0, +\infty]$:

- $\mu(\{M_i\}) = r_i$, $\mu(\emptyset) = 0$, $\mu(M) = +\infty$
- Let $A \subseteq M$ then $\mu(A) = \sum_{a \in A} \mu(a)$

We need to check that μ is HL-measure - μ is positive and for exactly one atom M_1 we have that $\mu(\{M_1\}) = r_1 = +\infty$. We will verify the last condition - $\mu(a) = \mu(b) = +\infty$ from here we have that $M_1 \in a$ and $M_1 \in b$ so $M_1 \in a \sqcap b$. It follows that $\mu(a \sqcap b) = +\infty$. Then μ is HL-measure. So $\mathcal{C} = \langle \langle \mathcal{B}, C \rangle, \mu \rangle$ is a HL-structure. Now we need to define valuation $v : BoolVars \to \mathcal{P}(M)$:

- $v(0) = \emptyset$, $v(1) = M$
- $v(p) = \{M_i \mid k\text{-}monom\ m_i\ contains\ p\}$

For this definition of v we have: $v(m_i) = \begin{cases} \{M_i\},\ if\ i \in I^{neg} \\ \emptyset,\ if\ i \in I^{pos} \end{cases}$

In order to show it we will check the value for an arbitrary *k-monom* $m_i \neq 0$ $v(m_i) = v(p_1)^{\epsilon_1} \cap v(p_2)^{\epsilon_2} \cap \ldots \cap v(p_k)^{\epsilon_k}$. We start with proving that $M_i \in v(p_1)^{\epsilon_1}$. We have to consider two cases:

- $m_i = p_1 \sqcap p_2^{\epsilon_2} \sqcap \ldots \sqcap p_k^{\epsilon_k}$ then by definition $M_i \in v(p_1)$
- $m_i = p_1^* \sqcap p_2^{\epsilon_2} \sqcap \ldots \sqcap p_k^{\epsilon_k}$ then we have that $M_i \notin v(p_1)$ so $M_i \in M \setminus v(p_1)$ which by definition is the value for $v(p_1^*)$

So, we proved that in both cases $M_i \in v(p_1)^{\epsilon_1}$. Using the same arguments for other propositional variables we will show that $M_i \in v(m_i)$. The next step is to prove that the set $v(m_i)$ contains only one element M_i. We will use similar approach let $M_j \in M$ and let a *k-monom* m_j be different from m_i we have that $M_j \in v(m_j)$. Now we will prove that $M_j \notin v(m_i)$. We know that $m_j \neq m_i$ then they are different at some position l. We assume that p_l is in m_j and p_l^* is in m_i. Then $M_j \in v(p_l)$ and $M_j \notin M \setminus v(p_l)$. So, $M_j \notin v(m_i)$. We proved that for an arbitrary object $M_j \in M$. Therefore, $v(m_i) = \{M_i\}$. We need to prove that if $m_i = 0$, then $v(m_i) = \emptyset$. Let $m_i = 0$ and $m_j \neq 0$ so $v(m_j) = \{M_j\}$. We have that $m_i \neq m_j$ and let they differ at position l. Suppose that m_j contains p_l and m_i contains p_l^*. Then $M_j \in v(p_l)$ and $M_j \notin M \setminus v(p_l)$.
We have one more case when the algorithm finds a proof for $\neg\psi^E$. It is when the graph corresponding to the contact relation C_R is not connected. Let assume that the graph corresponding to C_R is not connected. So, there is a set $A \subseteq M$, $A \neq \emptyset$ and also $M \setminus A \neq \emptyset$ and there is no edge between them. We associate a term to the set A in the following way: $A = v(a)$, so $a = \bigsqcup_{i \in I} m_i$ where $I = \{i \mid M_i \in A\}$. Similarly, $a^* = \bigsqcup_{j \in J} m_j$ where $J = I^{neg} \setminus I$ corresponds to $M \setminus A$. So, $(a \neq 0) \wedge (a \neq 1)$ and by *(Con)* axiom $C_R(a, a^*)$. But by our assumption there is no edge between A and $M \setminus A$. So we have $(a \neq 0) \wedge (a \neq 1) \wedge \neg C_R(a, a^*)$. Thus, $\vdash \psi^E \wedge\ (Con) \Rightarrow \perp$ and we obtain $\vdash (Con) \Rightarrow (\psi^E \Rightarrow \perp)$. From the last formula and *(MP)* we infer $\psi^E \Rightarrow \perp$.

We use that $\Rightarrow$ is abbreviation for $\neg\psi^E \vee \perp$. Hence, we get $\neg\psi^E$.
We need to check that $\mathcal{M} \models \psi^E$. By definition $\psi^E = \varphi^P \wedge \varphi^C \wedge \varphi^M$ so we show that $\mathcal{M} \models \varphi^P$, $\mathcal{M} \models \varphi^C$ and $\mathcal{M} \models \varphi^M$:

- The formula φ^P has subformulae of the type $m_i = 0$ and $\neg(m_i = 0)$. We show that for all indices in $i \in I^{pos}$ $v(m_i) = \emptyset$ and for all indices $i \in I^{neg}$ $v(m_i) = \{M_i\}$. So, $v(m_i = 0)$ is equivalent to $v(m_i) = v(0) = \emptyset$. Similarly, for $i \in I^{neg}$ $\neg(\{M_i\} = \emptyset)$ which is correct. Hence, $\mathcal{M} \models \varphi^P$.
- The formula φ^C has subformulae $C(m_i, m_j)$ and $\neg C(m_i, m_j)$. Let us calculate $v(C(m_i, m_j))$ - by definition it is $C_R(v(m_i), v(m_j)) \leftrightarrow R(M_i, M_j) \leftrightarrow (i, j) \in J^{pos}$. We obtain $\neg C_R(m_i, m_j)$ for indices $(i, j) \in J^{neg}$. Therefore, $\mathcal{M} \models \varphi^C$.
- We need to prove that $\mathcal{M} \models \varphi^M$. The proof is similar to the one we did in Lemma 5.4. So, it is briefly mentioned. The formula is a boolean combination of formulae like $m_{i_1} \sqcup \ldots \sqcup m_{i_s} \leq_\mu m_{j_1} \sqcup \ldots \sqcup m_{j_t}$. So we get $\sum_{1 \leq x \leq s} \mu(\{M_{i_x}\}) \leq \sum_{1 \leq y \leq t} \mu(\{M_{j_y}\})$. It is equivalent to $\sum_{1 \leq x \leq s} r_{i_x} \leq \sum_{1 \leq y \leq t} r_{j_y}$. It is correct inequality since $(r_1, r_2, \ldots, r_l)$ is a solution of the corresponding S_n-system. Then, $\mathcal{M} \models \varphi^M$.

So, we have that $\mathcal{M} \models \psi^E$. □

Proposition 6.18. There exists an algorithm which takes as input an arbitrary formula φ from our language $\mathcal{L}$ and returns a model for φ over a finite relational HL-structure or a proof for $\neg\varphi$ for a finite number of steps.

Proof. Let φ be a formula from $\mathcal{L}$ and let $p_1, p_2, \ldots, p_k$ be all propositional variables in φ. According to the result from (Proposition 6.16) we have $\vdash \varphi \Leftrightarrow \Psi^E_\varphi$ where $\Psi^E_\varphi = \psi^E_1 \vee \psi^E_2 \vee \ldots \vee \psi^E_l$ and $for\ i = 1, 2, \ldots, l$ ψ^E_i is a good elementary formula. So, each ψ^E_i has the following representation $\varphi^P_i \wedge \varphi^C_i \wedge \varphi^M_i$. If we construct a model for one ψ^E_i, it will be a model for the whole Ψ^E_φ. We will process each disjunctive member one by one and check whether we could create a model for it or we will find a proof for its negation. We will apply Lemma 6.17 on each ψ^E_i and as a result we get $\mathcal{M}$ model for ψ^E_i or $\vdash \neg\psi^E_i$. We have two cases for each ψ^E_i:

Case 1: The Lemma 6.17 gives us a model $\mathcal{M}$ for ψ^E_i. So, $\mathcal{M} \models \psi^E_i$. Since we have a model for one disjunctive member then we have a model for the whole disjunction $\mathcal{M} \models \psi^E_i \vee \psi^E_2 \vee \ldots \vee \psi^E_l$. Thus, $\mathcal{M} \models \Psi^E_\varphi$ and the algorithm finishes.

Case 2: The Lemma 6.17 gives us a proof α_i for $\vdash \neg\psi^E_i$. Then the algorithm continues with the next ψ^E_i if such exists.

The algorithm did not construct a model for any of ψ^E_i so we have $\alpha_1, \alpha_2, \ldots, \alpha_l$ proofs for each $\neg\psi^E_i$. From here we consider the following propositional tautology $\psi_1 \Rightarrow \psi_2 \Rightarrow \psi_1 \wedge \psi_2$ we obtain that $\neg\psi^E_1 \wedge \neg\psi^E_2 \wedge \ldots \wedge \neg\psi^E_l$ is tautological consequence of $\neg\psi^E_1$, $\neg\psi^E_2$, $\ldots$, $\neg\psi^E_l$. By Tautology Theorem $\vdash \neg\psi^E_1 \wedge \neg\psi^E_2 \wedge \ldots \wedge \neg\psi^E_l$. It is actually $\vdash \neg\Psi^E_\varphi$. From the propositional tautology $(\psi_1 \Leftrightarrow \psi_2) \Leftrightarrow (\neg\psi_1 \Leftrightarrow \neg\psi_2)$ follows that $\vdash \neg\varphi \Leftrightarrow \neg\Psi^E_\varphi$. □

Theorem 6.19 (Completeness Theorem)**.** Let φ be an arbitrary formula from $\mathcal{L}$. Then the following conditions are equivalent:

(i) φ is a theorem of $\mathcal{L}$
(ii) φ is valid in any HL-structure
(iii) φ is valid in any finite relational HL-structure

Proof. The direction from *(i)* to *(ii)* follows from Theorem 6.3. The direction from *(ii)* to *(iii)* is clear because the finite relational HL-structures are subset of all HL-structures. We will prove the direction from *(iii)* to *(i)* by contraposition. So, we will suppose that $\not\vdash \varphi$. We construct $\Psi^E_{\neg\varphi}$ which is disjunction of good elementary formulae. From Proposition 6.16 we have $\vdash \neg\varphi \Leftrightarrow \Psi^E_{\neg\varphi}$. Now we apply the algorithm from Proposition 6.18 and it finishes with one of two possible results:

1. a proof of $\vdash \neg\Psi^E_{\neg\varphi}$. From tautology $\vdash (\neg\varphi \Leftrightarrow \Psi^E_{\neg\varphi}) \Leftrightarrow (\neg\neg\varphi \Leftrightarrow \neg\Psi^E_{\neg\varphi})$ and *(MP)* we obtain $\vdash (\neg\varphi \Leftrightarrow \Psi^E_{\neg\varphi}) \Rightarrow (\neg\neg\varphi \Leftrightarrow \neg\Psi^E_{\neg\varphi})$. From here we use $\vdash \neg\varphi \Leftrightarrow \Psi^E_{\neg\varphi}$ and *(MP)* to infer $\vdash \neg\neg\varphi \Leftrightarrow \neg\Psi^E_{\neg\varphi}$. From the last formula we obtain $\vdash \neg\Psi^E_{\neg\varphi} \Rightarrow \neg\neg\varphi$. Now we use the result of the algorithm and *(MP)* and infer $\vdash \neg\neg\varphi$. From here and the propositional tautology $\vdash \varphi \Leftrightarrow \neg\neg\varphi$ we have a proof of $\vdash \varphi$. It contradicts with our assumption. So, this result is not possible.

2. a model $\mathcal{M}$ such that $\mathcal{M} \models \Psi^E_{\neg\varphi}$. We infer $\vdash \Psi^E_{\neg\varphi} \Rightarrow \neg\varphi$ from $\vdash \neg\varphi \Leftrightarrow \Psi^E_{\neg\varphi}$ and *(MP)*. So that, we obtain $\mathcal{M} \models \Psi^E_{\neg\varphi} \Rightarrow \neg\varphi$ from the Soundness Theorem 6.3. The algorithm found a model for $\Psi^E_{\neg\varphi}$. Therefore, we get $\mathcal{M} \models \neg\varphi$ using that "$\Rightarrow$" is an abbreviation and the definition for truth in a model. Again we apply the definition for truth in a model and get $\mathcal{M} \models \neg\varphi \leftrightarrow \mathcal{M} \not\models \varphi$.

□

7 Finite Relational Structures and Polytopes

In this section we recall of the properties of *p-morphisms*. We define this notion for models of the desired type where the underlying Kripke frame is finite and connected. We also describe a mechanism which produces finite, connected and acyclic Kripke structure with HL-measure for a given finite, connected and cyclic one. At the end we describe a procedure that constructs polytopes for a given tree-like Kripke structure with HL-measure.

7.1 Notions

Definition 7.1. (P-morphism) Let $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{F}' = \langle W', R'\rangle$ be Kripke structures. Let f be a surjection from W onto W'. We say that f is p-morphism from $\mathcal{F}$ to $\mathcal{F}'$ if the following two conditions are satsified:

(p1) $(\forall x \in W)(\forall y \in W)(R(x,y) \to R'(f(x), f(y)))$

(p2) $(\forall x' \in W')(\forall y' \in W')(R'(x', y') \to (\exists x \in W)(\exists y \in W)(f(x) = x' \wedge f(y) = y' \wedge R(x,y)))$

Definition 7.2. If there exists a p-morphism from frame $\mathcal{F}$ to frame $\mathcal{F}'$ then $\mathcal{F}$ is called p-morphic preimage of $\mathcal{F}'$ and $\mathcal{F}'$ is said to be p-morphic image of $\mathcal{F}$.

Remark. Composition of p-morphisms is also a p-morphism.

Definition 7.3. Let $\mathcal{M} = \langle\langle\langle\mathcal{B}, C_R\rangle, \mu\rangle, v\rangle$ and $\mathcal{M}' = \langle\langle\langle\mathcal{B}', C'_R\rangle, \mu'\rangle, v'\rangle$ be a Kripke models where $\langle\langle\mathcal{B}, C_R\rangle, \mu\rangle$ and $\langle\langle\mathcal{B}', C'_R\rangle, \mu'\rangle$ are set-theoretic Contact algebras with measure obtained from Kripke frames $\mathcal{F}$ and $\mathcal{F}'$ accordingly. Both μ and μ' satisfy the conditions for HL-measure (Def. 5.1). We say that f is p-morphism from $\mathcal{M}$ to $\mathcal{M}'$ when the following conditions are satisfied:

(i) f is p-morphism from $\mathcal{F}$ to $\mathcal{F}'$

(ii) $(\forall p \in BoolVars)(\forall w \in W)(w \in v(p) \leftrightarrow f(w) \in v'(p))$

(iii) $\mu(v(p)) = \mu'(v'(p))$

In such cases, we will say that $\mathcal{M}$ is p-morphic preimage of $\mathcal{M}'$.

Lemma 7.4. Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke frame and $\mathcal{C} = \langle\mathcal{B}, C_R\rangle$ be the set-theoretic contact algebra obtained from $\mathcal{F}$. Then the following two conditions are equivalent:

(i) $\mathcal{C}$ is connected

(ii) $\mathcal{F}$ is connected

Proof. (i) ⇒ (ii) Let $\mathcal{C}$ be connected. Let x and y be elements from W such that there is no path in W between them. Let X_R and Y_R be all vertices accessible from x and y in W respectively. Let $X = X_R \cup \{x\}$ and $Y = Y_R \cup \{y\}$. Both sets X and Y are not empty because $x \in X$ and $y \in Y$. We have $X \cap Y = \emptyset$ since there is no path between x and y. Now we will use that $\mathcal{C}$ is connected so, $C_R(X, W \setminus X)$. In other words $(\exists x_0 \in X)(\exists y_0 \in W \setminus X)(R(x_0, y_0))$. The vertex x_0 is accessible from x and there is an edge between x_0 and y_0 so $y_0 \in X$. It is a contradiction, then $\mathcal{F}$ is also connected.

(ii) ⇒ (i) Let assume that $\mathcal{C}$ is not connected. Then there is a subset of W such that $X \neq \emptyset$ and $X \neq W$ and $\neg C_R(X, W \setminus X)$. So, $(\forall x \in X)(\forall y \in W \setminus X)(\neg R(x,y))$. Let x and y be arbitrary elements from X and $W \setminus X$ accordingly. Let $\{v_i\}_{i<k}$ be a path from x to y where $v_0 = x$ and $v_{k-1} = y$. There exists such path since $\mathcal{F}$ is connected. We chose element $x \in X$ so $v_i \in X$ for $i < k$. Then also $y \in X$ which is a contradiction. Thus, there is index $i < k - 1$ such that $v_i \in X$, $v_{i+1} \in W \setminus X$ and $R(v_i, v_{i+1})$. The last contradicts to $\neg C_R(X, W \setminus X)$. Therefore, $\mathcal{C}$ is also connected. □

Remark. We will use the standard notion for preimage $f^{-1}[A]$ to denote the set $\{x \mid f(x) \in A\}$.

We will define some notions from the graph theory for such Kripke structures with measure. We will use the definitions for *path* and *connected graph* from Section 2.4. Let $\mathcal{F} = \langle W, R\rangle$ be a Kripke structure. We will define *simple path* in $\mathcal{F}$ as a path and each node in this path appears only once. A simple cycle in $\mathcal{F}$ is a simple path $\{x_i\}_{i<k}$ such that $k > 2$ and $\langle x_0, x_{k-1}\rangle \in R$.

7.2 Untying

Let $\langle W, R\rangle$ be a finite connected Kripke structure and $\mathcal{C} = \langle\langle \mathcal{B}, C_R\rangle, \mu\rangle$ be the set-theoretic contact algebra with measure obtained from $\mathcal{F}$. We proved in the previous section that $\mathcal{C}$ is also connected. In this study we are interested in models from the following type $\mathcal{M} = \langle \mathcal{C}, v\rangle$ where $\mathcal{C}$ is a finite connected contact algebra with HL-measure. Through this section we will consider the tuple $\mathcal{F} = \langle W, R, \mu, v\rangle$ and $\mathcal{M}$ as interchangeable. We will examine some of its properties.

Definition 7.5. (Untying Step) Let π be a simple cycle in $\mathcal{F}$. Let π contains a and $\mu(a) \neq +\infty$. Let the node b be one of the two adjecent to a nodes and the element $a' \notin W$.

$$W' \overset{\text{def}}{=} W \cup \{a'\}$$
$$R' \overset{\text{def}}{=} (R \setminus \{\langle a, b\rangle, \langle b, a\rangle\}) \cup \{\langle a', b\rangle, \langle b, a'\rangle, \langle a', a'\rangle\}$$
$$\mu'(\{x\}) = \begin{cases} \mu(\{x\}),\ if\ x \notin \{a, a'\} \\ \frac{\mu(\{a\})}{2},\ otherwise \end{cases}$$
$$v'(p) = \begin{cases} v(p),\ if\ a \notin v(p) \\ v(p) \cup \{a'\},\ otherwise \end{cases}$$

We say that $\mathcal{F}' = \langle W', R', \mu', v'\rangle$ is obtained from $\mathcal{F}$ after applying *untying step*.

Remark. If μ is an HL-measure, then μ' is also an HL-measure.

Definition 7.6. (Untying) Let $\{\mathcal{F}_i\}_{i<\omega}$ be a sequence of finite connected Kripke models with HL-measure and valuation defined by the following procedure:

Base: $\mathcal{F}_0 \overset{\text{def}}{=} \mathcal{F}$

Step: $\mathcal{F}_{k+1} \overset{\text{def}}{=} \mathcal{F}_k$, if $\mathcal{F}_k$ is acyclic. Otherwise, we apply *untying step* over $\mathcal{F}_k$ to obtain $\mathcal{F}_{k+1}$ by breaking a simple cycle.

We will call such a sequence $\{\mathcal{F}_i\}_{i<\omega}$ untying of $\mathcal{F}$.

Lemma 7.7. Let $\{\mathcal{F}_i\}_{i<\omega}$ be untying of $\mathcal{F}$. Let $\mathcal{F}_k$ and $\mathcal{F}_{k+1}$ for $k < \omega$ be a consecutive elements in that sequence. If $\mathcal{F}_k$ has a cycle, then $\mathcal{F}_{k+1}$ has strictly less simple cycles than $\mathcal{F}_k$. Moreover, for each $\mathcal{F}$ there exists the least natural number K such that $\mathcal{F}_K$ is acyclic.

Proof. Let $\mathcal{F}_{k+1}$ be obtained from $\mathcal{F}_k$ by applying *untying step* and breaking the simple cycle π. Then, it does not appear in $\mathcal{F}_{k+1}$. We have to show that *untying step* does not introduce new cycles. Since the element a' is adjecent only to b it cannot appear in a simple cycle. So, every simple cycle of $\mathcal{F}_{k+1}$ is a simple cycle for $\mathcal{F}_k$. So that, on each *untying step* we remove an edge which is an element of at least one simple cycle. Hence, there is the least $K \in \mathbb{N}$ such that $\mathcal{F}_K$ is acyclic. □

Remark. We will use $\{\mathcal{F}_i\}_{i<K}$ to denote untying of $\mathcal{F}$ where K is the number from Lemma 7.7.

Lemma 7.8. Let $\{\mathcal{F}_i\}_{i<K}$ be untying of $\mathcal{F}$ and let $\mathcal{F}$ be connected. Then, for any $k \leq K$ $\mathcal{F}_k$ is connected.

Proof. We will prove the lemma by induction on k. The base case is $\mathcal{F}_0 = \mathcal{F}$ so $\mathcal{F}_0$ is connected. Let for all $i \leq k < K$ $\mathcal{F}_i$ is connected:

> $\mathcal{F}_{k+1} = \langle W_{k+1}, R_{k+1}, \mu_{k+1}, \nu_{k+1} \rangle$ is obtained by applying *untying step* on $\mathcal{F}_k$ and a vertex a. Let $(v_1, v_2, \ldots, v_j, a, b, v_{j+1}, \ldots, v_m)$ be the cycle we broke during the *untying step.* Let x and y are arbitrary distinct elements from W_{k+1}:
>
> *Case 1:* Let $x \neq a'$ and $y \neq a'$. Vertices x and y are elements of W_k so, they are connected. Let $\{x_i\}_{i<\ell}$ be the path from x to y. If a and b do not appear as consecutive elements in this path (we do not use the edge $\langle a, b\rangle$), then clearly $\{x_i\}_{i<\ell}$ is also a path in $\mathcal{F}_{k+1}$. We will consider the other case when the edge $\langle a, b\rangle$ is used in the path $\{x_i\}_{i<\ell}$. So $\{x_i\}_{i<\ell} = \{x_0, x_1, \ldots, a, b, \ldots x_{\ell-1}\}$. We will substitute in that path the edge $\langle a, b\rangle$ with the path $(a, v_j, v_{j-1}, \ldots v_1, v_m, \ldots, v_{j+2}, v_{j+1}, b)$. Therefore, we obtain $\{x_i\}_{i<\ell'} = \{x_0, x_1, \ldots, a, v_j, v_{j-1}, \ldots v_1, v_m, \ldots, v_{j+2}, v_{j+1}, b, \ldots x_{\ell'-1}\}$ is a path from x to y in $\mathcal{F}_{k+1}$.
>
> *Case 2:* Either x or y is a'. We will consider the case $x = a'$. We have $\langle a', b\rangle \in R_{k+1}$. We use the fact that $y, b \in W_k$ and the *(IH)*, thus y and b are connected. Therefore, a' and y are also connected. We could apply similar arguments in case when $y = a'$.

□

Lemma 7.9. (P-morphism, 1) Let $\mathcal{F}_k$ and $\mathcal{F}_{k+1}$ be consecutive elements from untying of $\mathcal{F}$. Then, $\mathcal{F}_{k+1}$ is a p-morphic preimage of $\mathcal{F}_k$.

Proof. Let $\mathcal{F}_{k+1}$ be obtained by applying untying step for $\mathcal{F}_k$ and vertex a. We will define the following surjective function $f : W_{k+1} \to W_k$ and $f = \{\langle a', a\rangle\} \cup \{(x, x) | x \in W_k\}$. We have to check that f satisfies the conditions for p-morphism:

> *(p1)* Let x and y be arbitrary elements from W_{k+1} and $x, y \notin \{a, a'\}$ so, x and y are also elements of W_k. Then, $f(x) = x$ and $f(y) = y$. Thus, from the definition of R_{k+1} follows that $R_{k+1}(x, y) \to R_k(x, y)$. We will consider

the cases when $x = a$ then using the definition of f we get that $f(x) = a$ and $f(y) = y$. Clearly by the definition of R_{k+1} we have that $R_{k+1}(a, y) \to R_k(a, y)$. In the case when $x = a'$ $f(x) = a$. In $\mathcal{F}_{k+1}$ there is only one element b such that $R_{k+1}(a', b)$. Now we use the definition of f and obtain $R_{k+1}(a', b) \to R_k(a, b)$.

(p2) Let x and y be arbitrary elements from W_k. If $x = a$, $y = b$, and $R_k(x, y)$ (or the other way around), then we have elements $a', b \in W_{k+1}$ such that $f(a') = a$, $f(b) = b$ and $R_{k+1}(a', b)$. For the rest of the elements of W_k $f(x) = x$ and $f(y) = y$ so, $R_{k+1}(x, y) \to R_k(x, y)$.

Now we have to prove that $f^{-1}[\nu_k(p)] = \nu_{k+1}(p)$ and $\mu_k(\nu_k(p)) = \mu_{k+1}(\nu_{k+1}(p))$. We start with $f^{-1}[\nu_k(p)] = \nu_{k+1}(p)$. Let suppose that $a \notin \nu_k(p)$ then from the definition of ν_{k+1} we have $\nu_{k+1}(p) = \nu_k(p)$. So, $f^{-1}[\nu_k(p)] = \nu_{k+1}(p)$. The other case is when $a \in \nu_k(p)$. Again using the definition we have that $\nu_{k+1}(p) = \nu_k(p) \cup \{a'\}$. Thus, $f^{-1}[\nu_k(p)] = \nu_k(p) \cup \{a'\} = \nu_{k+1}(p)$. We noticed that for any $p \in BoolVars$ either $a, a' \in \nu_{k+1}(p)$ or $a, a' \notin \nu_{k+1}(p)$.
Now we will show that $\mu_k(\nu_k(p)) = \mu_{k+1}(\nu_{k+1}(p))$. Let $a, a' \notin \nu_{k+1}(p)$. Then it is easy to see that $\mu_k(\nu_k(p)) = \mu_{k+1}(\nu_{k+1}(p))$ is true. Let assume that $a, a' \in \nu_{k+1}(p)$. So, $\nu_{k+1}(p) = \{a, a', x_1, \ldots, x_\ell\}$. We know that the elements of that set are different and their singletons are pairwise disjoint so that, we could apply the additivity of the measure. Therefore, $\mu_{k+1}(\nu_{k+1}(p)) = \mu_{k+1}(\{a\}) + \mu_{k+1}(\{a'\}) + \mu_{k+1}(\{x_1\}) + \cdots + \mu_{k+1}(\{x_\ell\})$. We apply the definition and obtain $\mu_{k+1}(\nu_{k+1}(p)) = \frac{\mu_k(\{a\})}{2} + \frac{\mu_k(\{a\})}{2} + \mu_k(\{x_1\}) + \cdots + \mu_k(\{x_\ell\}) = \mu_k(\{a\}) + \mu_k(\{x_1\}) + \cdots + \mu_k(\{x_\ell\}) = \mu_k(\nu_k(p))$. □

Lemma 7.10. Every finite connected Kripke structure with HL-measure is a p-morphic preimage of finite connected acyclic Kripke structure with HL-measure.

Proof. The proof follows from the above lemmas. □

Lemma 7.11. (P-morphism, 2) Let t be a term from $\mathcal{L}$ and let f be a p-morphism from $\mathcal{F}_{k+1}$ to $\mathcal{F}_k$. Then:

(i) $f^{-1}[\nu_k(t)] = \nu_{k+1}(t)$
(ii) $\mu_k(\nu_k(t)) = \mu_{k+1}(\nu_{k+1}(t))$

Proof. (i) We will prove by induction on the construction of terms. The base of the induction is when $t = p$ and $p \in BoolVars$ then it is true by Lemma 7.9. We check the cases when $t = 0$ and $t = 1$:

In the case when $t = 0$ we have that $\nu_k(0) = \emptyset$ we also defined that $\nu_{k+1}(0) = \emptyset$. In the other case when $t = 1$ we have that $\nu_k(1) = W_k$. We also defined that $\nu_{k+1}(1) = W_{k+1}$.

We continue with considering the case $t = t_1 \sqcup t_2$, $t = t_1 \sqcap t_2$ and $t = t_1^*$ and for t_1 and t_2 *(IH)* holds.

We start with $t = t_1 \sqcup t_2$. We will use the definitions for preimage and valuation $f^{-1}[v_k(t_1 \sqcup t_2)] = f^{-1}[v_k(t_1) \cup v_k(t_2)] = f^{-1}[v_k(t_1)] \cup f^{-1}[v_k(t_2)] = v_{k+1}(t_1) \cup v_{k+1}(t_2) = v_{k+1}(t)$.
The case $t = t_1 \sqcap t_2$ is similar to the first one.
We consider the case when $t = t_1^*$. We will apply the definitions $f^{-1}[v_k(t_1^*)] = f^{-1}[W_k \setminus v_k(t_1)] = f^{-1}[W_k] \setminus f^{-1}[v_k(t_1)] = W_{k+1} \setminus v_{k+1}(t_1) = v_{k+1}(t)$.

(ii) Let $\mathcal{F}_{k+1}$ be obtained by applying *untying step* on $\mathcal{F}_k$ and vertex a. First of all we will check the measures when $t = 0$ and $t = 1$:

In the case when $t = 0$ we have that $v_k(0) = \emptyset$ and so $\mu_k(\emptyset) = 0$. Similarly, we defined $v_{k+1}(0) = \emptyset$ and $\mu_{k+1}(\emptyset) = 0$.
The other case when $t = 1$ we have that $v_k(1) = W_k$ so, $\mu_k(W_k) = +\infty$. So that, $v_{k+1}(1) = W_{k+1}$ and $\mu_{k+1}(W_{k+1}) = +\infty$.

We consider the following cases:

If $a \notin v_k(t)$ then $a, a' \notin v_{k+1}(t)$. Then by definitions of v_{k+1} and μ_{k+1} we obtain that $v_k(t) = v_{k+1}(t)$ and $m_k(v_k(t)) = \mu_{k+1}(v_{k+1}(t))$.
If $a \in v_k(t)$ then $a, a' \in v_{k+1}(t)$. Without loss of generality we will consider that $v_k(t) = \{a, x_1, \ldots, x_\ell\}$. So, $v_{k+1}(t) = \{a, a', x_1, \ldots, x_\ell\}$. We calculate the measure of $v_{k+1}(t)$: $\mu_{k+1}(v_{k+1}(t)) = \mu_{k+1}(\{a, a', x_1, \ldots, x_\ell\})$. We again use that the elements of that set are different and their singletons are pairwise disjoint so that, we could apply the additivity of the measure: $\mu_{k+1}(\{a, a', x_1, \ldots, x_\ell\}) = \mu_{k+1}(\{a\}) + \mu_{k+1}(\{a'\}) + \mu_{k+1}(\{x_1\}) + \cdots + \mu_{k+1}(\{x_\ell\})$. It follows from the definition of μ_{k+1} that:

$\mu_{k+1}(\{x_i\}) = \mu_k(\{x_i\})$, for i from 1 to ℓ
$\mu_{k+1}(\{a\}) + \mu_{k+1}(\{a'\}) = \mu_k(\{a\})$

Then, $\mu_k(v_k(t)) = \mu_{k+1}(v_{k+1}(t))$.

□

Lemma 7.12. (P-morphism, 3) Let s and t be terms of $\mathcal{L}$ and let for some k $\mathcal{F}_{k+1}$ be p-morphic preimage of $\mathcal{F}_k$. Relations C_{R_k} and $C_{R_{k+1}}$ are defined in standard way in terms of R_k and R_{k+1}. Then:

(i) $v_k(s) \subseteq v_k(t) \leftrightarrow v_{k+1}(s) \subseteq v_{k+1}(t)$
(ii) $C_{R_k}(v_k(s), v_k(t)) \leftrightarrow C_{R_{k+1}}(v_{k+1}(s), v_{k+1}(t))$
(iii) $\mu_k(v_k(s)) \leq \mu_k(v_k(t)) \leftrightarrow \mu_{k+1}(v_{k+1}(s)) \leq \mu_{k+1}(v_{k+1}(t))$

Proof. Let f be p-morphism from $\mathcal{F}_{k+1}$ to $\mathcal{F}_k$.
(i) We consider the direction "$\rightarrow$". We use that $v_k(s) \subseteq v_k(t)$ so, $f^{-1}[v_k(s)] \subseteq f^{-1}[v_k(t)]$. Using the result from Lemma 7.11 we obtain $v_{k+1}(s) \subseteq v_{k+1}(t)$. The opposite direction $v_{k+1}(s) \subseteq v_{k+1}(t) \rightarrow v_k(s) \subseteq v_k(t)$. We assume that $v_{k+1}(s) =$

$\{x_1, x_2, \ldots, x_{\ell-1}\}$ and $\nu_{k+1}(t) = \{x_1, x_2, \ldots, x_{\ell-1}, x_\ell\}$. So, $\{f(x_1), f(x_2), \ldots, f(x_{\ell-1})\} \subseteq \{f(x_1), f(x_2), \ldots, f(x_{\ell-1}), f(x_\ell)\}$. Therefore, $\nu_k(s) \subseteq \nu_k(t)$.
(ii) We consider the direction "$\rightarrow$". Then, $(\exists x \in \nu_k(s))(\exists y \in \nu_k(t))(R_k(x, y))$. We use the condition *(p2)* so, there exist x' and y' in W_{k+1} such that $f(x') = x$, $f(y') = y$ and $R_{k+1}(x', y')$. We use that $x' \in \nu_{k+1}(s)$ it follows from $x \in \nu_k(s)$, $x' \in f^{-1}[\nu_k(s)]$ and Lemma 7.11. We could prove $y' \in \nu_{k+1}(t)$. Therefore, $C_{R_{k+1}}(\nu_{k+1}(s), \nu_{k+1}(t))$.
The opposite direction "$\leftarrow$". We use the definitions for $C_{R_{k+1}}(\nu_{k+1}(s), \nu_{k+1}(t))$ and obtain that $x \in \nu_{k+1}(s) \subseteq W_{k+1}$, $y \in \nu_{k+1}(t) \subseteq W_{k+1}$ and $R_{k+1}(x, y)$. Then, by condition *(p1)* $R_k(f(x), f(y))$. As in the other direction $f(x) \in \nu_k(s)$ and $f(y) \in \nu_k(t)$. Therefore, $C_{R_k}(\nu_k(s), \nu_k(t))$.
(iii) In Lemma 7.11 we proved that for an arbitrary term t from $\mathcal{L}$ such that $\mu_k(\nu_k(t)) = \mu_{k+1}(\nu_{k+1}(t))$. We will apply this result in the following way:

$$\mu_k(\nu_k(t)) = \mu_{k+1}(\nu_{k+1}(t))$$
$$\mu_k(\nu_k(s)) = \mu_{k+1}(\nu_{k+1}(s))$$

If we have that $\mu_k(\nu_k(s)) \leq \mu_k(\nu_k(t))$, then it is also true that $\mu_{k+1}(\nu_{k+1}(s)) \leq \mu_{k+1}(\nu_{k+1}(t))$ and vice versa. □

Lemma 7.13. (P-morphism, 4) Let φ be a formula of $\mathcal{L}$ and let for some k the model $\mathcal{F}_{k+1}$ be p-morphic preimage of the model $\mathcal{F}_k$. Then, $\mathcal{F}_k \models \varphi \leftrightarrow \mathcal{F}_{k+1} \models \varphi$.

Proof. The proof follows from Lemma 7.12. □

7.3 Finite connected acyclic Kripke structures and polytopes

In this section we will explore an algorithm which associates each node from the finite connected acyclic Kripke structure $\mathcal{F} = \langle W, R, \mu \rangle$ with HL-measure to a polytope.

For this section we assume that v_0 is the unique vertex from W which has a measure $+\infty$ ($\mu(\{v_0\}) = +\infty$). We start with introducing the following abbreviation $L^n_{v_0}$. It is the set of all vertices reachable from v_0 with simple path with length n. We will also use $L^N_{v_0}$ to denote all vertices reachable from v_0 with path with length N and vertices from $L^N_{v_0}$ do not have any other directly accessible vertices except the ones from $L^{N-1}_{v_0}$.

Construction 7.14. (Weight Function) We define inductively *weight function* $L_M : W \rightarrow \mathbb{R}^+$ in the following way:

- For $v \in L^N_{v_0}$, $L_M(v) = \mu_K(\{v\})$
- Let $v \in L^n_{v_0}$ and $V' = \{v_1, v_2, \ldots, v_s\}$ be all vertices from $L^{n+1}_{v_0}$ which are directly accessible from v. Then $L_M(v) = \sum_{v' \in V'} L_M(v') + \mu_K(\{v\})$.
- $L_M(v_0) = +\infty$

Next step is to develop a procedure which constructs a corresponding polytope for a vertex. Since we have two different types of vertices we will show two procedures. We will give one method for the vertex v with measure $+\infty$. The other one will be for vertex v with measure positive real number.

Construction 7.15. Let $\mathscr{F} = \langle W, R, \mu\rangle$ be a finite acyclic Kripke structure with HL-measure. Let the vertex v_0 from W has a measure $+\infty$ and let the vertex v from W be different from v_0 and is with measure positive real number. We show the constructions for both vertices:

Construction for v_0*:* We will construct from v_0 corresponding polytope P_{v_0} as a finite union of basis polytopes (Def. 3.4). We use the following procedure:

Base: Without loss of generality we will assume that all vertices directly accessible from v_0 are $\{v_1, v_2, \ldots, v_s\}$ given by the abbreviation $L^1_{v_0}$. We further use two variables $left$ and $right$ with indices to denote the beginning and the end of intervals which will be included in the corresponding polytope to v_0. We start with initializing $left_0 = 0$ and $right_0 = 1$.

Step: We are processing a vertex v_i which is an element from $L^1_{v_0}$. We calculate an interval that will be included in the representation of v_0 as polytope:

$$left_i = right_{i-1} + L_M(v_i)$$
$$right_i = \begin{cases} left_i + 1, & i < s \\ +\infty, & i = s \end{cases}$$

We apply these steps for all s elements in $L^1_{v_0}$. We define P_{v_0} as follows:

$$P_{v_0} = \begin{cases} [0, +\infty), & s = 0 \\ \bigcup_{i=0}^{s-1}[left_i; right_i] \cup [right_s; +\infty), & s > 0 \end{cases}$$

The polytope P_{v_0} corresponds to v_0.

Construction for v*:* The procedure works over a fixed interval $[left; right]$ with length $L_M(v)$. We will show a mechanism to build its corresponding polytope P_v:

Base: Without loss of generality we assume that v appears in some level $L^n_{v_0}$ for $n >= 1$ and $V' = \{v_1, v_2, \ldots, v_s\}$ are all vertices from $L^{n+1}_{v_0}$ which are directly accessible from v. We initialize $step = \mu_K(\{v\})/(s+1)$. We also initialize $left_0 = left$ and $right_0 = left_0 + step$. Similarly to the previous procedure we define $P^0 = [left_0; right_0]$.

Step: We are processing a vertex v_j which is an element from V'. We define an interval that will be included in the corresponding to v polytope:

$$left_j = right_{j-1} + L_M(v_j)$$
$$right_j = left_j + step$$

We apply these steps for all elements in V'. So, we obtain the polytope $P_v = \bigcup_{i=0}^{s}[left_i; right_i]$ We constructed the polytope P_v corresponding to the vertex v.

So that, we have a map from a vertex v to the corresponding polytope P_v. Thus, this map is from W onto $Pol(\mathbb{R}^+)$ such that:

(i) $Int(P_v) \neq \emptyset$

(ii) $R(v_1, v_2) \leftrightarrow C(P_{v_1}, P_{v_2})$
(iii) $\mu(v) = \mu_L(v)$
(iv) $v1 \neq v_2 \rightarrow P_{v_1} \cap P_{v_2}$ is finite set of real numers

Remark. We remind that the contact relation for polytopes is defined as non-empty set-theoretic intersection and m_L is the Lebesgue measure.

By means of this map we define a function $h : \mathscr{P}(W) \rightarrow Pol(\mathbb{R}^+)$ as follows:

$h(A) = \bigsqcup_{v \in A} P_v$

Since W is finite function h is well-defined. It is easy to prove the basic properties of h summarized in the following lemma:

Lemma 7.16. Let A, A_1 and A_2 be subsets of W. Then:

(i) $h(\emptyset) = \emptyset$
(ii) $h(W) = \mathbb{R}^+$ $(= [0; +\infty))$
(iii) $h(W \setminus A) = (h(A))^*$
(iv) $h(A_1 \cup A_2) = h(A_1) \sqcup h(A_2)$
(v) $A_1 \neq A_2 \rightarrow h(A_1) \neq h(A_2)$
(vi) $A_1 \subseteq A_2 \leftrightarrow h(A_1) \subseteq h(A_2)$
(vii) $C_R(A_1, A_2) \leftrightarrow C(h(A_1), h(A_2))$ $(\leftrightarrow h(A_1) \cap h(A_2))$
(viii) $\mu(A) = \mu_L(h(A))$

Proof. The proof is straightforward verification. □

Now we are ready to prove the main theorem:

Theorem 7.17. Let $\mathscr{B}$ be the contact algebra of polytopes in $\mathbb{R}^+$ and μ_L be the Lebesgue measure on $\mathbb{R}$. For any formula φ from $\mathscr{L}$ the following conditions are equivalent:

(i) $\mathscr{L}_{HL} \vdash \varphi$
(ii) $\langle \mathscr{B}, \mu_L \rangle \models \varphi$

Proof. *(i)* $\Rightarrow$ *(ii)* We have already mentioned that $\langle \mathscr{B}, \mu_L \rangle$ is an HL-structure. Therefore, $\langle \mathscr{B}, \mu_L \rangle \models \varphi$ by the Theorem 6.3.

(ii) $\Rightarrow$ *(i)* We prove this direction by contraposition. So, we assume that φ is not a theorem of $\mathscr{L}_{HL}$, $\nvdash \varphi$. Then, there exist a finite connected acyclic Kripke frame $\mathscr{F} = \langle W, R \rangle$ and an HL-measure μ such that $\langle \mathscr{F}, \mu \rangle \not\models \varphi$. Let v be a valuation such that $\langle \langle \mathscr{F}, \mu \rangle, v \rangle \models \neg\varphi$. Now we consider a valuation v' in $\langle \mathscr{B}, \mu_L \rangle$ defined as $v'(p) = h(v(p))$ for any variable $p \in BoolVars$. Then by the results from Lemma 7.16 for all Boolean terms a and b it follows:

- $\langle\langle\mathcal{F},\mu\rangle,\nu\rangle \models (a \leq b) \leftrightarrow \nu(a) \subseteq \nu(b) \leftrightarrow h(\nu(a)) \subseteq h(\nu(b)) \leftrightarrow \nu'(a) \subseteq \nu'(b) \leftrightarrow \langle\langle\mathcal{B},\mu_L\rangle,\nu'\rangle \models (a \leq b)$
- $\langle\langle\mathcal{F},\mu\rangle,\nu\rangle \models C(a,b) \leftrightarrow C_R(\nu(a),\nu(b)) \leftrightarrow C(h(\nu(a)),h(\nu(b))) \leftrightarrow C(\nu'(a),\nu'(b)) \leftrightarrow \langle\langle\mathcal{B},\mu_L\rangle,\nu'\rangle \models C(a,b)$
- $\langle\langle\mathcal{F},\mu\rangle,\nu\rangle \models (a \leq_\mu b) \leftrightarrow \mu(\nu(a)) \leq \mu(\nu(b)) \leftrightarrow \mu_L(h(\nu(a))) \leq \mu_L(h(\nu(b))) \leftrightarrow \mu_L(\nu'(a)) \leq \mu_L(\nu'(b)) \leftrightarrow \langle\langle\mathcal{B},\mu_L\rangle,\nu'\rangle \models (a \leq_\mu b)$

Now an induction on the construction of the formulae shows that for any formula ψ it holds:

$$\langle\langle\mathcal{F},\mu\rangle,\nu\rangle \models \psi \leftrightarrow \langle\langle\mathcal{B},\mu_L\rangle,\nu'\rangle \models \psi$$

Therefore, $\langle\langle\mathcal{B},\mu_L\rangle,\nu'\rangle \models \neg\psi$. Hence, $\langle\langle\mathcal{B},\mu_L\rangle,\nu'\rangle \not\models \psi$. □

8 Open problems

We would like to mention some problems that had not been part of this study:

- Is it possible to find a finite axiomatic system equivalent to $\mathcal{L}_{HL}$?
- What is the complexity of $\mathcal{L}_{HL}$? *Tip:* Our conjecture is PSpace-complete.

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