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Universal Levenshtein Automata. Building and Properties

A thesis submitted for the degree of Master of Computer Science

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Contents

1 Introduction.

One possible measure for the proximity of two strings is the so-called *Levenshtein distance* (known also as *edit distance*), based on *primitive edit operations*. Primitive edit operations are replacement of one symbol with another (*substitution*), deletion of a symbol, insertion of a symbol and others. The distance between two strings w and v is defined as the minimal number of the primitive edit operations that transform w into v.

This master thesis gives a detailed formal review of the so-called *universal Levenshtein automaton*. The input word for this automaton is a sequence of bit vectors $i(w,v)$ which is computed by given two words w and v. The automaton recognizes $i(w,v)$ iff the distance between w and v is not greater than n.

The greatest advantage of the universal Levenshtein automata $A_n^{\forall,\chi}$ is obtained when we have to extract from a dictionary all words v that are close enough to a given word w. If the dictionary is repesented as a finite deterministic automaton D we can traverse parallelly the two automata $A_n^{\forall,\chi}$ and D to find all these words. Description of this algorithm and its modified version called *forward-backward method*, which is extremely fast in practice, can be found in [MSFASLD].

Short review of the contents

Section 2 - definition of three different *Levenshtein distances* based on the number of edit operations. Section 3 - definition of the nondeterministic Levenshtein automaton $A_n^{ND,\chi}(w)$ and proof that the language of $A_n^{ND,\chi}(w)$ consists of all strings x such that the distance between w and x is not greater than n. Section 4 - definition of the deterministic Levenshtein automaton $A_n^{D,\chi}(w)$ and proof that the languages of $A_n^{ND,\chi}(w)$ and $A_n^{D,\chi}(w)$ are equal. The universal Levenshtein automaton $A_n^{\forall,\chi}$ is defined in section 5. Section 6 - the algorithm for its building. Section 7 - proof that $A_n^{\forall,\chi}$ is minimal. Section 8 - some properties of $A_n^{\forall,\epsilon}$.

Remarks

The aim of this master thesis is to review the deterministic Levenshtein automata and the universal Levenshtein automata presented by their authors Mihov and Schulz in [SMFSCLA] and [MSFASLD]. The main efforts in this master thesis are concentrated on the strict proofs and the details.

This paper is a draft transation of the original text with additional comments and more figures. The original can be found at [ORIG].

The term *Levenshtein distances* is used in the text for d_L^{ϵ}, d_L^t and d_L^{ms}, although for the words $w_1 = abcd$, $w_2 = abdc$ and $w_3 = bdac$ the triangle inequality is not satisfied for d_L^t. $d_L^t(abcd, abdc) = 1$, $d_L^t(abdc, bdac) = 2$, but $d_L^t(abcd, bdac) = 4$.

2 Levenshtein distances. Properties.

Let Σ be a finite set of letters.

Definition 1 $d_L^\epsilon : \Sigma^* \times \Sigma^* \to N$
Let $v, w, v', w' \in \Sigma^*$ and $a, b \in \Sigma$.
1) $v = \epsilon$ or $w = \epsilon$
$d_L^\epsilon(v, w) \stackrel{def}{=} max(|v|, |w|)$
2) $|v| \geq 1$ and $|w| \geq 1$
Let $v = av'$ and $w = bw'$.

$$\begin{array}{rcll} d_L^\epsilon(v,w) & \stackrel{def}{=} & min(& if(a=b, d_L^\epsilon(v',w'), \infty), \\ & & & 1 + d_L^\epsilon(v', bw'), \\ & & & 1 + d_L^\epsilon(av', w'), \\ & & & 1 + d_L^\epsilon(v', w') \quad) \end{array}$$

Notations Here and in what follows the value of the expression $if(Condition, ValueIfConditionIsTrue, ValueIfConditionIsFalse)$ is $ValueIfConditionIsTrue$ if $Condition$ is satisfied and $ValueIfConditionIsFalse$ otherwise. $|x|$ denotes the length of x.

The function d_L^ϵ is called *Levenshtein distance*. $d_L^\epsilon(v, w)$ is called *Levenshtein distance between the words v and w*. The Levenshtein distance between the words v and w is the minimal number of primitive edit operations that transform v into w. The primitive edit operations are deletion of a letter, insertion of a letter and substitution of one letter with another.

Definition 2′ $\hookrightarrow : \Sigma^* \times N \to \Sigma^*$
Let $k \in N$, $x_1, x_2, ..., x_k \in \Sigma$ and $t \in N$.

$$x_1x_2...x_k \hookrightarrow t \stackrel{def}{=} \begin{cases} \epsilon & \text{if } t \geq k \\ x_{t+1}x_{t+2}...x_k & \text{otherwise} \end{cases}$$

Treating the transposition of two letters also as a primitive edit operation we receive the following definition of *Levenshtein distance extended with transposition*:

Definition 2 $d_L^t : \Sigma^* \times \Sigma^* \to N$
Let $v, w, v', w' \in \Sigma^*$ and $a, b, a_1, b_1 \in \Sigma$.
1) $v = \epsilon$ or $w = \epsilon$
$d_L^t(v, w) \stackrel{def}{=} max(|v|, |w|)$
2) $|v| \geq 1$ and $|w| \geq 1$
Let $v = av'$ and $w = bw'$.

$$
\begin{aligned}
d_L^t(v,w) \overset{def}{=} \quad min(\quad & if(a=b, d_L^t(v',w'), \infty), \\
& 1 + d_L^t(v', bw'), \\
& 1 + d_L^t(av', w'), \\
& 1 + d_L^t(v', w'), \\
& if(a_1 < v' \ \& \ b_1 < w' \ \& \ a = b_1 \ \& \ a_1 = b, 1 + d_L^t(v \hookrightarrow 2, w \hookrightarrow 2), \infty) \quad)
\end{aligned}
$$

Notations We use $c < d$ to denote that c is a prefix of d if c and d are words.

The function d_L^t is called *Levenshtein distance extended with transposition.*

When merging of two letters into one and splitting of one letter into two other letters are considered as primitive edit operations we use the following definition of *Levenshtein distance extended with merge and split*:

Definition 3 $d_L^{ms} : \Sigma^* \times \Sigma^* \to N$
Let $v, w, v', w' \in \Sigma^*$ and $a, b \in \Sigma$.
1) $v = \epsilon$ or $w = \epsilon$
$d_L^{ms}(v,w) \overset{def}{=} max(|v|, |w|)$
2) $|v| \geq 1$ and $|w| \geq 1$
Let $v = av'$ and $w = bw'$.

$$
\begin{aligned}
d_L^{ms}(v,w) \overset{def}{=} \quad min(\quad & if(a=b, d_L^{ms}(v',w'), \infty), \\
& 1 + d_L^{ms}(v', bw'), \\
& 1 + d_L^{ms}(av', w'), \\
& 1 + d_L^{ms}(v', w'), \\
& if(|w| \geq 2, 1 + d_L^{ms}(v', w \hookrightarrow 2), \infty), \\
& if(|v| \geq 2, 1 + d_L^{ms}(v \hookrightarrow 2, w'), \infty) \quad)
\end{aligned}
$$

The function d_L^{ms} is called *Levenshtein distance extended with merge and split.*

Notations We use χ as a metasymbol. For example d_L^χ denotes d_L^ϵ, d_L^t or d_L^{ms} if $\chi \in \{\epsilon, t, ms\}$.

Proposition 1 Let $\chi \in \{\epsilon, t, ms\}$ and $v, w \in \Sigma^*$. Then $d_L^\chi(v,w) = 0 \Leftrightarrow v = w$.

Proof
$\Leftarrow$) Let $v = w = x$. Using induction on $|x|$ we prove that $d_L^\chi(x,x) = 0$.
1) $|x| = 0$
$d_L^\chi(x,x) = d_L^\chi(\epsilon,\epsilon) = 0$
2) Induction hypothesis: $d_L^\chi(x,x) = 0$
Let $a \in \Sigma$. We prove that $d_L^\chi(ax,ax) = 0$:

$$
\begin{aligned}
d_L^\chi(ax,ax) = \quad min(\quad & if(a=a, d_L^\chi(x,x), \infty), \\
& ... \ ) =
\end{aligned}
$$

$$min(\quad if(a=a,0,\infty), \; ... \;) = 0$$

$\Rightarrow$) With induction on $|v|$ we prove that $d_L^\chi(v,w)=0 \Rightarrow v=w$.
1) $v=\epsilon$. Let $d_L^\chi(v,w)=0$. $d_L^\chi(v,w)=max(|v|,|w|)=0$. Hence $w=\epsilon$.
2) Induction hypothesis: $\forall w \in \Sigma^*(d_L^\chi(v,w)=0 \Rightarrow v=w)$
Let $a \in \Sigma$ and $w \in \Sigma^*$. We have to prove that $d_L^\chi(av,w)=0 \Rightarrow av=w$. Let $d_L^\chi(av,w)=0$. From the definition of d_L^χ it follows that $|w| \geq 1$. Let $b \in \Sigma$, $w' \in \Sigma^*$ and $w=bw'$. From the definition of d_L^χ it follows that $a=b$ and $d_L^\chi(v,w')=0$. The induction hypothesis implies that $v=w'$. Therefore $av=w$.

Proposition 2 Let $\chi \in \{\epsilon,t,ms\}$ and $v,w \in \Sigma^*$. Then $d_L^\chi(v,w)=d_L^\chi(w,v)$.

The proof of the *Proposition* 2 is straightforward.

Remark As we know *Proposition* 1 and *Proposition* 2, it remains to prove the triangle inequality for d_L^χ ($d_L^\chi(v,w) \leq d_L^\chi(v,x)+d_L^\chi(x,w)$) to show that d_L^χ is *distance*. But this property is used nowhere in this paper. That's why we don't prove it.

Definition 4 Let $\chi \in \{\epsilon,t,ms\}$.
$L_{Lev}^\chi : N \times \Sigma^* \rightarrow P(\Sigma^*)$
$L_{Lev}^\chi(n,w) \stackrel{def}{=} \{v|d_L^\chi(v,w) \leq n\}$
We can find the definitions of L_{Lev}^ϵ, L_{Lev}^t and L_{Lev}^{ms} in [SMFSCLA].

Proposition 3 Let $\chi \in \{\epsilon,t,ms\}$. Let $a \in \Sigma$ and $v,w \in \Sigma^*$. Then $d_L^\chi(v,w)=k \Rightarrow d_L^\chi(av,w) \leq k+1$.

Proof Let $d_L^\chi(v,w)=k$.
1) $w=\epsilon$
$d_L^\chi(av,w)=d_L^\chi(av,\epsilon)=k+1$
2) $|w| \geq 1$
Form the definition of d_L^χ it follows that $d_L^\chi(av,w) \leq 1+d_L^\chi(v,w)=k+1$.

Proposition 4 Let $\chi \in \{\epsilon,t,ms\}$. Let $a,w_1 \in \Sigma$ and $v,w \in \Sigma^*$. Then $d_L^\chi(v,w)=k \Rightarrow d_L^\chi(av,w_1w) \leq k+1$.

Proof Let $d_L^\chi(v,w)=k$. From the definition of d_L^χ it follows that $d_L^\chi(av,w_1w) \leq 1+d_L^{ms}(v,w)=k+1$.

Proposition 5 Let $\chi \in \{\epsilon,t,ms\}$. Let $w_1 \in \Sigma$ and $v,w \in \Sigma^*$. Then $d_L^\chi(v,w)=k \Rightarrow d_L^\chi(v,w_1w) \leq k+1$.

Proof *Proposition* 5 follows directly from *Proposition* 3 and *Proposition* 2.

Proposition 6 Let $\chi \in \{\epsilon, t, ms\}$. Let $w_1 \in \Sigma$ and $v, w \in \Sigma^*$. Then $d_L^\chi(v, w) = k \Rightarrow d_L^\chi(w_1 v, w_1 w) \leq k$.

Proof Let $d_L^\chi(v, w) = k$. From the definition of d_L^χ it follows that $d_L^\chi(w_1 v, w_1 w) \leq d_L^{ms}(v, w) = k$.

Proposition 7 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $w = w_1 w_2 ... w_p$, $p \geq 1$ and $n > 0$. Then

$$\begin{aligned} L_{Lev}^\chi(n, w) \supseteq \quad & \Sigma.L_{Lev}^\chi(n-1, w) \bigcup \\ & \Sigma.L_{Lev}^\chi(n-1, w_2 w_3 ... w_p) \bigcup \\ & L_{Lev}^\chi(n-1, w_2 w_3 ... w_p) \bigcup \\ & w_1.L_{Lev}^\chi(n, w_2 w_3 ... w_p). \end{aligned}$$

Proof From *Properties* 3, 4, 5 and 6 it follows respectively that
$L_{Lev}^\chi(n, w) \supseteq \Sigma.L_{Lev}^\chi(n-1, w)$,
$L_{Lev}^\chi(n, w) \supseteq \Sigma.L_{Lev}^\chi(n-1, w_2 w_3 ... w_p)$,
$L_{Lev}^\chi(n, w) \supseteq L_{Lev}^\chi(n-1, w_2 w_3 ... w_p)$ and
$L_{Lev}^\chi(n, w) \supseteq w_1.L_{Lev}^\chi(n, w_2 w_3 ... w_p)$.
Therefore

$$\begin{aligned} L_{Lev}^\chi(n, w) \supseteq \quad & \Sigma.L_{Lev}^\chi(n-1, w) \bigcup \\ & \Sigma.L_{Lev}^\chi(n-1, w_2 w_3 ... w_p) \bigcup \\ & L_{Lev}^\chi(n-1, w_2 w_3 ... w_p) \bigcup \\ & w_1.L_{Lev}^\chi(n, w_2 w_3 ... w_p). \end{aligned}$$

We show how to extend

$$\begin{aligned} A = \quad & \Sigma.L_{Lev}^\chi(n-1, w) \bigcup \\ & \Sigma.L_{Lev}^\chi(n-1, w_2 w_3 ... w_p) \bigcup \\ & L_{Lev}^\chi(n-1, w_2 w_3 ... w_p) \bigcup \\ & w_1.L_{Lev}^\chi(n, w_2 w_3 ... w_p) \end{aligned}$$

to $L_{Lev}^\chi(n, w)$. First we define R^χ as an extension of A and afterwards we prove that $R^\chi = L_{Lev}^\chi$.

Definition 5 Let $\chi \in \{\epsilon, t, ms\}$.
$R^\chi : N^+ \times \Sigma^+ \rightarrow P(\Sigma^*)$
Let $w \in \Sigma^*$, $w = w_1 w_2 ... w_p$, $p \geq 1$ and $n \geq 1$.
1) $\chi = \epsilon$

$$\begin{aligned} R^\epsilon(n, w) \stackrel{def}{=} \quad & \Sigma.L_{Lev}^\epsilon(n-1, w) \bigcup \\ & \Sigma.L_{Lev}^\epsilon(n-1, w_2 w_3 ... w_p) \bigcup \\ & L_{Lev}^\epsilon(n-1, w_2 w_3 ... w_p) \bigcup \\ & w_1.L_{Lev}^\epsilon(n, w_2 w_3 ... w_p) \end{aligned}$$

2) $\chi = t$

$$\begin{aligned} R^t(n,w) \overset{def}{=} \quad & \Sigma.L^t_{Lev}(n-1,w) \bigcup \\ & \Sigma.L^t_{Lev}(n-1,w_2w_3...w_p) \bigcup \\ & L^t_{Lev}(n-1,w_2w_3...w_p) \bigcup \\ & w_1.L^t_{Lev}(n,w_2w_3...w_p) \bigcup \\ & if(|w| \geq 2, w_2w_1.L^t_{Lev}(n-1,w_3...w_p),\phi). \end{aligned}$$

3) $\chi = ms$

$$\begin{aligned} R^{ms}(n,w) \overset{def}{=} \quad & \Sigma.L^{ms}_{Lev}(n-1,w) \bigcup \\ & \Sigma.L^{ms}_{Lev}(n-1,w_2w_3...w_p) \bigcup \\ & L^{ms}_{Lev}(n-1,w_2w_3...w_p) \bigcup \\ & w_1.L^{ms}_{Lev}(n,w_2w_3...w_p) \bigcup \\ & \Sigma.\Sigma.L^{ms}_{Lev}(n-1,w_2w_3...w_p) \bigcup \\ & if(|w| \geq 2, \Sigma.L^{ms}_{Lev}(n-1,w \hookrightarrow 2),\phi). \end{aligned}$$

Proposition 8 Let $w \in \Sigma^*$, $w = w_1w_2...w_p$, $p \geq 1$ and $n \geq 1$. Then $L^{\chi}_{Lev}(n,w) = R^{\chi}(n,w)$.

Proof
$\supseteq$)
1) $\chi = \epsilon$
From *Proposition* 7 it follows that $L^{\epsilon}_{Lev}(n,w) \supseteq R^{\epsilon}(n,w)$.
2) $\chi = t$
We have to prove that
$(*^t)$ $|w| \geq 2 \Rightarrow L^t_{Lev}(n,w) \supseteq w_2w_1.L^t_{Lev}(n-1,w_3...w_p)$.
Let $|w| \geq 2$ and $v \in L^t_{Lev}(n-1,w_3...w_p)$. Hence $d^t_L(v,w_3...w_p) \leq n-1$. From the definition of d^t_L it follows that $d^t_L(w_2w_1v,w_1w_2w_3...w_p) \leq 1 + d^t_L(v,w_3...w_p) \leq n$. From $(*^t)$ and *Proposition* 7 it directly follows that $L^t_{Lev}(n,w) \supseteq R^t(n,w)$.
3) $\chi = ms$
We have to prove that
$(*^{ms}_1)$ $L^{ms}_{Lev}(n,w) \supseteq \Sigma.\Sigma.L^{ms}_{Lev}(n-1,w_2...w_p)$ and
$(*^{ms}_2)$ $|w| \geq 2 \Rightarrow L^{ms}_{Lev}(n,w) \supseteq \Sigma.L^{ms}_{Lev}(n-1,w_3...w_p)$.
3.1) First we prove $(*^{ms}_1)$
Let $v \in L^{ms}_{Lev}(n-1,w_2...w_p)$ and $a,b \in \Sigma$. Hence $d^{ms}_L(v,w_2...w_p) \leq n-1$. From the definition of d^{ms}_L it follows that $d^{ms}_L(abv,w_1w_2...w_p) \leq 1 + d^{ms}_L(v,w_2...w_p) \leq n$.

3.2) We prove $(*^{ms}_2)$
Let $|w| \geq 2$, $v \in L^{ms}_{Lev}(n-1,w_3...w_p)$ and $a \in \Sigma$. Hence $d^{ms}_L(v,w_3...w_p) \leq n-1$. From the definition of d^{ms}_L it follows that $d^{ms}_L(av,w_1w_2w_3...w_p) \leq 1 + d^{ms}_L(v,w_3...w_p) \leq n$.
From $(*^{ms}_1)$, $(*^{ms}_2)$ and *Proposition* 7 it directly follows that $L^{ms}_{Lev}(n,w) \supseteq R^{ms}(n,w)$.

Therefore $L^{\chi}_{Lev}(n,w) \supseteq R^{\chi}(n,w)$.
$\subseteq$)
Let $v \in L^{\chi}_{Lev}(n,w)$ and $d^{\chi}_{L}(v,w) = k \leq n$.
1) $v = \epsilon$
$d^{\chi}_{L}(v,w) = |w| = k$
$d^{\chi}_{L}(v,w_2...w_p) = |w_2...w_p| = k-1 \leq n-1$
Therefore $v \in L^{\chi}_{Lev}(n-1,w_2...w_p)$.
2) $|v| \geq 1$
Let $|v| = t$ and $v = v_1v_2...v_t$. Hence

$$\begin{aligned} d^{\chi}_{L}(v,w) = \quad min(\quad & if(v_1 = w_1, d^{\chi}_{L}(v_2...v_t, w_2...w_p), \infty), \\ & 1 + d^{\chi}_{L}(v_2...v_t, w), \\ & 1 + d^{\chi}_{L}(v, w_2...w_p), \\ & 1 + d^{\chi}_{L}(v_2...v_t, w_2...w_p), \\ & ...) = k \end{aligned}$$

2.1) $v_1 = w_1$ & $d^{\chi}_{L}(v_2...v_t, w_2...w_p) = k \leq n$
In this case $v \in w_1.L^{\chi}_{Lev}(n, w_2...w_p)$.
2.2) $d^{\chi}_{L}(v_2...v_t, w) = k-1 \leq n-1$
In this case $v \in \Sigma.L^{\chi}_{Lev}(n-1, w)$.
2.3) $d^{\chi}_{L}(v, w_2...w_p) = k-1 \leq n-1$
In this case $v \in L^{\chi}_{Lev}(n-1, w_2...w_p)$.
2.4) $d^{\chi}_{L}(v_2...v_t, w_2...w_p) = k-1 \leq n-1$
In this case $v \in \Sigma.L^{\chi}_{Lev}(n-1, w_2...w_p)$.
Therefore $L^{\epsilon}_{Lev}(n,w) \subseteq R^{\epsilon}(n,w)$.
2.5) $\chi = t$ and $|w| \geq 2$ & $|v| \geq 2$ & $v_1 = w_2$ & $v_2 = w_1$ & $d^{t}_{L}(v \hookrightarrow 2, w \hookrightarrow 2) = k-1 \leq n-1$
In this case $v \in \Sigma.\Sigma.L^{t}_{Lev}(n-1, w_2...w_p)$.
Therefore $L^{t}_{Lev}(n,w) \subseteq R^{t}(n,w)$.
2.6) $\chi = ms$ and $|w| \geq 2$ & $d^{ms}_{L}(v_2...v_t, w_3...w_p) = k-1 \leq n-1$
In this case $v \in \Sigma.L^{ms}_{Lev}(n-1, w \hookrightarrow 2)$.
2.7) $\chi = ms$ and $|v| \geq 2$ & $d^{ms}_{L}(v_3...v_t, w_2...w_p) = k-1 \leq n-1$
In this case $v \in \Sigma.\Sigma.L^{ms}_{Lev}(n-1, w_2...w_p)$.
Therefore $L^{ms}_{Lev}(n,w) \subseteq R^{ms}(n,w)$.
So $L^{\chi}_{Lev}(n,w) \subseteq R^{\chi}(n,w)$.

3 Nondeterministic finite Levenshtein automata for fixed word.

Notations We denote the tuples $<< i,0>,e>$, $<< i,1>,e>$ and $<< i,2>,e>$ with $i^{\#e}$, $i^{\#e}_{t}$ and $i^{\#e}_{s}$ correspondingly.

Definition 6 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$ and $n \in N$. We define the nondeterministic finite Levenshtein automaton $A^{ND,\chi}_{n}(w)$.

$$A^{ND,\chi}_{n}(w) \stackrel{def}{=} < \Sigma, Q^{ND,\chi}_{n}, I^{ND,\chi}, F^{ND,\chi^*}_{n}, \delta^{ND,\chi}_{n} >$$

Notations Suppose that $\gamma : A \multimap B$ is partial function. We use the expression $!\gamma(\pi)$ in order to denote that $\gamma(\pi)$ is defined and $\neg!\gamma(\pi)$ - to denote that $\gamma(\pi)$ is not defined. The special expression $< \pi_1, a, \pi_2 > \in \delta_n^{ND,\chi}$ used for the transition partial function $\delta_n^{ND,\chi} : Q_n^{ND,\epsilon} \times \Sigma \bigcup \{\epsilon\} \multimap P(Q_n^{ND,\epsilon})$ means that $!\delta_n^{ND,\chi^*}(\pi_1, a) \,\&\, \pi_2 \in \delta_n^{ND,\chi^*}(\pi_1, a)$.

Let $|w| = p$ and $w = w_1 w_2 ... w_p$.
1) $\chi = \epsilon$
$Q_n^{ND,\epsilon} \stackrel{def}{=} \{i^{\#e} | 0 \le i \le p \,\&\, 0 \le e \le n\}$
$I^{ND,\epsilon} \stackrel{def}{=} \{0^{\#0}\}$
$F_n^{ND,\epsilon*} \stackrel{def}{=} \{p^{\#e} | 0 \le e \le n\}$
Let $a \in \Sigma \bigcup \{\epsilon\}$ and $q_1, q_2 \in Q_n^{ND,\epsilon}$.

$$\begin{aligned}
&< q_1, a, q_2 > \in \delta_n^{ND,\epsilon} \stackrel{def}{\Leftrightarrow} \\
&q_1 = i^{\#e} \,\&\, q_1 = i^{\#e+1} \,\&\, a \in \Sigma \text{ or} \\
&q_1 = i^{\#e} \,\&\, q_1 = i + 1^{\#e+1} \text{ or} \\
&q_1 = i^{\#e} \,\&\, q_1 = i + 1^{\#e} \,\&\, a = w_{i+1}
\end{aligned}$$

2) $\chi = t$
$Q_n^{ND,t} \stackrel{def}{=} Q_n^{ND,\epsilon} \bigcup \{i_t^{\#e} | 0 \le i \le p - 2 \,\&\, 1 \le e \le n\}$
$I^{ND,t} \stackrel{def}{=} \{0^{\#0}\}$
$F_n^{ND,t*} \stackrel{def}{=} F_n^{ND,\epsilon*}$
Let $a \in \Sigma \bigcup \{\epsilon\}$ and $q_1, q_2 \in Q_n^{ND,t}$.

$$\begin{aligned}
&< q_1, a, q_2 > \in \delta_n^{ND,t} \stackrel{def}{\Leftrightarrow} \\
&< q_1, a, q_2 > \in \delta_n^{ND,\epsilon} \text{ or} \\
&q_1 = i^{\#e} \,\&\, q_2 = i_t^{\#e+1} \,\&\, a = w_{i+2} \text{ or} \\
&q_1 = i_t^{\#e} \,\&\, q_2 = i + 2^{\#e} \,\&\, a = w_{i+1}
\end{aligned}$$

3) $\chi = ms$
$Q_n^{ND,ms} \stackrel{def}{=} Q_n^{ND,\epsilon} \bigcup \{i_s^{\#e} | 0 \le i \le p - 1 \,\&\, 1 \le e \le n\}$
$I^{ND,ms} \stackrel{def}{=} \{0^{\#0}\}$
$F_n^{ND,ms*} \stackrel{def}{=} F_n^{ND,\epsilon*}$
Let $a \in \Sigma \bigcup \{\epsilon\}$ and $q_1, q_2 \in Q_n^{ND,ms}$.

$$\begin{aligned}
&< q_1, a, q_2 > \in \delta_n^{ND,ms} \stackrel{def}{\Leftrightarrow} \\
&< q_1, a, q_2 > \in \delta_n^{ND,\epsilon} \text{ or} \\
&q_1 = i^{\#e} \,\&\, q_2 = i + 2^{\#e+1} \,\&\, a \in \Sigma \text{ or} \\
&q_1 = i^{\#e} \,\&\, q_2 = i + 1_s^{\#e} \,\&\, a \in \Sigma \text{ or} \\
&q_1 = i_s^{\#e} \,\&\, q_2 = i + 1^{\#e} \,\&\, a \in \Sigma
\end{aligned}$$

The extended transition function δ_n^{ND,χ^*} for $A_n^{ND,\chi}$ is defined as usual. First we define the ϵ-closure $Cl_\epsilon : Q_n^{ND,\epsilon} \to P(Q_n^{ND,\epsilon})$:

$$Cl_\epsilon(q) \stackrel{def}{=} \{q\} \bigcup \{\pi | \exists k \ge 0 \exists \eta_1, \eta_2, ..., \eta_k (< q, \epsilon, \eta_1 >, < \eta_1, \epsilon, \eta_2 >, ..., < \eta_k, \epsilon, \pi > \in \delta_n^{ND,\chi})\}$$

We define ϵ-closure for set of states ($Cl_\epsilon : P(Q_n^{ND,\epsilon}) \to P(Q_n^{ND,\epsilon})$) in the following way:

$$Cl_\epsilon(A) \stackrel{def}{=} \bigcup_{\pi \in A} Cl_\epsilon(\pi)$$

Let $v \in \Sigma^*$ and $a \in \Sigma$. We define recursively the partial function $\delta_n^{ND,\chi^*} : Q_n^{ND,\epsilon} \times \Sigma^* \multimap P(Q_n^{ND,\epsilon})$:

$$\delta_n^{ND,\chi^*}(q,\epsilon) \stackrel{def}{=} Cl_\epsilon(q)$$

$$\delta_n^{ND,\chi^*}(q,va) \stackrel{def}{=} \begin{cases} \neg ! & \text{if } \neg !\delta_n^{ND,\chi^*}(q,v) \\ \neg ! & \text{if } !\delta_n^{ND,\chi^*}(q,v)\ \& \\ & \bigcup_{\pi \in \delta_n^{ND,\chi^*}(q,v)} \delta_n^{ND,\chi}(\pi,a) = \phi \\ Cl_\epsilon(\bigcup_{\pi \in \delta_n^{ND,\chi^*}(q,v)} \delta_n^{ND,\chi}(\pi,a)) & \text{otherwise} \end{cases}$$

In what follows we use the expression $< \pi_1, v, \pi_2 > \in \delta_n^{ND,\chi^*}$ to denote that $!\delta_n^{ND,\chi^*}(\pi_1,v)\ \&\ \pi_2 \in \delta_n^{ND,\chi^*}(\pi_1,v)$.

Remark $Q_n^{ND,\chi}$, F_n^{ND,χ^*} and $\delta_n^{ND,\chi}$ depend on the word w. When we use these notations the word w will be clear from the context.

Description of $A_n^{ND,\epsilon}$ can be found in [MSFASLD].

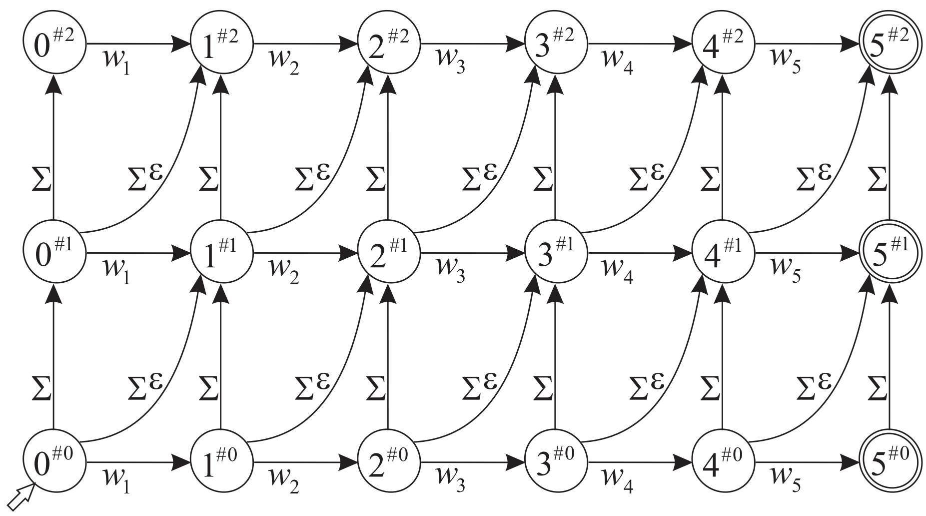

Fig. 1 $A_2^{ND,\epsilon}(w_1 w_2 ... w_5)$
(Σ^ϵ denotes $\Sigma \bigcup \{\epsilon\}$.)

The transitions in the automaton $A_n^{ND,\chi}(w)$ correspond with the definition of R^χ: in $A_n^{ND,\epsilon}$ the transitions $< i^{\#e}, a, i^{\#e+1} >$ $(a \in \Sigma)$ correspond to

$\Sigma.L^{\epsilon}_{Lev}(n-1,w)$. The transitions $< i^{\#e}, a, i+1^{\#e+1} >$ $(a \in \Sigma)$ - to $\Sigma.L^{\epsilon}_{Lev}(n-1, w_2w_3...w_p)$. The transitions $< i^{\#e}, \epsilon, i^{\#e+1} >$ - to $L^{\epsilon}_{Lev}(n-1, w_2w_3...w_p)$. And the transitions $< i^{\#e}, w_i, i+1^{\#e} >$ - to $w_1.L^{\epsilon}_{Lev}(n, w_2w_3...w_p)$. Later we prove that $L(A_n^{ND,\chi}(w)) = L^{\epsilon}_{Lev}(n,w)$.

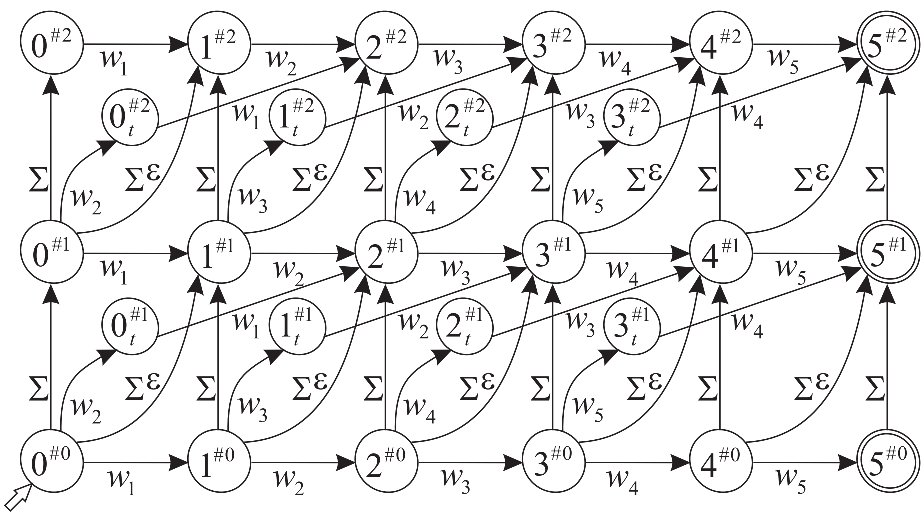

Fig. 2 $A_2^{ND,t}(w_1w_2...w_5)$

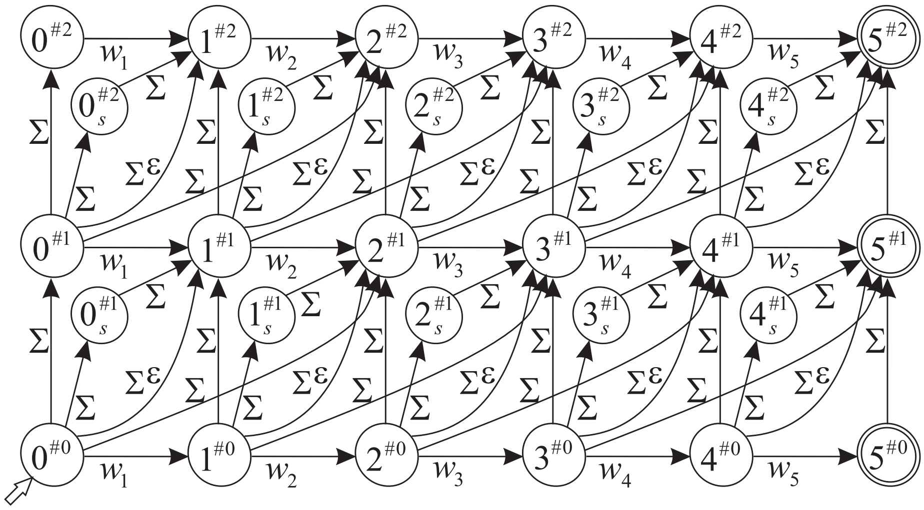

Fig. 3 $A_2^{ND,ms}(w_1w_2...w_5)$

Proposition 9 Let $\chi \in \{\epsilon, t, ms\}$. Let $n \in N$ and $w \in \Sigma^*$. Let $i^{\#e} \in Q_n^{ND,\chi}$. Then $L(i^{\#e}) = L_{Lev}^{\chi}(n-e, w_{i+1}...w_p)$.

($L(\pi) = \{w | \exists \pi' \in F_n^{ND,\chi}(< \pi, w, \pi' > \in \delta_n^{ND,\chi^*})\}$)

The properties $L(i^{\#e}) = L_{Lev}^{\epsilon}(n-e, w_{i+1}...w_p)$, $L(i^{\#e}) = L_{Lev}^{t}(n-e, w_{i+1}...w_p)$ and $L(i^{\#e}) = L_{Lev}^{ms}(n-e, w_{i+1}...w_p)$ are formulated in [SMFSCLA].

Proof Induction on i.
1) $i = p$
$L(p^{\#e}) = \{x | x \in \Sigma^* \ \& \ |x| \leq n-e\} = L_{Lev}^{\chi}(n-e, \epsilon)$
2) $0 \leq i \leq p-1$
Induction hypothesis:
(IH_1) $\forall j \geq 1 \forall e (L(i+j^{\#e}) = L_{Lev}^{\chi}(n-e, w_{i+j+1}...w_p))$
We prove with induction on e that $L(i^{\#e}) = L_{Lev}^{\chi}(n-e, w_{i+1}...w_p)$.
2.1) $e = n$
$L(i^{\#n}) = w_{i+1}.L(i+1^{\#n}) =_{IH_1} w_{i+1}.L_{Lev}^{\chi}(0, w_{i+2}...w_p) = w_{i+1}...w_p = L_{Lev}^{\chi}(0, w_{i+1}...w_p) = L_{Lev}^{\chi}(n-e, w_{i+1}...w_p)$
2.2) $0 \leq e \leq n-1$
Induction hypothesis:
(IH_2) $L(i^{\#e+1}) = L_{Lev}^{\chi}(n-e-1, w_{i+1}...w_p)$
2.2.1) $\chi = \epsilon$
$L(i^{\#e}) = \Sigma.L(i^{\#e+1}) \bigcup \Sigma.L(i+1^{\#e+1}) \bigcup L(i+1^{\#e+1}) \bigcup w_{i+1}.L(i+1^{\#e}) =_{IH_{1,2}}$

$\Sigma.L_{Lev}^{\epsilon}(n-e-1, w_{i+1}...w_p) \bigcup$
$\Sigma.L_{Lev}^{\epsilon}(n-e-1, w_{i+2}...w_p) \bigcup$
$L_{Lev}^{\epsilon}(n-e-1, w_{i+2}...w_p) \bigcup$
$w_{i+1}.L_{Lev}^{\epsilon}(n-e, w_{i+2}...w_p) =$
$R^{\epsilon}(n-e, w_{i+1}...w_p) =_{Proposition\ 8}$
$L_{Lev}^{\epsilon}(n-e, w_{i+1}...w_p)$

2.2.2) $\chi = t$
$L(i^{\#e}) = \Sigma.L(i^{\#e+1}) \bigcup \Sigma.L(i+1^{\#e+1}) \bigcup L(i+1^{\#e+1}) \bigcup w_{i+1}.L(i+1^{\#e}) \bigcup if(i \leq p-2, w_{i+2}w_{i+1}.L(i+1^{\#e+1}), \phi) =_{IH_{1,2}}$

$\Sigma.L_{Lev}^{t}(n-e-1, w_{i+1}...w_p) \bigcup$
$\Sigma.L_{Lev}^{t}(n-e-1, w_{i+2}...w_p) \bigcup$
$L_{Lev}^{t}(n-e-1, w_{i+2}...w_p) \bigcup$
$w_{i+1}.L_{Lev}^{t}(n-e, w_{i+2}...w_p) \bigcup$
$if(|w_{i+1}...w_p| \geq 2, w_{i+2}w_{i+1}.L_{Lev}^{t}(n-e-1, w_{i+3}...w_p), \phi) =$
$R^{t}(n-e, w_{i+1}...w_p) =_{Proposition\ 8}$
$L_{Lev}^{t}(n-e, w_{i+1}...w_p)$

2.2.3) $\chi = ms$

$L(i^{\#e}) = \Sigma.L(i^{\#e+1}) \bigcup \Sigma.L(i+1^{\#e+1}) \bigcup L(i+1^{\#e+1}) \bigcup w_{i+1}.L(i+1^{\#e}) \bigcup$
$\Sigma.\Sigma.L(i+1^{\#e+1}) \bigcup if(i \leq p-2, \Sigma.L(i+2^{\#e+1}), \phi) =_{IH_{1,2}}$

$$\begin{array}{l}
\Sigma.L^{ms}_{Lev}(n-e-1, w_{i+1}...w_p) \bigcup \\
\Sigma.L^{ms}_{Lev}(n-e-1, w_{i+2}...w_p) \bigcup \\
L^{ms}_{Lev}(n-e-1, w_{i+2}...w_p) \bigcup \\
w_{i+1}.L^{ms}_{Lev}(n-e, w_{i+2}...w_p) \bigcup \\
\Sigma.\Sigma.L^{ms}_{Lev}(n-e-1, w_{i+2}...w_p) \bigcup \\
if(|w_{i+1}...w_p| \geq 2, \Sigma.L^{ms}_{Lev}(n-e-1, w_{i+3}...w_p), \phi) = \\
R^{ms}(n-e, w_{i+1}...w_p) =_{\mathit{Proposition\ 8}} \\
L^{ms}_{Lev}(n-e, w_{i+1}...w_p)
\end{array}$$

Corollary Let $\chi \in \{\epsilon, t, ms\}$, $w \in \Sigma^*$ and $n \in N$. *Proposition* 9 implies that $L(A_n^{ND,\chi}(w)) = L(0^{\#0}) = L^{\chi}_{Lev}(n, w)$.

4 Deterministic finite Levenshtein automata for fixed word.

In this section we show a special way for determinization of $A_n^{ND,\chi}(w)$. As a result we receive the deterministic automaton $A_n^{D,\chi}(w)$.

Definition 7 Let $\chi \in \{\epsilon, t, ms\}$.
$Q^{ND,\epsilon} \stackrel{def}{=} \{i^{\#e} | i, e \in Z\}$
$Q^{ND,t} \stackrel{def}{=} Q^{ND,\epsilon} \bigcup \{i_t^{\#e} | i, e \in Z\}$
$Q^{ND,ms} \stackrel{def}{=} Q^{ND,\epsilon} \bigcup \{i_s^{\#e} | i, e \in Z\}$
Let $n \in N$. We define $\delta_e^{D,\chi} : Q^{ND,\chi} \times \{0,1\}^* \rightarrow P(Q^{ND,\chi})$.
Let $b \in \{0,1\}^*$, $k \in N$ and $b = b_1 b_2 ... b_k$.
1) $\chi = \epsilon$

$$\delta_e^{D,\epsilon}(i^{\#e}, b) \stackrel{def}{=} \begin{cases} \{i+1^{\#e}\} & \text{if } 1 < b \\ \{i^{\#e+1}, i+1^{\#e+1}\} & \text{if } b = 0^k \ \&\ b \neq \epsilon \ \&\ e < n \\ \{i^{\#e+1}, i+1^{\#e+1}, i+j^{\#e+j-1}\} & \text{if } 0 < b \ \&\ j = \mu z[b_z = 1] \\ \{i^{\#e+1}\} & \text{if } b = \epsilon \ \&\ e < n \\ \phi & \text{otherwise} \end{cases}$$

$\mu z[A]$ denotes the least z such that A is satisfied.

2) $\chi = t$
2.1)

$$\delta_e^{D,t}(i^{\#e},b) \stackrel{def}{=} \begin{cases} \{i+1^{\#e}\} & \text{if } 1 < b \\ \{i^{\#e+1}, i+1^{\#e+1}, i+2^{\#e+1}, i_t^{\#e+1}\} & \text{if } 01 < b \\ \{i^{\#e+1}, i+1^{\#e+1}, i+j^{\#e+j-1}\} & \text{if } 00 < b \,\&\, j = \mu z[b_z = 1] \\ \{i^{\#e+1}, i+1^{\#e+1}\} & \text{if } b = 0^k \,\&\, b \neq \epsilon \,\&\, e < n \\ \{i^{\#e+1}\} & \text{if } b = \epsilon \,\&\, e < n \\ \phi & \text{otherwise} \end{cases}$$

2.2)

$$\delta_e^{D,t}(i_t^{\#e},b) \stackrel{def}{=} \begin{cases} \{i+2^{\#e}\} & \text{if } 1 < b \\ \phi & \text{otherwise} \end{cases}$$

3) $\chi = ms$

3.1)

$$\delta_e^{D,ms}(i^{\#e},b) \stackrel{def}{=} \begin{cases} \{i+1^{\#e}\} & \text{if } 1 < b \\ \{i^{\#e+1}, i_s^{\#e+1}, i+1^{\#e+1}, i+2^{\#e+1}\} & \text{if } 00 < b \,\vee\, 01 < b \\ \{i^{\#e+1}, i_s^{\#e+1}, i+1^{\#e+1}\} & \text{if } 0 = b \,\&\, e < n \\ \{i^{\#e+1}\} & \text{if } \epsilon = b \,\&\, e < n \\ \phi & \text{otherwise} \end{cases}$$

3.2) $\delta_e^{D,ms}(i_s^{\#e},b) \stackrel{def}{=} \{i+1^{\#e}\}$

The function $\delta_e^{D,\chi}$ is called *function of the elementary transitions.*

Definition 8 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $n \in N$ and $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi^*}, \delta_n^{ND,\chi} >$.

We define $w_{[\,]} : Q_n^{ND,\chi} \to \Sigma^*$.

Let $w = w_1 w_2 ... w_p$ and $\pi \in Q_n^{ND,\chi}$.

1) $\pi = i^{\#e} \in Q_n^{ND,\chi}$

$w_{[i^{\#e}]} = w_{i+1} w_{i+2} ... w_{i+k}$ where $k = min(n-e+1, p-i)$

2) $\pi = i_t^{\#e} \in Q_n^{ND,t}$

$w_{[i_t^{\#e}]} = w_{[i^{\#e}]}$

3) $\pi = i_s^{\#e} \in Q_n^{ND,ms}$

$w_{[i_s^{\#e}]} = w_{[i^{\#e}]}$

The word $w_{[\pi]}$ is called *relevant subword of* w *for* π ([SMFSCLA]).

Definition 9 $\beta : \Sigma \times \Sigma^* \to \{0,1\}^*$

$\beta(x, w_1 w_2 ... w_p) = b_1 b_2 ... b_p$ where $b_i = 1 \Leftrightarrow x = w_i$.

$\beta(x, w_1 w_2 ... w_p)$ is called *characteristic vector of* x *with respect to the word* $w_1 w_2 ... w_p$.

Definition 10 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $n \in N$ and $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi^*}, \delta_n^{ND,\chi} >$.

We define $\delta_e^{D,\chi} : Q_n^{ND,\chi} \times \Sigma \to P(Q_n^{ND,\chi})$.
$\delta_e^{D,\chi}(\pi, x) \stackrel{def}{=} \delta_e^{D,\chi}(\pi, \beta(x, w_{[\pi]}))$

The definitions of $\delta_e^{D,\epsilon}$, $\delta_e^{D,t}$ and $\delta_e^{D,ms}$ are given in [SMFSCLA].

Definition 11 Let $\chi \in \{\epsilon, t, ms\}$.
We define $<_s^\chi \subseteq Q^{ND,\epsilon} \times Q^{ND,\chi}$.
1) $\chi = \epsilon$
$i^{\#e} <_s^\epsilon j^{\#f} \stackrel{def}{\Leftrightarrow} f > e \,\&\, |j - i| \leq f - e$
2) $\chi = t$
$i^{\#e} <_s^t j^{\#f} \stackrel{def}{\Leftrightarrow} i^{\#e} <_s^\epsilon j^{\#f}$
$i^{\#e} <_s^t j_t^{\#f} \stackrel{def}{\Leftrightarrow} f > e \,\&\, |j + 1 - i| \leq f - e$
3) $\chi = ms$
$i^{\#e} <_s^{ms} j^{\#f} \stackrel{def}{\Leftrightarrow} i^{\#e} <_s^\epsilon j^{\#f}$
$i^{\#e} <_s^{ms} j_s^{\#f} \stackrel{def}{\Leftrightarrow} i^{\#e} <_s^\epsilon j^{\#f}$

The relation $<_s^\chi$ is defined in such way that $\pi_1 <_s^\chi \pi_2 \Rightarrow L(\pi_1) \supset L(\pi_2)$. That's why, when we determinize $A_n^{ND,\chi}$, for each state A of the received deterministic automaton it will be true that (*) $\forall q_1, q_2 \in A(q_1 \not<_s^\chi q_2)$. As we take into account that $q' \in \delta_e^{D,\chi}(q, x) \Rightarrow < q, x, q' > \in \delta_n^{ND,\chi^*}$ and $q_1 \leq_s^\chi q_2 \,\&\, < q_2, x, q_2' > \in \delta_n^{ND,\chi^*} \Rightarrow \exists q_1' \in \delta_e^{D,\chi}(q_1, x)(q_1' \leq_s^\chi q_2')$ and also (*), we can define the transition function $\delta_n^{D,\chi}$ for the deterministic automaton: $\delta_n^{D,\chi}(A, x) = \bigsqcup_{q \in A} \delta_e^{D,\chi}(q, x)$ where $\bigsqcup B$ removes from $\bigcup B$ each π for which there is such q in $\bigcup B$ that $q <_s^\chi \pi$.

Remark $\pi_1 <_s^\chi \pi_2$ corresponds to π_1 *subsumes* π_2 from [SMFSCLA]. We don't define when $i_t^{\#e} <_s^t \pi$ or $i_s^{\#e} <_s^{ms} \pi$, i.e. our definition of $<_s^\chi$ implies that $i_t^{\#e} \not<_s^t \pi$ and $i_s^{\#e} \not<_s^{ms} \pi$ for each $i_{t|s}^{\#e}$ and π. We don't define when $i_t^{\#e} <_s^t \pi$ or $i_s^{\#e} <_s^{ms} \pi$ because every "good" definition of $i_t^{\#e} <_s^\chi \pi$ or $i_s^{\#e} <_s^\chi \pi$ will satisfy the property $i_t^{\#e} <_s^t \pi \Rightarrow i + 1^{\#e} <_s^\chi \pi$ or $i_s^{\#e} <_s^{ms} \pi \Rightarrow i^{\#e} <_s^\chi \pi$ correspondingly. If we keep in mind that $i_t^{\#e} \in \delta_e^{D,t}(A, x) \Rightarrow i + 1^{\#e} \in \delta_e^{D,t}(A, x)$, $i_s^{\#e} \in \delta_e^{D,t}(A, x) \Rightarrow i^{\#e} \in \delta_e^{D,t}(A, x)$ and look at the definition of $\bigsqcup$, we shall see easily that each "good" definition of $i_t^{\#e} <_s^t \pi$ or $i_s^{\#e} <_s^{ms} \pi$ leads to the same automata $A_n^{D,\chi}$ and $A_n^{\forall,\chi}$ as our definition.

The set $\{\pi | \pi \in Q_2^{ND,\epsilon} \,\&\, 3^{\#0} \leq_s^\epsilon \pi\}$ when
$A_2^{ND,\epsilon}(w_1 w_2 ... w_5) = < \Sigma, Q_2^{ND,\epsilon}, I^{ND,\epsilon}, F_2^{ND,\epsilon^*}, \delta_2^{ND,\epsilon} >$ is depicted on fig. 4.

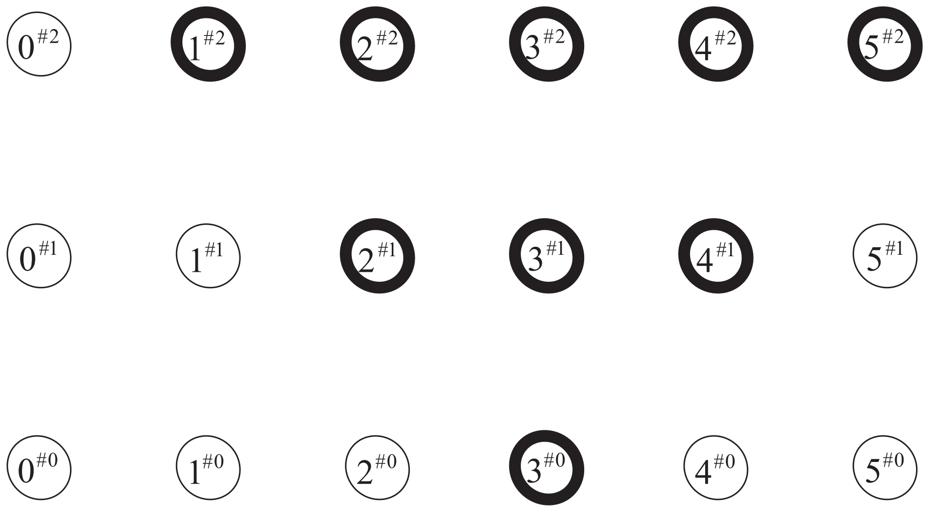

Fig. 4 $A_2^{ND,\epsilon}(w_1w_2...w_5) =< \Sigma, Q_2^{ND,\epsilon}, I^{ND,\epsilon}, F_2^{ND,\epsilon^*}, \delta_2^{ND,\epsilon} >$,
The elements of $\{\pi | \pi \in Q_2^{ND,\epsilon} \,\&\, 3^{\#0} \leq_s^\epsilon \pi\}$ are bold.

We can easily prove the following proposition.

Proposition 10 Let $\chi \in \{\epsilon, t, ms\}$. Then $\leq_s^\chi$ a partial order.

Definition 12 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$ and $n \in N$ and $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi^*}, \delta_n^{ND,\chi} >$.
$\bigsqcup : P(P(Q_n^{ND,\chi})) \rightarrow P(Q_n^{ND,\chi})$
$\bigsqcup A \stackrel{def}{=} \{\pi | \pi \in \bigcup A \,\&\, \neg\exists \pi' \in \bigcup A (\pi' <_s^\chi \pi)\}$

$\bigsqcup$ is defined in [SMFSCLA].

Proposition 11 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $|w| = p$ and $n \in N$. Then $L(A_n^{ND,\chi}(w)) = L(< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >)$ where $F_n^{ND,\chi} = \{i^{\#e} | p - i \leq n - e\}$.

Proof Let $i^{\#e}$ be such that $p - i \leq n - e$. It follows from the definition of $\delta_n^{ND,\chi}$ that
$< i^{\#e}, \epsilon, i + 1^{\#e+1} > \in \delta_n^{ND,\chi}$,
$< i + 1^{\#e+1}, \epsilon, i + 2^{\#e+2} > \in \delta_n^{ND,\chi}$,
...
$< p - 1^{\#e+p-i-1}, \epsilon, p^{\#e+p-i} > \in \delta_n^{ND,\chi}$.
Hence $< i^{\#e}, \epsilon, p^{\#e+p-i} > \in \delta_n^{ND,\chi^*}$.
Therefore $L(A_n^{ND,\chi}(w)) \supseteq L(< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >)$.
Obviously $F_n^{ND,\chi^*} \subseteq F_n^{ND,\chi}$.
Hence $L(A_n^{ND,\chi}(w)) \subseteq L(< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >)$.

Therefore $L(A_n^{ND,\chi}(w)) = L(< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >)$.

In what follows we presume that $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >$.

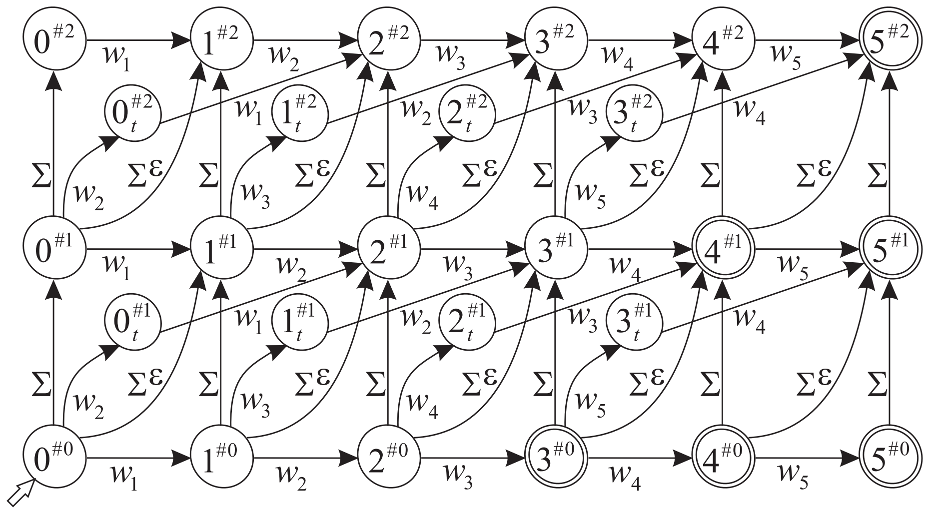

Fig. 5 $\quad A_2^{ND,t}(w_1 w_2 ... w_5) =< \Sigma, Q_2^{ND,t}, I^{ND,t}, F_2^{ND,t}, \delta_2^{ND,t} >$

Definition 13 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$ and $n \in N$ and $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >$. Let $M \subseteq Q_n^{ND,\chi}$ and $\pi \in Q_n^{ND,\epsilon}$. M is called *state with base position* π iff $\forall \pi' \in M(\pi \leq_s^\chi \pi')$ & $\forall \pi_1, \pi_2 \in M(\pi_1 \not<_s^\chi \pi_2)$.

Definition 14 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$ and $n \in N$. We define the deterministic finite automaton $A_n^{D,\chi}(w)$.

$$A_n^{D,\chi}(w) \stackrel{def}{=} < \Sigma, Q_n^{D,\chi}, I^{D,\chi}, F_n^{D,\chi}, \delta_n^{D,\chi} >$$

Let $|w| = p$ and $w = w_1 w_2 ... w_p$.
$\rho : [0,p] \to P(P(Q_n^{ND,\chi}))$
$\rho(i) \stackrel{def}{=} \{M | M \textit{ is state with base position } i^{\#0}\}$

$$Q_n^{D,\chi} \stackrel{def}{=} (\bigcup_{0 \leq i \leq p} \rho(i)) \backslash \{\phi\}$$

$I^{D,\chi} \stackrel{def}{=} \{0^{\#0}\}$
$F_n^{D,\chi} \stackrel{def}{=} \{M | M \in Q_n^{D,\chi}$ & $\exists \pi \in M(\pi \in F_n^{ND,\chi})\}$
$\delta_n^{D,\chi} : Q_n^{D,\chi} \times \Sigma \,-\!\!\!\ominus\!\!\!\rightarrow Q_n^{D,\chi}$

$$\delta_n^{D,\chi}(M,x) \stackrel{def}{=} \begin{cases} \bigsqcup_{\pi\in M} \delta_e^{D,\chi}(\pi,x) & if \bigcup_{\pi\in M} \delta_e^{D,\chi}(\pi,x) \neq \phi \\ \neg ! & \text{otherwise} \end{cases}$$

The finite automata $A_n^{D,\epsilon}(w)$, $A_n^{D,t}(w)$ and $A_n^{D,ms}(w)$ are defined in [SMF-SCLA].

Correctness of the definition

1) We prove that

$$M \in \rho(i) \,\&\, 0 \leq i \leq p-1 \,\&\, x \in \Sigma \Rightarrow \forall \pi \in M(\delta_e^{D,\chi}(\pi,x) \in \rho(i+1))$$

Let $M \in \rho(i)$, $0 \leq i \leq p-1$, $x \in \Sigma$ and $\pi \in M$. We prove that $\delta_e^{D,\chi}(\pi,x) \in \rho(i+1)$:

1.1) $\chi = \epsilon$

Let $\pi = j^{\#f}$. Hence $i^{\#0} \leq_s^\epsilon j^{\#f}$ and $|j-i| \leq f$.

1.1.1) $\delta_e^{D,\epsilon}(\pi,x) = \{j+1^{\#f}\}$

$|j+1-(i+1)| \leq f$. Hence $i+1^{\#0} \leq_s^\epsilon j+1^{\#f}$. Therefore $\delta_n^{D,\epsilon}(\pi,x) \in \rho(i+1)$.

1.1.2) $\delta_e^{D,\epsilon}(\pi,x) = \{j^{\#f+1}, j+1^{\#f+1}, j+z^{\#f+z-1}\}$ for some z such that $z > 1$

Obviously $\forall \pi_1,\pi_2 \in \delta_e^{D,\epsilon}(\pi,x)(\pi_1 \not<_s^\epsilon \pi_2)$. It follows from $|j-i| \leq f$ that $|j-(i+1)| \leq f+1$, $|j+1-(i+1)| \leq f+1$ and $|j+z-(i+1)| \leq f+z-1$. Therefore $i+1^{\#0} \leq_s^\epsilon j^{\#f+1}$, $i+1^{\#0} \leq_s^\epsilon j+1^{\#f+1}$ and $i+1^{\#0} \leq_s^\epsilon j+z^{\#f+z-1}$. Hence $\delta_e^{D,\epsilon}(\pi,x) \in \rho(i+1)$.

1.1.3) $\delta_e^{D,\epsilon}(\pi,x) = \{j^{\#f+1}, j+1^{\#f+1}\}$ or $\delta_e^{D,\epsilon}(\pi,x) = \{j^{\#f+1}\}$ or $\delta_e^{D,\epsilon}(\pi,x) = \phi$

Obviously $\delta_e^{D,\epsilon}(\pi,x) \in \rho(i+1)$.

1.2) $\chi = t$

1.2.1) $\pi = j^{\#f}$

In this case $i^{\#0} \leq_s^t j^{\#f}$ and $|j-i| \leq f$.

1.2.1.1) $\delta_e^{D,t}(\pi,x) = \{j+1^{\#f}\}$

$|j+1-(i+1)| \leq f$. Hence $i+1^{\#0} \leq_s^t j+1^{\#f}$ and $\delta_e^{D,t}(\pi,x) \in \rho(i+1)$.

1.2.1.2) $\delta_e^{D,t}(\pi,x) = \{j^{\#f+1}, j+1^{\#f+1}, j+2^{\#f+1}, j_t^{\#f+1}\}$

In this case $\forall \pi' \in \delta_e^{D,t}(\pi,x)(\pi' = k_{(t)}^{\#l} \Rightarrow l = f+1)$. Therefore $\forall \pi_1,\pi_2 \in \delta_e^{D,t}(\pi,x)(\pi_1 \not<_s^t \pi_2)$. It follows from $|j-i| \leq f$ that $|j-(i+1)| \leq f+1$, $|j+1-(i+1)| \leq f+1$, $|j+2-(i+1)| \leq f+1$ and $|j+1-i| \leq f+1$. Hence $i+1^{\#0} \leq_s^t j^{\#f+1}$, $i+1^{\#0} \leq_s^t j+1^{\#f+1}$, $i+1^{\#0} \leq_s^t j+2^{\#f+1}$ and $i+1^{\#0} \leq_s^t j_t^{\#f+1}$. Therefore $\delta_e^{D,t}(\pi,x) \in \rho(i+1)$.

1.2.1.3) $\delta_e^{D,t}(\pi,x) = \{j^{\#f+1}, j+1^{\#f+1}, j+z^{\#f+z-1}\}$ for some z such that $z > 2$

Obviously $\forall \pi_1,\pi_2 \in \delta_e^{D,t}(\pi,x)(\pi_1 \not<_s^t \pi_2)$. It follows from $|j-i| \leq f$ that $|j+z-(i+1)| \leq f+z-1$. Therefore $\delta_e^{D,t}(\pi,x) \in \rho(i+1)$.

1.2.1.4) $\delta_e^{D,t}(\pi,x) = \{j^{\#f+1}, j+1^{\#f+1}\}$ or $\delta_e^{D,t}(\pi,x) = \phi$

Obviously $\delta_e^{D,t}(\pi,x) \in \rho(i+1)$.

1.2.2) $\pi = j_t^{\#f}$

In this case $i^{\#0} \leq_s^t j_t^{\#f}$ and $|j+1-i| \leq f$.

1.2.2.1) $\delta_e^{D,t}(\pi, x) = \{j + 2^{\#f}\}$

$|j+2-(i+1)| \leq f$. Hence $i+1^{\#0} \leq_s^t j+2^{\#f}$ and $\delta_e^{D,t}(\pi, x) \in \rho(i+1)$.

1.2.2.2) $\delta_e^{D,t}(\pi, x) = \phi$

Obviosly $\phi \in \rho(i+1)$.

1.3) $\chi = ms$

1.3.1) $\pi = j^{\#f}$

In this case $i^{\#0} \leq_s^{ms} j^{\#f}$ and $|j-i| \leq f$.

1.3.1.1) $\delta_e^{D,ms}(\pi, x) = \{j + 1^{\#f}\}$

$|j+1-(i+1)| \leq f$. Hence $i+1^{\#0} \leq_s^{ms} j+1^{\#f}$ and $\delta_e^{D,ms}(\pi, x) \in \rho(i+1)$.

1.3.1.2) $\delta_e^{D,ms}(\pi, x) = \{j^{\#f+1}, j_s^{\#f+1}, j+1^{\#f+1}, j+2^{\#f+1}\}$

In this case $\forall \pi' \in \delta_e^{D,ms}(\pi, x)(\pi' = k_{(s)}^{\#l} \Rightarrow l = f+1)$. Therefore $\forall \pi_1, \pi_2 \in \delta_e^{D,ms}(\pi, x)(\pi_1 \not<_s^{ms} \pi_2)$. It follows from $|j-i| \leq f$ that $|j-(i+1)| \leq f+1$, $|j+1-(i+1)| \leq f+1$ and $|j+2-(i+1)| \leq f+1$. Hence $i+1^{\#0} \leq_s^{ms} j_{(s)}^{\#f+1}$, $i+1^{\#0} \leq_s^{ms} j+1^{\#f+1}$ and $i+1^{\#0} \leq_s^{ms} j+2^{\#f+1}$. Therefore $\delta_e^{D,ms}(\pi, x) \in \rho(i+1)$.

1.3.1.3) $\delta_e^{D,ms}(\pi, x) = \{j^{\#f+1}, j_s^{\#f+1}, j+1^{\#f+1}\}$ or $\delta_e^{D,ms}(\pi, x) = \{j^{\#f+1}\}$ or $\delta_e^{D,ms}(\pi, x) = \phi$

Obviosly $\delta_e^{D,ms}(\pi, x) \in \rho(i+1)$.

1.3.2) $\pi = j_s^{\#f}$.

In this case $i^{\#0} \leq_s^{ms} j_s^{\#f}$ and $|j-i| \leq f$. $\delta_e^{D,ms}(\pi, x) = \{j+1^{\#f}\}$. $|j+1-(i+1)| \leq f$. Hence $i+1^{\#0} \leq_s^{ms} j+1^{\#f}$ and $\delta_e^{D,ms}(\pi, x) \in \rho(i+1)$.

2) We prove that

M is state with base position $p^{\#e}$ & $0 \leq e \leq n-1$ & $x \in \Sigma \Rightarrow$
$\forall \pi \in M(\delta_e^{D,\chi}(\pi, x)$ *is state with base position* $p^{\#e+1})$.

Let M *be state with base position* $p^{\#e}$, $0 \leq e \leq n-1$ and $x \in \Sigma$. Let $\pi \in M$. We have to prove that $\delta_e^{D,\chi}(\pi, x)$ *is state with base position* $p^{\#e+1}$.

2.1) $\chi = \epsilon$

Let $\pi = j^{\#f}$. Hence $p^{\#e} \leq_s^\epsilon j^{\#f}$, $|j-p| \leq f-e$ and $f \geq e$.

2.1.1) $\delta_e^{D,\epsilon}(\pi, x) = \{j + 1^{\#f}\}$

It follows from $|j-p| \leq f-e$ that $|j+1-p| \leq f-(e+1)$. Therefore $p^{\#e+1} \leq_s^\epsilon j+1^{\#f}$ and $\delta_e^{D,\epsilon}(\pi, x)$ *is state with base position* $p^{\#e+1}$.

2.1.2) $\delta_e^{D,\epsilon}(\pi, x) = \{j^{\#f+1}, j+1^{\#f+1}, j+z^{\#f+z-1}\}$ for some z such that $z > 1$

Obviously $\forall \pi_1, \pi_2 \in \delta_e^{D,\epsilon}(\pi, x)(\pi_1 \not<_s^\epsilon \pi_2)$. It follows from $|j-p| \leq f-e$ that $|j-p| \leq f+1-(e+1)$, $|j+1-p| \leq f+1-(e+1)$ and $|j+z-p| \leq f+z-1-(e+1)$. Hence $p^{\#e+1} \leq_s^\epsilon j^{\#f+1}$, $p^{\#e+1} \leq_s^\epsilon j+1^{\#f+1}$ and $p^{\#e+1} \leq_s^\epsilon j+z^{\#f+z-1}$. Therefore $\delta_e^{D,\epsilon}(\pi, x)$ *is state with base position* $p^{\#e+1}$.

2.1.3) $\delta_e^{D,\epsilon}(\pi, x) = \{j^{\#f+1}, j+1^{\#f+1}\}$ or $\delta_e^{D,\epsilon}(\pi, x) = \{j^{\#f+1}\}$ or $\delta_e^{D,\epsilon}(\pi, x) = \phi$

Obviously $\delta_e^{D,\epsilon}(\pi, x)$ *is state with base position* $p^{\#e+1}$.

2.2) $\chi = t$

2.2.1) $\pi = j^{\#f}$

In this case $p^{\#e} \leq_s^t j^{\#f}$, $|j-p| \leq f-e$ and $f \geq e$.

2.2.1.1) $\delta_e^{D,t}(\pi, x) = \{j + 1^{\#f}\}$

It follows from $|j-p| \le f-e$ that $|j+1-p| \le f-(e+1)$. Therefore $p^{\#e+1} \le_s^t j+1^{\#f}$ and $\delta_e^{D,t}(\pi,x)$ *is state with base position* $p^{\#e+1}$.

2.2.1.2) $\delta_e^{D,t}(\pi,x) = \{j^{\#f+1}, j+1^{\#f+1}, j+2^{\#f+1}, j_t^{\#f+1}\}$

In this case $\forall \pi' \in \delta_e^{D,t}(\pi,x)(\pi' = k_{(t)}^{\#l} \Rightarrow l = f+1)$. Therefore $\forall \pi_1, \pi_2 \in \delta_e^{D,t}(\pi,x)(\pi_1 \not<_s^t \pi_2)$. It follows from $|j-p| \le f-e$ that $|j-p| \le f+1-(e+1)$, $|j+1-p| \le f+1-(e+1)$, $|j+2-p| \le f+1-(e+1)$ and $|j+1-p| \le f+1-(e+1)$. Hence $p^{\#e+1} \le_s^t j^{\#f+1}$, $p^{\#e+1} \le_s^t j+1^{\#f+1}$,
$p^{\#e+1} \le_s^t j+2^{\#f+1}$ and $p^{\#e+1} \le_s^t j_t^{\#f+1}$.
Therefore $\delta_e^{D,t}(\pi,x)$ *is state with base position* $p^{\#e+1}$.

2.2.1.3) $\delta_e^{D,t}(\pi,x) = \{j^{\#f+1}, j+1^{\#f+1}, j+z^{\#f+z-1}\}$ for some z such that $z > 2$

Obviously $\forall \pi_1, \pi_2 \in \delta_e^{D,t}(\pi,x)(\pi_1 \not<_s^t \pi_2)$. It follows from $|j-p| \le f-e$ that $|j+z-p| \le f+z-1-(e+1)$.
Therefore $\delta_e^{D,t}(\pi,x)$ *is state with base position* $p^{\#e+1}$.

2.2.1.4) $\delta_e^{D,t}(\pi,x) = \{j^{\#f+1}, j+1^{\#f+1}\}$ or $\delta_e^{D,t}(\pi,x) = \phi$

Obviously $\delta_e^{D,t}(\pi,x)$ *is state with base position* $p^{\#e+1}$.

2.2.2) $\pi = j_t^{\#f}$.

Therefore $p^{\#e} \le_s^t j_t^{\#f}$, $|j+1-p| \le f-e$ and $f \ge e$.

2.2.2.1) $\delta_e^{D,t}(\pi,x) = \{j+2^{\#f}\}$

It follows from $|j+1-p| \le f-e$ that $|j+2-p| \le f-(e+1)$. Therefore $p^{\#e+1} \le_s^t j+2^{\#f}$ and $\delta_e^{D,t}(\pi,x)$ *is state with base position* $p^{\#e+1}$.

2.2.2.2) $\delta_e^{D,t}(\pi,x) = \phi$

Obviously ϕ *is state with base position* $p^{\#e+1}$.

2.3) $\chi = ms$

2.3.1) $\pi = j^{\#f}$

In this case $p^{\#e} \le_s^{ms} j^{\#f}$ and $|j-p| \le f-e$ and $f \ge e$.

2.3.1.1) $\delta_e^{D,ms}(\pi,x) = \{j+1^{\#f}\}$

It follows from $|j-p| \le f-e$ that $|j+1-p| \le f-(e+1)$. Therefore $p^{\#e+1} \le_s^{ms} j+1^{\#f}$ and $\delta_e^{D,ms}(\pi,x)$ *is state with base position* $p^{\#e+1}$.

2.3.1.2) $\delta_e^{D,ms}(\pi,x) = \{j^{\#f+1}, j_s^{\#f+1}, j+1^{\#f+1}, j+2^{\#f+1}\}$

In this case $\forall \pi' \in \delta_e^{D,ms}(\pi,x)(\pi' = k_{(s)}^{\#l} \Rightarrow l = f+1)$. Therefore $\forall \pi_1, \pi_2 \in \delta_e^{D,ms}(\pi,x)(\pi_1 \not<_s^{ms} \pi_2)$. It follows from $|j-p| \le f-e$ that $|j-p| \le f+1-(e+1)$, $|j+1-p| \le f+1-(e+1)$ and $|j+2-p| \le f+1-(e+1)$. Hence $p^{\#e+1} \le_s^{ms} j_{(s)}^{\#f+1}$, $p^{\#e+1} \le_s^{ms} j+1^{\#f+1}$ and $p^{\#e+1} \le_s^{ms} j+2^{\#f+1}$. Therefore $\delta_e^{D,ms}(\pi,x)$ *is state with base position* $p^{\#e+1}$.

2.3.1.3) $\delta_e^{D,ms}(\pi,x) = \{j^{\#f+1}, j_s^{\#f+1}, j+1^{\#f+1}\}$ or $\delta_e^{D,ms}(\pi,x) = \{j^{\#f+1}\}$ or $\delta_e^{D,ms}(\pi,x) = \phi$

Obviously $\delta_e^{D,ms}(\pi,x)$ *is state with base position* $p^{\#e+1}$.

2.3.2) $\pi = j_s^{\#f}$

In this case $p^{\#e} \le_s^{ms} j_s^{\#f}$, $|j-p| \le f-e$, $f \ge e$ and $\delta_e^{D,ms}(\pi,x) = \{j+1^{\#f}\}$. It follows from $|j-p| \le f-e$ that $|j+1-p| \le f-(e+1)$. Therefore $p^{\#e+1} \le_s^{ms} j+1^{\#f}$ and $\delta_e^{D,ms}(\pi,x)$ *is state with base position* $p^{\#e+1}$.

3) We prove that

$A \subseteq \{M | M$ *is state with base position* $i^{\#e}\} \Rightarrow$

$\bigsqcup A$ *is state with base position* $i^{\#e}$.

Let $A \subseteq \{M | M$ *is state with base position* $i^{\#e}\}$.

It follows from the definiton of $\bigsqcup$ that $\bigsqcup A \subseteq \bigcup A$. Therefore $\forall \pi \in \bigsqcup A(i^{\#e} \leq_s^\chi \pi)$. It follows from the definiton of $\bigsqcup$ that $\forall \pi_1, \pi_2 \in \bigsqcup A(\pi_1 \not<_s^\chi \pi_2)$. Therefore $\bigsqcup A$ *is state with base position* $i^{\#e}$.

1), 2) and 3) imply that $\delta_n^{D,\chi}$ is well defined.

The properties for correctness of $\delta_n^{D,\epsilon}$, $\delta_n^{D,t}$ and $\delta_n^{D,ms}$ can be found in [SMF-SCLA].

Proposition 12 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $|w| = p$, $n \in N$ and $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >$. Then $i^{\#e} \in F_n^{ND,\chi} \,\&\, \pi \leq_s^\chi i^{\#e} \Rightarrow \pi \in F_n^{ND,\chi}$.

Proof Let $i^{\#e} \in F_n^{ND,\chi}$ and $\pi = j^{\#f} \leq_s^\chi i^{\#e}$. Hence $|j - i| \leq e - f$ and $p - i \leq n - e$. Therefore $p - j \leq n - f - (e - f - (i - j))$. Therefore $p - j \leq n - f$ and $\pi \in F_n^{ND,\chi}$.

Proposition 13 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $|w| = p$, $n \in N$ and $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >$. Let $x \in \Sigma$, $s \in N$, $\xi_0 = j_{(s)}^{\#f}$ and $\xi_1, \xi_2 ... \xi_s, \eta_2' \in Q_n^{ND,\chi}$. Then

$j < p \,\&$

$< \xi_0, \epsilon, \xi_1 > \in \delta_n^{ND,\chi} \,\&\, < \xi_1, \epsilon, \xi_2 > \in \delta_n^{ND,\chi} \,\&\, ... \,\&\, < \xi_{s-1}, \epsilon, \xi_s > \in \delta_n^{ND,\chi} \,\&$

$< \xi_s, x, \eta_2' > \in \delta_n^{ND,\chi} \Rightarrow$

$j + 1^{\#f} \leq_s^\chi \eta_2'$.

Remark *Proposition* 13 does not hold for $\xi_0 = j_t^{\#f}$ because we may have $< j_t^{\#f}, x, j + 2^{\#f} > \in \delta_n^{ND,t}$ and $j + 1^{\#f} \not\leq_s^t j + 2^{\#f}$.

Proof

Let $j < p$, $< j_{(s)}^{\#f}, \epsilon, \xi_1 > \in \delta_n^{ND,\chi}$, $< \xi_1, \epsilon, \xi_2 > \in \delta_n^{ND,\chi}$,..., $< \xi_{s-1}, \epsilon, \xi_s > \in \delta_n^{ND,\chi}$ and $< \xi_s, x, \eta_2' > \in \delta_n^{ND,\chi}$.

1.1) $\chi = \epsilon$

$\xi_0 = j^{\#f}$

It follows from the definition of $\delta_n^{ND,\epsilon}$ that $\eta_2' \in \{j + s^{\#f+1+s}, j + 1 + s^{\#f+1+s}, j + 1 + s^{\#f+s}\}$. Therefore $j + 1^{\#f} \leq_s^\epsilon \eta_2'$.

1.2) $\chi = t$

$\xi_0 = j^{\#f}$

It follows from the definition of $\delta_n^{ND,t}$ that $\eta_2' \in \{j + s^{\#f+1+s}, j + 1 + s^{\#f+1+s}, j + 1 + s^{\#f+s}, j + s_t^{\#f+1+s}\}$. Therefore $j + 1^{\#f} \leq_s^t \eta_2'$

1.3) $\chi = ms$

1.3.1) $\xi_0 = j^{\#f}$

It follows from the definition of $\delta_n^{ND,ms}$ that $\eta_2' \in \{j + s^{\#f+1+s}, j + 1 + s^{\#f+1+s}, j + 1 + s^{\#f+s}, j + s_s^{\#f+1+s}, j + 2 + s^{\#f+s}\}$. Therefore $j + 1^{\#f} \leq_s^{ms} \eta_2'$.

1.3.2) $\xi_0 = j_s^{\#f}$

In this case we have that $s = 0$ and $\eta'_2 = j+1^{\#f}$. Therefore $j+1^{\#f} \leq_s^{ms} \eta'_2$.

Proposition 14 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $n \in N$ and $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >$. Let $\eta_1, \eta_2 \in Q_n^{ND,\chi}, x \in \Sigma, s \in N$, $\xi_0 = \eta_2$ and $\xi_1, \xi_2 ... \xi_s, \eta'_2 \in Q_n^{ND,\chi}$. Then
$\eta_1 \leq_s^\chi \eta_2$ &
$< \xi_0, \epsilon, \xi_1 > \in \delta_n^{ND,\chi}$ & $< \xi_1, \epsilon, \xi_2 > \in \delta_n^{ND,\chi}$ & ... & $< \xi_{s-1}, \epsilon, \xi_s > \in \delta_n^{ND,\chi}$ &
$< \xi_s, x, \eta'_2 > \in \delta_n^{ND,\chi} \Rightarrow$
$\exists \eta'_1 \in \delta_e^{D,\chi}(\eta_1, x)(\eta'_1 \leq_s^\chi \eta'_2)$.

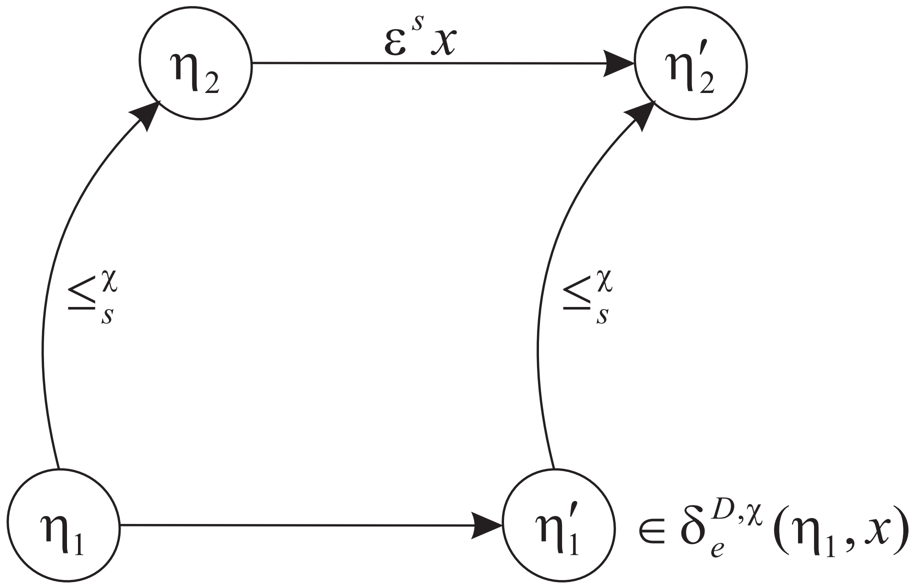

Fig. 6

Remark Using *Proposition* 14 we can easily prove that $\eta_1 \leq_s^\chi \eta_2 \Rightarrow L(\eta_1) \supset L(\eta_2)$.

Proof Let $|w| = p$ and $\eta_1 = i^{\#e}$.
1) $\chi = \epsilon$
Let $\eta_2 = j^{\#f}$.
1.1) $\delta_e^{D,\epsilon}(i^{\#e}, x) = \{i+1^{\#e}\}$
1.1.1) $j < p$
We have from *Proposition* 13 that $j+1^{\#f} \leq_s^\epsilon \eta'_2$. It follows from $i^{\#e} \leq_s^\epsilon j^{\#f}$ that $i+1^{\#e} \leq_s^\epsilon j+1^{\#f}$. Therefore $i+1^{\#e} \leq_s^\epsilon \eta'_2$.
1.1.2) $j = p$

We have from the definition of $\delta_n^{ND,\epsilon}$ that $f < n$, $\eta_2 = p^{\#f}$ and $\eta_2' = p^{\#f+1}$. Therefore $|p-i| \leq f-e$ and $f \geq e$. Hence $|p-(i+1)| \leq (f+1)-e$. Therefore $i+1^{\#e} \leq_s^\epsilon p^{\#f+1} = \eta_2'$.

1.2) $\delta_e^{D,\epsilon}(i^{\#e}, x) = \{i^{\#e+1}, i+1^{\#e+1}, i+z^{\#e+z-1}\}$ and $0 < \beta(x, w_[i^{\#e}])$ and $z = \mu z'[\beta(x, w_[i^{\#e}])_{z'} = 1]$

1.2.1) $j < p$

We have from *Proposition* 13 that $j+1^{\#f} \leq_s^\epsilon \eta_2'$. It follows from $i^{\#e} \leq_s^\epsilon j^{\#f}$ that $i+1^{\#e} \leq_s^\epsilon j+1^{\#f}$. Therefore $i+1^{\#e} \leq_s^\epsilon \eta_2$. $0 < \beta(x, w_[i^{\#e}])$ implies that $i+1^{\#e} \neq \eta_2'$. Therefore $i+1^{\#e} <_s^\epsilon \eta_2'$.

Obviously $\forall \gamma \in Q_n^{ND,\epsilon}(i+1^{\#e} <_s^\epsilon \gamma \leq_s^\epsilon \eta_2' \;\&\; \neg\exists\pi(i+1^{\#e} <_s^\epsilon \pi <_s^\epsilon \gamma) \Rightarrow \gamma \in \{i^{\#e+1}, i+1^{\#e+1}, i+2^{\#e+1}\})$.

1.2.1.1) $\exists \gamma \in Q_n^{ND,\epsilon}(i+1^{\#e} <_s^\epsilon \gamma \leq_s^\epsilon \eta_2' \;\&\; \neg\exists\pi(i+1^{\#e} <_s^\epsilon \pi <_s^\epsilon \gamma) \;\&\; \gamma \in \{i^{\#e+1}, i+1^{\#e+1}\})$.

$\{i^{\#e+1}, i+1^{\#e+1}\} \subset \delta_e^{D,\epsilon}(i^{\#e}, x)$. Therefore there is $\gamma \in \delta_e^{D,\epsilon}(i^{\#e}, x)$ such that $\gamma \leq_s^\epsilon \eta_2'$.

1.2.1.2) $\forall \gamma \in Q_n^{ND,\epsilon}(i+1 <_s^\epsilon \gamma \leq_s^\epsilon \eta_2' \;\&\; \neg\exists\pi(i+1^{\#e} <_s^\epsilon \pi <_s^\epsilon \gamma) \Rightarrow \gamma \notin \{i^{\#e+1}, i+1^{\#e+1}\})$.

Therefore $\forall \gamma \in Q_n^{ND,\epsilon}(i+1 <_s^\epsilon \gamma \leq_s^\epsilon \eta_2' \;\&\; \neg\exists\pi(i+1^{\#e} <_s^\epsilon \pi <_s^\epsilon \gamma) \Rightarrow \gamma = i+2^{\#e+1})$. Therefore $\eta_2' = i+m^{e+m-1}$ for some m such that $m > 1$ and $\beta(x, w_[i^{\#e}])_m = 1$. $z = \mu z'[\beta(x, w_[i^{\#e}])_{z'} = 1]$ implies that $z \leq m$. Therefore $i+z^{\#e+z-1} \leq_s^\epsilon \eta_2'$.

1.2.2) $j = p$

We have from the definition of $\delta_n^{ND,\epsilon}$ that $f < n$, $\eta_2 = p^{\#f}$ and $\eta_2' = p^{\#f+1}$. It follows from $i^{\#e} \leq_s^\epsilon p^{\#f}$ that $i^{\#e+1} \leq_s^\epsilon p^{\#f+1}$.

1.3) $\delta_e^{D,\epsilon}(i^{\#e}, x) = \{i^{\#e+1}, i+1^{\#e+1}\}$ and $\beta(x, w_[i^{\#e}]) = 0^k$ for some k such that $k > 0$.

We prove that $\exists \eta_1' \in \delta_e^{D,\epsilon}(\eta_1, x)(\eta_1' \leq_s^\epsilon \eta_2')$ in a way analogous to 1.2).

1.4) $\delta_e^{D,\epsilon}(i^{\#e}, x) = \{i^{\#e+1}\}$ and $i = p$ and $e < n$.

1.4.1) $\eta_2 = p^{\#f}$ and $f < n$

Obviously $i^{\#e+1} \leq_s^\epsilon p^{\#f+1} = \eta_2'$.

1.4.2) $\eta_1 <_s^\epsilon \eta_2$ and $\eta_2 = j^{\#f}$ and $j < p$

We have from *Proposition* 13 that $j+1^{\#f} \leq_s^\epsilon \eta_2'$. It follows from $p^{\#e} <_s^\epsilon j^{\#f}$ that $|j-p| \leq f-e$ and $f > e$. Therefore $|j+1-p| \leq f-(e+1)$ and $p^{\#e+1} \leq_s^\epsilon j+1^{\#f} \leq_s^\epsilon \eta_2'$.

1.5) $\delta_e^{D,\epsilon}(i^{\#e}, x) = \phi$

Obviously $e = n$ and $\eta_1 = \eta_2$ and ($i = p$ or $0 = \beta(x, w_[i^{\#e}])$). Therefore $\neg\exists\eta_2'(<\xi_0, \epsilon, \xi_1> \in \delta_n^{ND,\epsilon} \;\&\; <\xi_1, \epsilon, \xi_2> \in \delta_n^{ND,\epsilon} \;\&\; ... \;\&\; <\xi_{s-1}, \epsilon, \xi_s> \in \delta_n^{ND,\epsilon} \;\&\; <\xi_s, x, \eta_2'>)$. Contradiction. (This case is impossible.)

2) $\chi = t$

2.1) $\eta_2 = j^{\#f}$

2.1.1) $\delta_e^{D,t}(i^{\#e}, x) = \{i+1^{\#e}\}$

2.1.1.1) $j < p$

We have from *Proposition* 13 that $j+1^{\#f} \leq_s^t \eta_2'$. It follows from $i^{\#e} \leq_s^t j^{\#f}$ that $i+1^{\#e} \leq_s^t j+1^{\#f}$. Therefore $i+1^{\#e} \leq_s^t \eta_2'$.

2.1.1.2) $j = p$

We have from the definition of $\delta_n^{ND,t}$ that $f < n$, $\eta_2 = p^{\#f}$ and $\eta_2' = p^{\#f+1}$. Therefore $|p-i| \le f-e$ and $f \ge e$. Hence $|p-(i+1)| \le (f+1)-e$. Therefore $i+1^{\#e} \le_s^t p^{\#f+1} = \eta_2'$.

2.1.2) $\delta_e^{D,ms}(i^{\#e}, x) = \{i^{\#e+1}, i{+}1^{\#e+1}, i{+}2^{\#e+1}, i_t^{\#e+1}\}$ and $01 < \beta(x, w_[i^{\#e}])$

2.1.2.1) $j < p$

We have from the *Proposition* 13 that $j+1^{\#f} \le_s^t \eta_2'$. It follows from $i^{\#e} \le_s^t j^{\#f}$ that $i+1^{\#e} \le_s^t j+1^{\#f}$. And from $01 < \beta(x, w_[i^{\#e}])$ - that $\eta_2' \ne i+1^{\#f}$. Therefore $i+1^{\#e} <_s^t \eta_2'$. Let $\eta_1' \in Q_n^{ND,t}$ be such that $i+1 <_s^t \eta_1' \le_s^t \eta_2'$ and $\neg\exists\pi(i+1^{\#e} <_s^t \pi <_s^t \eta_1')$ (obviously such η_1' exists). It follows from the definition of $<_s^t$ that $\eta_1' \in \{i^{\#e+1}, i+1^{\#e+1}, i+2^{\#e+1}, i_t^{\#e+1}\}$ (if we suppose that $\eta_1' \in \{i-2_t^{\#e+1}, i-1_t^{\#e+1}\}$ then $s=0$, $\eta_2 \in \{i-2^{\#e}, i-1^{\#e}\}$, $\eta_1 \not\le_s^t \eta_2$, contradiction).

2.1.2.2) $j = p$

We have from the definition of $\delta_n^{ND,t}$ that $f < n$, $\eta_2 = p^{\#f}$ and $\eta_2' = p^{\#f+1}$. It follows from $i^{\#e} \le_s^t p^{\#f}$ that $i^{\#e+1} \le_s^t p^{\#f+1}$.

2.1.3) $\delta_e^{D,t}(i^{\#e}, x) = \{i^{\#e+1}, i+1^{\#e+1}, i+z^{\#e+z-1}\}$ and $00 < \beta(x, w_[i^{\#e}])$ and $z = \mu z'[\beta(x, w_[i^{\#e}])_{z'} = 1]$

The proof that $\exists\eta_1' \in \delta_e^{D,t}(\eta_1, x)(\eta_1' \le_s^t \eta_2')$ is analogous to the proof in 1.2).

2.1.4) $\delta_e^{D,t}(i^{\#e}, x) = \{i^{\#e+1}, i+1^{\#e+1}\}$

The proof that $\exists\eta_1' \in \delta_e^{D,t}(\eta_1, x)(\eta_1' \le_s^t \eta_2')$ is analogous to the proof in 1.2).

2.1.5) $\delta_e^{D,t}(i^{\#e}, x) = \{i^{\#e+1}\}$ and $i = p$ and $e < n$

In this case the proof is analogous to the one in 1.4).

2.1.6) $\delta_e^{D,ms}(i^{\#e}, x) = \phi$

Like in 1.5) we prove that this case is impossible.

2.2) $\eta_2 = j_t^{\#f}$

We have from the definition of $\delta_n^{ND,t}$ that $s = 0$, $\eta_2' = j+2^{\#f}$ and $1 < \beta(x, w_[j_t^{\#f}])$. It follows from $i^{\#e} <_s^t j_t^{\#f}$ that $|j+1-i| \le f-e$ and $f > e$.

2.2.1) $\delta_e^{D,t}(i^{\#e}, x) = \{i+1^{\#e}\}$

We have from $|j+1-i| \le f-e$ that $|j+2-(i+1)| \le f-e$. Therefore $i+1^{\#e} \le_s^t j+2^{\#f}$.

2.2.2) $\delta_e^{D,t}(i^{\#e}, x) = \{i^{\#e+1}, i+1^{\#e+1}, i+2^{\#e+1}, i_t^{\#e+1}\}$

It follows from $|j+1-i| \le f-e$ that $|j+2-i| \le f-(e+1)$ $(j+1 < i)$ or $|j+2-(i+1)| \le f-(e+1)$ $(i = j+1)$ or $|j+2-(i+2)| \le f-(e+1)$ $(i < j+1)$. Therefore $i^{\#e+1} \le_s^t j+2^{\#f}$ or $i+1^{\#e+1} \le_s^t j+2^{\#f}$ or $i+2^{\#e+1} \le_s^t j+2^{\#f}$.

2.2.3) $\delta_e^{D,t}(i^{\#e}, x) = \{i^{\#e+1}, i+1^{\#e+1}, i+z^{\#e+z-1}\}$ and $00 < \beta(x, w_[i^{\#e}])$ and $z = \mu z'[\beta(x, w_[i^{\#e}])_{z'} = 1]$

It follows from $|j+1-i| \le f-e$ that $|j+2-(i+1)| \le f-e$. Hence $i+1^{\#e} <_s^t j+2^{\#f}$. The proof that $\exists\eta_1' \in \delta_e^{D,t}(\eta_1, x)(\eta_1' \le_s^t \eta_2')$ is analogous to 1.2).

2.2.4) $\delta_e^{D,t}(i^{\#e}, x) = \{i^{\#e+1}, i+1^{\#e+1}\}$ and $\beta(x, w_[i^{\#e}]) = 0^k$ for some k such that $k > 0$

Like in 2.2.2) we prove that $i^{\#e+1} \le_s^t j+2^{\#f}$ or $i+1^{\#e+1} \le_s^t j+2^{\#f}$. (It follows from $\beta(x, w_[i^{\#e}]) = 0^k$ and $1 < \beta(x, w_[j_t^{\#f}])$ that $j+1 \le i$).

2.2.5) $\delta_e^{D,t}(i^{\#e}, x) = \{i^{\#e+1}\}$ and $i = p < n$

We have from $|j+1-i| \le f-e$ that $|j+2-i| \le f-(e+1)$

2.2.6) $\delta_e^{D,t}(i^{\#e}, x) = \phi$

Obviously this case is impossible.

3) $\chi = ms$

Let $\eta_2 = j_{(s)}^{\#f}$.

3.1) $\delta_e^{D,ms}(i^{\#e}, x) = \{i+1^{\#e}\}$

3.1.1) $j < p$

We have from *Proposition* 13 that $j+1^{\#f} \le_s^{ms} \eta_2'$. It follows from $i^{\#e} \le_s^{ms} j_{(s)}^{\#f}$ that $i+1^{\#e} \le_s^{ms} j+1^{\#f}$. Therefore $i+1^{\#e} \le_s^{ms} \eta_2'$.

3.1.2) $j = p$

We have from the defintion of $\delta_n^{ND,ms}$ that $f < n$, $\eta_2 = p^{\#f}$ and $\eta_2' = p^{\#f+1}$. Therefore $|p-i| \le f-e$ and $f \ge e$. Hence $|p-(i+1)| \le (f+1)-e$. Therefore $i+1^{\#e} \le_s^{ms} p^{\#f+1} = \eta_2'$.

3.2) $\delta_e^{D,ms}(i^{\#e}, x) = \{i^{\#e+1}, i_s^{\#e+1}, i{+}1^{\#e+1}, i{+}2^{\#e+1}\}$ and ($00 < \beta(x, w_[i^{\#e}])$ or $01 < \beta(x, w_[i^{\#e}])$)

3.2.1) $j < p$

We have from *Proposition* 13 that $j+1^{\#f} \le_s^{ms} \eta_2'$. It follows from $i^{\#e} \le_s^{ms} j_{(s)}^{\#f}$ that $i+1^{\#e} \le_s^{ms} j+1^{\#f}$. And from $00 < \beta(x, w_[i^{\#e}])$ or $01 < \beta(x, w_[i^{\#e}])$ - that $\eta_2' \ne i+1^{\#f}$. Therefore $i+1^{\#e} <_s^{ms} \eta_2'$. Let $\eta_1' \in Q_n^{ND,ms}$ be such that $i+1 <_s^{ms} \eta_1' \le_s^{ms} \eta_2'$ and $\neg\exists\pi(i+1^{\#e} <_s^{ms} \pi <_s^{ms} \eta_1')$ (obviously such η_1' exists). It follows from the definition of $<_s^{ms}$ that $\eta_1' \in \{i^{\#e+1}, i_s^{\#e+1}, i+1^{\#e+1}, i+2^{\#e+1}\}$. (if we suppose that $\eta_1' \in \{i-1_s^{\#e+1}, i+1_s^{\#e+1}\}$ then $s=0$, $\eta_2 \in \{i-1^{\#e}, i+1^{\#e}\}$, $\eta_1 \not\le_s^{ms} \eta_2$, contradiction).

3.2.2) $j = p$

We have from the definition of $\delta_n^{ND,ms}$ that $f < n$, $\eta_2 = p^{\#f}$ and $\eta_2' = p^{\#f+1}$. It follows from $i^{\#e} \le_s^{ms} p^{\#f}$ that $i^{\#e+1} \le_s^{ms} p^{\#f+1}$.

3.3) $\delta_e^{D,ms}(i^{\#e}, x) = \{i^{\#e+1}, i_s^{\#e+1}, i+1^{\#e+1}\}$ and $0 < \beta(x, w_[i^{\#e}])$

The proof that $\exists\eta_1' \in \delta_e^{D,ms}(\eta_1, x)(\eta_1' \le_s^{ms} \eta_2')$ is analogous to the proof in 3.2).

3.4) $\delta_e^{D,ms}(i^{\#e}, x) = \{i^{\#e+1}\}$ and $i = p$ and $e < n$.

Like in 1.4) we prove that $\exists\eta_1' \in \delta_e^{D,ms}(\eta_1, x)(\eta_1' \le_s^{ms} \eta_2')$.

3.5) $\delta_e^{D,ms}(i^{\#e}, x) = \phi$

Obviously this case is impossible.

Proposition 15 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $n \in N$, $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >$ and $A_n^{D,\chi}(w) =< \Sigma, Q_n^{D,\chi}, I^{D,\chi}, F_n^{D,\chi}, \delta_n^{D,\chi} >$. Then $L(A_n^{ND,\chi}(w)) \subseteq L(A_n^{D,\chi}(w))$.

Proof

Let $x_1...x_k \in L(A_n^{ND,\chi}(w))$

1) $x_1...x_k = \epsilon$.

Let $\pi_0, \pi_1, ..., \pi_r \in Q_n^{ND,\chi}$, $r \in N$ be such states that $\pi_0 = 0^{\#0}$, $< \pi_0, \epsilon, \pi_1 >\in \delta_n^{ND,\chi}$, $< \pi_1, \epsilon, \pi_2 >\in \delta_n^{ND,\chi}$, ..., $< \pi_{r-1}, \epsilon, \pi_r >\in \delta_n^{ND,\chi}$ and $\pi_r \in F_n^{ND,\chi}$ (it follows from $\epsilon \in L(A_n^{ND,\chi}(w))$ that such states exist). The definition of $\delta_n^{ND,\chi}$ implies that $\pi_i = i^{\#i}$ for $0 \le i \le r$. Therefore $r^{\#r} \in F_n^{ND,\chi}$. Obviously

$0^{\#0} \leq_s^\chi r^{\#r}$. We have from *Proposition* 12 that $0^{\#0} \in F_n^{ND,\chi}$. Therefore $\{0^{\#0}\} = I^{D,\chi} \in F_n^{D,\chi}$ and $\epsilon = x_1...x_k \in L(A_n^{D,\chi}(w))$.

2) $x_1...x_k \neq \epsilon$.

Let $\pi_0, \pi_1, ..., \pi_{r'} \in Q_n^{ND,\chi}$, $r' \in N$ be such states that $\pi_0 = 0^{\#0}$, $< \pi_0, p_1, \pi_1 > \in \delta_n^{ND,\chi}$, $< \pi_1, p_2, \pi_2 > \in \delta_n^{ND,\chi}$, ..., $< \pi_{r'-1}, p_{r'}, \pi_{r'} > \in \delta_n^{ND,\chi}$, $p_i \in \Sigma \bigcup \{\epsilon\}$ for $0 \leq i \leq r'$, $\pi_{r'} \in F_n^{ND,\chi}$ and $p_1 p_2 ... p_{r'} = x_1 x_2 ... x_k$ (it follows from $x_1 x_2 ... x_k \in L(A_n^{ND,\chi}(w))$ that such states exist). Let r be such that $r \leq r'$ and $p_r = x_k$ and $p_{r+1} = p_{r+2} = ... = p_{r'} = \epsilon$. Obviously $\pi_r \leq_s^\chi \pi_{r'}$. It follows from *Proposition* 12 that $\pi_r \in F_n^{ND,\chi}$. Let $M_0 = \{0^{\#0}\}$ and $M_{i+1} = \delta_n^{D,\chi}(M_i, x_{i+1})$ for $i = 0, 1, ..., k-1$. We have to prove that $M_k \in F_n^{D,\chi}$. Let $j_1 < j_2 < ... < j_k$ be such that $p_{j_1} p_{j_2} ... p_{j_k} = x_1 x_2 ... x_k$ and $p_{j_i} \in \Sigma$ for $1 \leq i \leq k$. Using induction on i we prove that $\exists \pi \in M_i (\pi \leq_s^\chi \pi_{j_i})$ if $1 \leq i \leq k$.

2.1) i = 1

$M_1 = \delta_n^{D,\chi}(\{0^{\#0}\}, x_1) = \delta_e^{D,\chi}(0^{\#0}, x_1)$

Let $\eta_1 = \eta_2 = 0^{\#0}$ and $s = j_1 - 1$. Therefore $< 0^{\#0}, \epsilon, \pi_1 > \in \delta_n^{ND,\chi}$, $< \pi_1, \epsilon, \pi_2 > \in \delta_n^{ND,\chi}$, ..., $< \pi_{s-1}, \epsilon, \pi_s > \in \delta_n^{ND,\chi}$ and $< \pi_s, x_1, \pi_{j_1} > \in \delta_n^{ND,\chi}$. It follows from *Proposition* 14 that $\exists \pi \in M_1 (\pi \leq_s^\chi \pi_{j_1})$.

2.2) Induction hypothesis: $\exists \pi \in M_i (\pi \leq_s^\chi \pi_{j_i})$. We have to prove that $\exists \pi' \in M_{i+1} (\pi' \leq_s^\chi \pi_{j_{i+1}})$. Let $\eta_1 \in M_i$ and $\eta_1 \leq_s^\chi \pi_{j_i}$. Let $\eta_2 = \pi_{j_i}$. Obviously we can find such $s \in N$ that $< \eta_2, \epsilon, \pi_{j_i+1} >$, $< \pi_{j_i+1}, \epsilon, \pi_{j_i+2} >$, ..., $< \pi_{j_i+s-1}, \epsilon, \pi_{j_i+s} > < \pi_{j_i+s}, x_{i+1}, \pi_{j_{i+1}} > \in \delta_n^{ND,\chi}$. Let $\pi'' \in \delta_e^{D,\chi}(\eta_1, x_{i+1})$ be such that $\pi'' \leq_s^\chi \pi_{j_{i+1}}$ (it follows from *Proposition* 14 that such π'' exists).

$$\pi'' \in \bigcup_{q \in M_i} \delta_e^{D,\chi}(q, x_{i+1})$$

$$M_{i+1} = \bigsqcup_{q \in M_i} \delta_e^{D,\chi}(q, x_{i+1})$$

Therefore $\exists \pi' \in M_{i+1} (\pi' \leq_s^\chi \pi'')$. Therefore $\exists \pi' \in M_{i+1} (\pi' \leq_s^\chi \pi_{j_{i+1}})$.

We proved that $\exists \pi \in M_i (\pi \leq_s^\chi \pi_{j_i})$ if $1 \leq i \leq k$. So $\exists \pi \in M_k (\pi \leq_s^\chi \pi_{j_k})$. $\pi_{j_k} = \pi_r \in F_n^{ND,\chi}$. Hence $\exists \pi \in M_k (\pi \in F_n^{ND,\chi})$. Therefore $M_k \in F_n^{D,\chi}$ and $x_1 x_2 ... x_k \in L(A_n^{D,\chi}(w))$.

Proposition 16 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $n \in N$, $A_n^{ND,\chi}(w) = < \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >$ and $\pi \in Q_n^{ND,\chi}$, $x \in \Sigma$ and $q \in \delta_e^{D,\chi}(\pi, x)$. Then $\exists s \in N \exists \eta_0 \eta_1 ... \eta_s \in Q_n^{ND,\chi} (\eta_0 = \pi \ \& \ < \eta_0, \epsilon, \eta_1 > \in \delta_n^{ND,\chi} \ \& \ < \eta_1, \epsilon, \eta_2 > \in \delta_n^{ND,\chi} \ \& \ ... \ \& \ < \eta_{s-1}, \epsilon, \eta_s > \in \delta_n^{ND,\chi} \ \& \ < \eta_s, x, q > \in \delta_n^{ND,\chi})$.

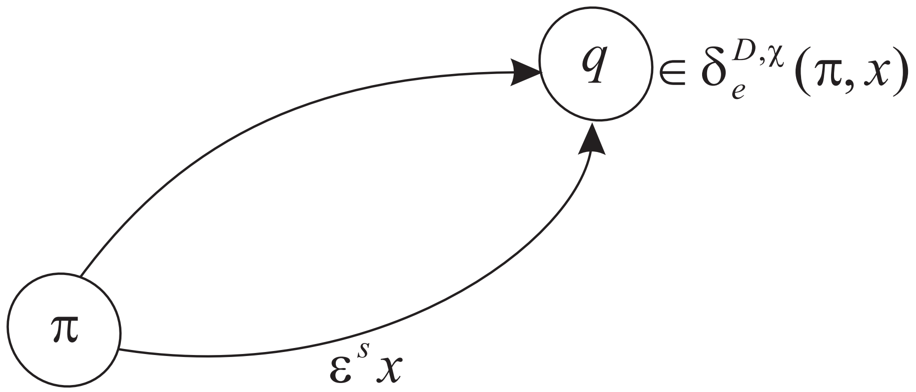

Fig. 7

Proof *Proposition* 16 follows directly from the definitions of $\delta_e^{D,\chi}$ and $\delta_n^{ND,\chi}$.

Proposition 17 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $n \in N$, $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >$ and $A_n^{D,\chi}(w) =< \Sigma, Q_n^{D,\chi}, I^{D,\chi}, F_n^{D,\chi}, \delta_n^{D,\chi} >$. Then $L(A_n^{ND,\chi}(w)) \supseteq L(A_n^{D,\chi}(w))$.

Proof Let $x_1x_2...x_k \in L(A_n^{D,\chi}(w))$. Let $M_0 = \{0^{\#0}\}$, $M_{i+1} = \delta_n^{D,\chi}(M_i, x_{i+1})$ for $0 \leq i \leq k-1$ and $M_k \in F_n^{D,\chi}$. We prove with induction on i that $\forall \pi \in M_i(< 0^{\#0}, x_1x_2...x_i, \pi > \in \delta_n^{ND,\chi^*})$ if $0 \leq i \leq k$.

1) $i = 0$

$< 0^{\#0}, \epsilon, 0^{\#0} > \in \delta_n^{ND,\chi}$

2) Induction hypothesis: $\forall \pi \in M_i(< 0^{\#0}, x_1x_2...x_i, \pi > \in \delta_n^{ND,\chi^*})$

We have to prove that $\forall \pi' \in M_{i+1}(< 0^{\#0}, x_1x_2...x_{i+1}, \pi' > \in \delta_n^{ND,\chi^*})$.

$$M_{i+1} = \bigsqcup_{q \in M_i} \delta_e^{D,\chi}(q, x_{i+1}) \subseteq \bigcup_{q \in M_i} \delta_e^{D,\chi}(q, x_{i+1})$$

Let $\pi' \in M_{i+1}$. Therefore $\exists q \in M_i(\pi' \in \delta_e^{D,\chi}(q, x_{i+1}))$. Let q be such that $q \in M_i$ and $\pi' \in \delta_e^{D,\chi}(q, x_{i+1})$. Therefore $< 0^{\#0}, x_1x_2...x_i, q > \in \delta_n^{ND,\chi^*}$. It follows from *Proposition* 16 that $< q, x_{i+1}, \pi' > \in \delta_n^{ND,\chi^*}$. Therefore $< 0^{\#0}, x_1x_2...x_{i+1}, \pi' > \in \delta_n^{ND,\chi^*}$.

We proved that $\forall \pi \in M_i(< 0^{\#0}, x_1x_2...x_i, \pi > \in \delta_n^{ND,\chi^*})$ if $0 \leq i \leq k$. Let π be such that $\pi \in M_k \bigcap F_n^{ND,\chi}$ (since $M_k \in F_n^{D,\chi}$ such π exists). Therefore $< 0^{\#0}, x_1x_2...x_k, \pi > \in \delta_n^{ND,\chi^*}$ and $x_1x_2...x_k \in L(A_n^{D,\chi}(w))$.

Corollary Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $n \in N$, $A_n^{ND,\chi}(w) =< \Sigma, Q_n^{ND,\chi}, I^{ND,\chi}, F_n^{ND,\chi}, \delta_n^{ND,\chi} >$ and $A_n^{D,\chi}(w) =< \Sigma, Q_n^{D,\chi}, I^{D,\chi}, F_n^{D,\chi}, \delta_n^{D,\chi} >$. It follows from *Proposition* 17 and *Proposition* 15 that $L_{Lev}^{\chi}(n, w) = L(A_n^{ND,\chi}(w)) = L(A_n^{D,\chi}(w))$.

In [SMFSCLA] we can also find proof that $L^{\chi}_{Lev}(n,w) = L(A^{D,\chi}_n(w))$.

Proposition 18 Let $\chi \in \{\epsilon, t, ms\}$. Let $n \in N$ and $b \in \{0,1\}^*$. Then
1) $\delta^{D,\chi}_e(i + t^{\#e}, b) = \{j + t^{\#f}_{(t|s)} | j^{\#f}_{(t|s)} \in \delta^{D,\chi}_e(i^{\#e}, b)\}$
2) $\delta^{D,\chi}_e(i + t^{\#e}_t, b) = \{j + t^{\#f}_{(t)} | j^{\#f}_{(t)} \in \delta^{D,\chi}_e(i^{\#e}_t, b)\}$
3) $\delta^{D,\chi}_e(i + t^{\#e}_s, b) = \{j + t^{\#f}_{(s)} | j^{\#f}_{(s)} \in \delta^{D,\chi}_e(i^{\#e}_s, b)\}$

Proof *Proposition* 18 follows directly from the definition of $\delta^{D,\chi}_e$.

5 Universal Levenshtein automata.

We show that for each $n \in N$ we can build finite deterministic automaton $A^{\forall,\chi}_n$ in such way that:

1) when $\chi = \epsilon$ every nonfinal state of $A^{\forall,\chi}_n$ is finite set that consists of elements of the type $I + i^{\#e}$ and every final state is finite set that consists of elements of the type $M + j^{\#f}$. (When $\chi = t$ there are in the states also elements of the type $I_t + i^{\#e}$ and $M_t + j^{\#f}$, when $\chi = ms$ - of the type $I_s + i^{\#e}$ and $M_s + j^{\#f}$);

2) each symbol from the input alphabet for $A^{\forall,\chi}_n$ is binary vector, i.e. word from the language $\{0,1\}^*$;

3) for every two words $v_1 v_2 ... v_k$ and w from Σ^* we can build $b = b_1 b_2 ... b_k$ such that $b_i \in \{0,1\}^*$ and $b \in L(A^{\forall,\chi}_n) \Leftrightarrow v \in L(A^{D,\chi}_n(w))$, i.e. $b \in L(A^{\forall,\chi}_n) \Leftrightarrow v \in L^{\chi}_{Lev}(n,w)$ (v_i is called *symbol corresponding to the word* b_i). Let $q^{\forall}_0$, $q^{\forall}_1$, ..., $q^{\forall}_k$ be the states of $A^{\forall,\chi}_n$ that we visit traversing $A^{\forall,\chi}_n$ with b and let q^D_0, q^D_1, ..., q^D_k be the states of $A^{D,\chi}_n(w)$ that we visit traversing $A^{D,\chi}_n(w)$ with v. We build $A^{\forall,\chi}_n$ in such way that we receive q^D_j when we replace the parameters I in $q^{\forall}_j$ with j (if $q^{\forall}_j$ is nonfinal) or the parameters M in $q^{\forall}_j$ with k (if $q^{\forall}_j$ is final). And also: $q^{\forall}_j$ is final only if q^D_j is final.

Notations We use expressions of the type $F(I)^{\#e}$, $F(I_t)^{\#e}$, $F(I_s)^{\#e}$, $F(M)^{\#e}$, $F(M_t)^{\#e}$ and $F(M_s)^{\#e}$ to denote correspondingly tuples of the type $<< \lambda I.F(I), 0 >, e >$, $<< \lambda I.F(I), 1 >, e >$, $<< \lambda I.F(I), 2 >, e >$, $<< \lambda M.F(M), 3 >, e >$, $<< \lambda M.F(M), 4 >, e >$ and $<< \lambda M.F(M), 5 >, e >$.

Definition 15 Let $\chi \in \{\epsilon, t, ms\}$. Let $n \in N$. We define the finite automaton $A^{\forall,\chi}_n$.

$$A^{\forall,\chi}_n \stackrel{def}{=} < \Sigma^{\forall}_n, Q^{\forall,\chi}_n, I^{\forall,\chi}, F^{\forall,\chi}_n, \delta^{\forall,\chi}_n >$$

$\Sigma^{\forall}_n \stackrel{def}{=} \{x | x \in \{0,1\}^+ \;\&\; |x| \leq 2n+2\}$

We define I^{χ}_s:
1) $\chi = \epsilon$
$I^{\epsilon}_s \stackrel{def}{=} \{I + t^{\#k} \mid |t| \leq k \;\&\; -n \leq t \leq n \;\&\; 0 \leq k \leq n\}$

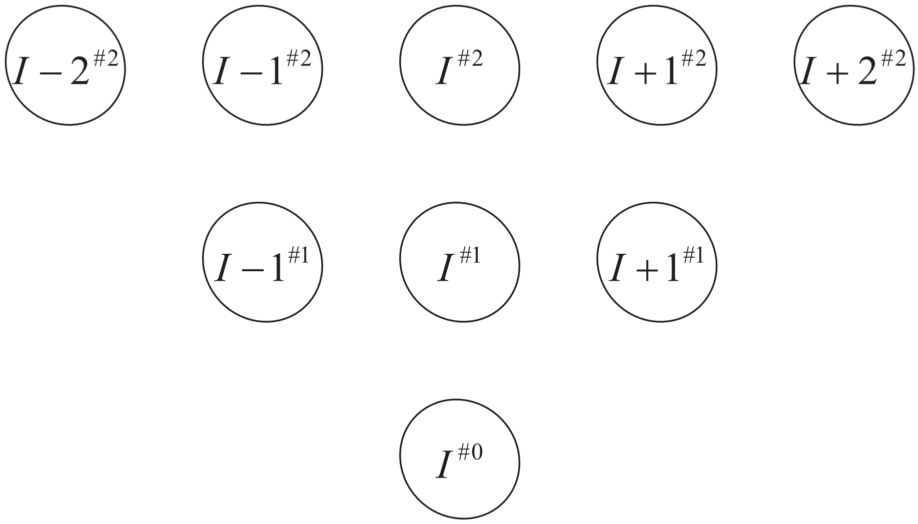

Fig. 8 $I_s^\epsilon, n=2$

2) $\chi = t$

$I_s^t \stackrel{def}{=} I_s^\epsilon \bigcup \{I_t + t^{\#k} \mid |t+1|+1 \leq k \,\&\, -n \leq t \leq n-2 \,\&\, 1 \leq k \leq n\}$

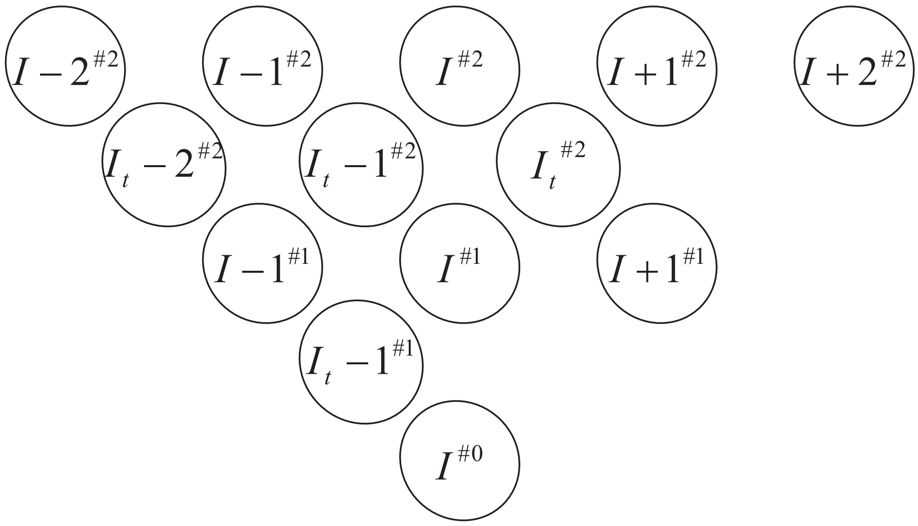

Fig. 9 $I_s^t, n=2$

3) $\chi = ms$

$I_s^{ms} \stackrel{def}{=} I_s^\epsilon \bigcup \{I_s + t^{\#k} \mid |t+1|+1 \leq k \,\&\, -n \leq t \leq n-2 \,\&\, 1 \leq k \leq n\}$

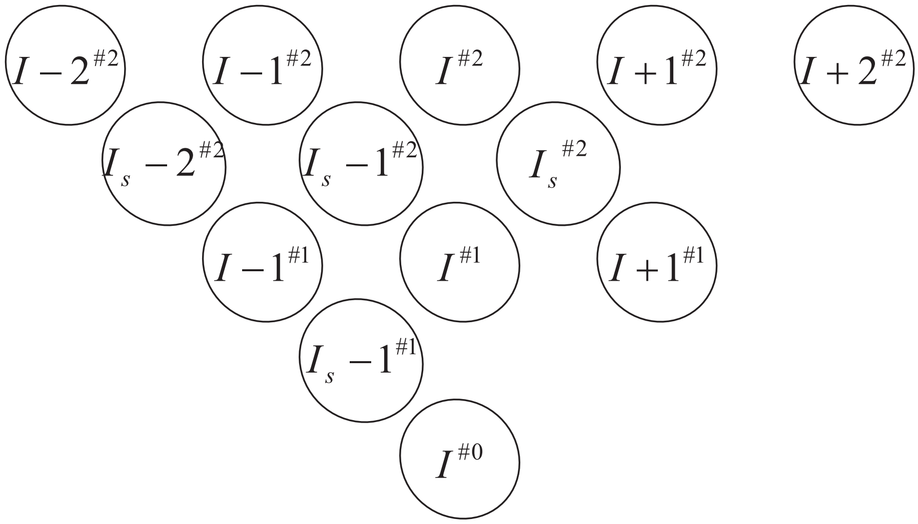

Fig. 10 $I_s^{ms}, n=2$

We define M_s^{χ}:
1) $\chi = \epsilon$
$M_s^{\epsilon} \overset{def}{=} \{M + t^{\#k} \mid k \geq -t-n \;\&\; -2n \leq t \leq 0 \;\&\; 0 \leq k \leq n\}$

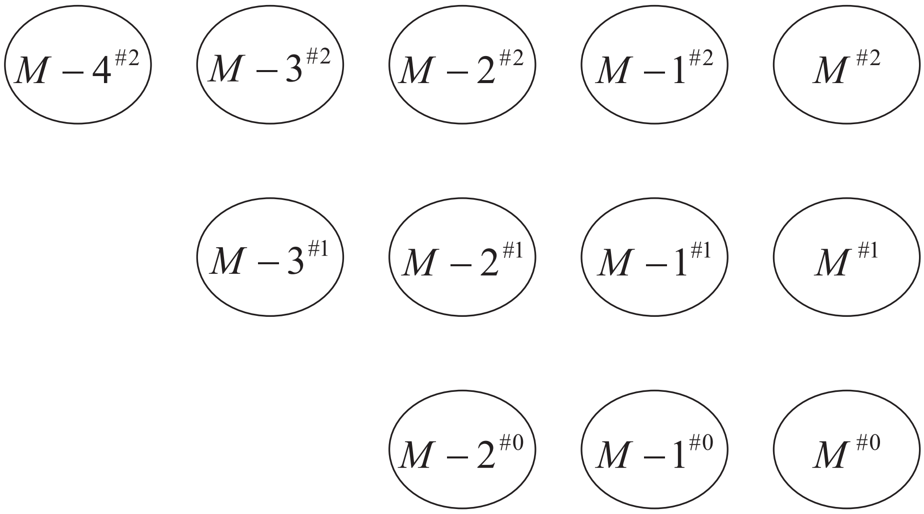

Fig. 11 $M_s^{\epsilon}, n=2$

2) $\chi = t$

$M_s^t \stackrel{def}{=} M_s^\epsilon \bigcup \{M_t + t^{\#k} \mid k \geq -t-n \;\&\; -2n \leq t \leq -2 \;\&\; 1 \leq k \leq n\}$

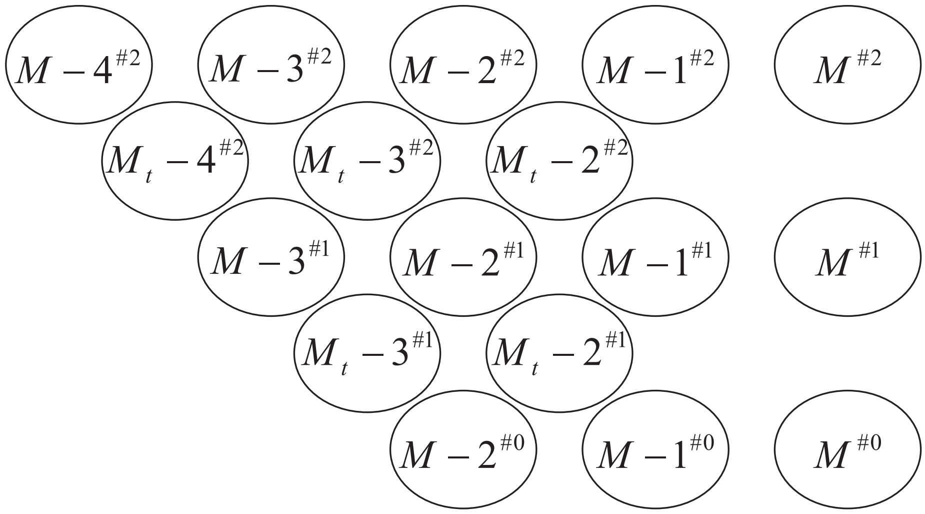

Fig. 12 $M_s^t, n = 2$

3) $\chi = ms$

$M_s^{ms} \stackrel{def}{=} M_s^\epsilon \bigcup \{M_s + t^{\#k} \mid k \geq -t-n \;\&\; -2n \leq t \leq -1 \;\&\; 1 \leq k \leq n\}$

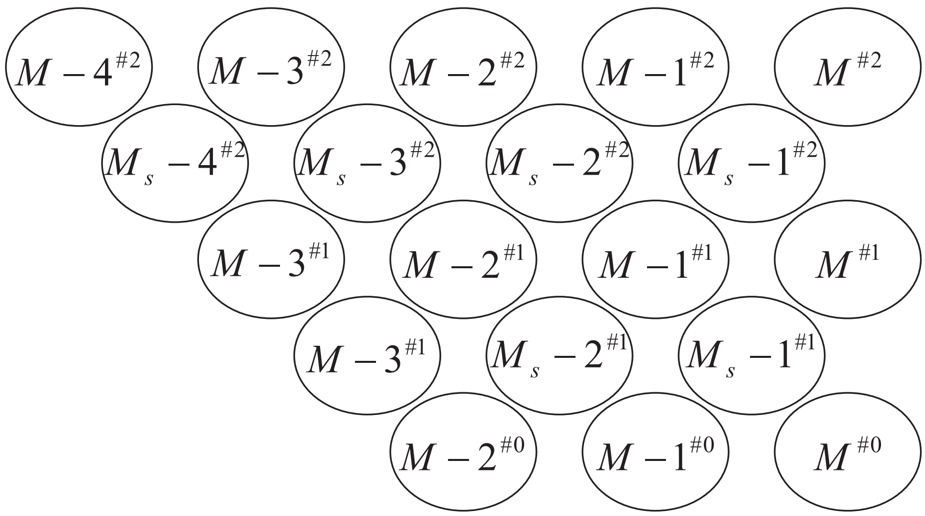

Fig. 13 $M_s^{ms}, n = 2$

We define $<_s^\chi \subseteq (I_s^\chi \bigcup M_s^\chi) \times (I_s^\chi \bigcup M_s^\chi)$:

1) $\chi = \epsilon$

$I + i^{\#e} <^{\epsilon}_{s} I + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <^{\epsilon}_{s} j^{\#f}$

$M + i^{\#e} <^{\epsilon}_{s} M + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <^{\epsilon}_{s} j^{\#f}$

2) $\chi = t$

$I + i^{\#e} <^{t}_{s} I + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <^{t}_{s} j^{\#f}$

$I + i^{\#e} <^{t}_{s} I_t + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <^{t}_{s} j^{\#f}_{t}$

$M + i^{\#e} <^{t}_{s} M + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <^{t}_{s} j^{\#f}$

$M + i^{\#e} <^{t}_{s} M_t + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <^{t}_{s} j^{\#f}_{t}$

3) $\chi = ms$

$I + i^{\#e} <^{ms}_{s} I + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <^{ms}_{s} j^{\#f}$

$I + i^{\#e} <^{ms}_{s} I_s + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <^{ms}_{s} j^{\#f}_{s}$

$M + i^{\#e} <^{ms}_{s} M + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <^{ms}_{s} j^{\#f}$

$M + i^{\#e} <^{ms}_{s} M_s + j^{\#f} \overset{def}{\Leftrightarrow} i^{\#e} <^{ms}_{s} j^{\#f}_{s}$

We are ready to define I^{χ}_{states} and M^{χ}_{states}.

$I^{\chi}_{states} \overset{def}{=} \{Q | Q \subseteq I^{\chi}_{s} \,\&\, \forall q_1, q_2 \in Q(q_1 \not<^{\chi}_{s} q_2)\} \backslash \{\phi\}$

$M^{\chi}_{states} \overset{def}{=} \{Q | Q \subseteq M^{\chi}_{s} \,\&\, \forall q_1, q_2 \in Q(q_1 \not<^{\chi}_{s} q_2) \,\&\, \exists q \in Q(q \leq^{\chi}_{s} M^{\#n}) \,\&\, \exists i \in [-n, 0] \forall q \in Q(M + i^{\#0} \leq^{\chi}_{s} q)\}$

$Q^{\forall,\chi}_{n} \overset{def}{=} I^{\chi}_{states} \bigcup M^{\chi}_{states}$

$I^{\forall,\chi} \overset{def}{=} \{I^{\#0}\}$

$F^{\forall,\chi}_{n} \overset{def}{=} M^{\chi}_{states}$

We define $r_n : (I^{\chi}_{s} \bigcup M^{\chi}_{s}) \times \Sigma^{\forall}_{n} \longrightarrow \{0,1\}^*$. $r_n(S, x)$ represents the characteristic vetor determined by the state of $A^{ND,\chi}_{n}$ corresponding to S and the symbol corresponding to x.

1) $S = I + i^{\#e}$ or $S = I_t + i^{\#e}$ or $S = I_s + i^{\#e}$

$$r_n(S, x_1 x_2 ... x_k) \overset{def}{=} \begin{cases} x_{n+i+1} x_{n+i+2} ... x_{n+i+h} & \text{if } h > 0 \\ \epsilon & \text{if } h = 0 \\ \neg ! & \text{if } h < 0 \end{cases}$$

where $h = min(n - e + 1, k - n - i)$

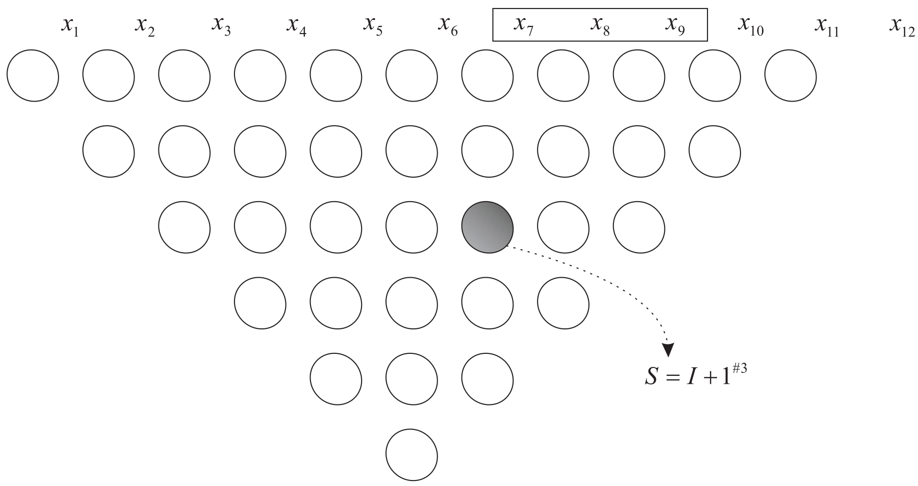

Fig. 14 $r_5(I + 1^{\#3}, x_1x_2...x_{12}) = x_7x_8x_9$

2) $S = M + i^{\#e}$ or $S = M_t + i^{\#e}$ or $S = M_s + i^{\#e}$

$$r_n(S, x_1x_2...x_k) \stackrel{def}{=} \begin{cases} x_{k+i+1}x_{k+i+2}...x_{k+i+h} & \text{if } h > 0 \\ \epsilon & \text{if } h = 0 \\ \neg ! & \text{if } h < 0 \end{cases}$$

where $h = min(n - e + 1, -i)$

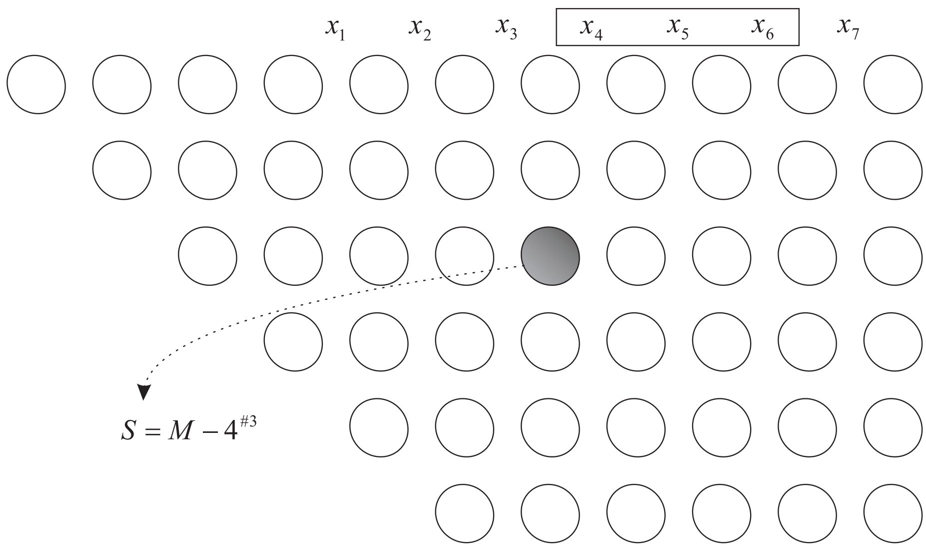

Fig. 15 $r_5(M - 4^{\#3}, x_1x_2...x_7) = x_4x_5x_6$

$P^\epsilon \stackrel{def}{=} \{I + i^{\#e} | i, e \in Z\} \bigcup \{M + i^{\#e} | i, e \in Z\}$
$P^t \stackrel{def}{=} P^\epsilon \bigcup \{I_t + i^{\#e} | i, e \in Z\} \bigcup \{M_t + i^{\#e} | i, e \in Z\}$
$P^{ms} \stackrel{def}{=} P^\epsilon \bigcup \{I_s + i^{\#e} | i, e \in Z\} \bigcup \{M_s + i^{\#e} | i, e \in Z\}$

We define $m_n : P^\chi \times N \to P^\chi$. When from some nonfinal state with word $b_1 b_2 ... b_k \in \Sigma_n^\forall$ we have to reach final state, we use the function m_n to convert the elements of the type $I + i^{\#e}$ into elements of the type $M + i^{\#e}$. And also, when from some final state we have to reach nonfinal state, we convert with the function m_n the elements of the type $M + i^{\#e}$ into elements of the type $I + i^{\#e}$.

1) $\chi = \epsilon$

$$m_n(S, k) \stackrel{def}{=} \begin{cases} M + i + n + 1 - k^{\#e} & \text{if } S = I + i^{\#e} \\ I + i - n - 1 + k^{\#e} & \text{if } S = M + i^{\#e} \end{cases}$$

2) $\chi = t$

$$m_n(S, k) \stackrel{def}{=} \begin{cases} M + i + n + 1 - k^{\#e} & \text{if } S = I + i^{\#e} \\ I + i - n - 1 + k^{\#e} & \text{if } S = M + i^{\#e} \\ M_t + i + n + 1 - k^{\#e} & \text{if } S = I_t + i^{\#e} \\ I_t + i - n - 1 + k^{\#e} & \text{if } S = M_t + i^{\#e} \end{cases}$$

3) $\chi = ms$

$$m_n(S, k) \stackrel{def}{=} \begin{cases} M + i + n + 1 - k^{\#e} & \text{if } S = I + i^{\#e} \\ I + i - n - 1 + k^{\#e} & \text{if } S = M + i^{\#e} \\ M_s + i + n + 1 - k^{\#e} & \text{if } S = I_s + i^{\#e} \\ I_s + i - n - 1 + k^{\#e} & \text{if } S = M_s + i^{\#e} \end{cases}$$

We define $m_n : P(P^\chi) \times N \to P(P^\chi)$:
$m_n(A, x) \stackrel{def}{=} \{m_n(a, x) | a \in A\}$

We define also $f_n : (I_s^\chi \bigcup M_s^\chi) \times N \to \{true, false\}$. Let A be some nonfinal state. We want to find the state A' reached from A with the symbol $b = b_1 b_2 ... b_k$. First using A and b we find new set B that consists of elements of the type $I + i^{\#e}$. B is a candidate to be A'. But if $f_n(S, k) = true$ for some element S from B, B is not good candidate and $A' = m_n(B, k)$. If for each element S from B we have that $f_n(S, k) = false$ B is good candidate and $A' = B$. In $A_n^{ND,\chi}(w)$ a state $i^{\#e}$ is final if $e <= i - (|w| - n)$ i.e. if the state is on the right of the diagonal $y = x - (|w| - n)$. For nonfinal state in $A_n^{\forall,\chi}(w)$ the diagonal corresponding to $y = x - (|w| - n)$ is $I + k - 2n + t^{\#t}$ $(0 \le t \le n)$, if $k < 2n + 2$ (k is the length of the input symbol b). In $A_n^{D,\chi}(w)$ there is no nonfinite state that has an element on the right of the diagonal $y = x - (|w| - n)$. Analogously in $A_n^{\forall,\chi}$ the transition function $\delta_n^{\forall,\chi}$ is not defined for nonfinal state and symbol $b_1 b_2 ... b_k$ if the nonfinal state has an element of the type $I + i^{\#e}$ on the right of the diagonal $I + k - 2n + t^{\#t}$. (The candidate B has such an element S only if $f_n(S, k) = true$.) The only one exception is $\{I^{\#0}\}$ - the initial state of $A_n^{\forall,\chi}$. This state is nonfinal but in $A_n^{D,\chi}(w)$ its corresponding state $\{0^{\#0}\}$ may be final. For each final state of $A_n^{\forall,\chi}$ the diagonal corresponding to $y = x - (|w| - n)$ consists of the elements $M - n + t^{\#t}$ $(0 \leq t \leq n)$ and the transition function $\delta_n^{\forall,\chi}$ is not defined for a final state whose elements of the type $M + i^{\#e}$ are on the right of the diagonal $M - n + t^{\#t}$. (S is on the right of the diagonal $M - n + t^{\#t}$ only if $f_n(S,k) = true$.)

1) $S = I + i^{\#e}$ or $S = I_t + i^{\#e}$ or $S = I_s + i^{\#e}$

$$f_n(S,k) \stackrel{def}{=} \begin{cases} true & \text{if } k \leq 2n+1 \;\&\; e \leq i + 2n + 1 - k \\ false & \text{otherwise} \end{cases}$$

2) $S = M + i^{\#e}$ or $S = M_t + i^{\#e}$ or $S = M_s + i^{\#e}$

$$f_n(S,k) \stackrel{def}{=} \begin{cases} true & \text{if } e > i + n \\ false & \text{otherwise} \end{cases}$$

We define $I^\chi : P(Q^{ND,\chi}) \to P(P^\chi)$:

1) $\chi = \epsilon$

$I^\epsilon(A) \stackrel{def}{=} \{I + i - 1^{\#e} | i^{\#e} \in A\}$

2) $\chi = t$

$I^t(A) \stackrel{def}{=} \{I + i - 1^{\#e} | i^{\#e} \in A\} \bigcup \{I_t + i - 1^{\#e} | i_t^{\#e} \in A\}$

3) $\chi = ms$

$I^{ms}(A) \stackrel{def}{=} \{I + i - 1^{\#e} | i^{\#e} \in A\} \bigcup \{I_s + i - 1^{\#e} | i_s^{\#e} \in A\}$

We define $M^\chi : P(Q^{ND,\chi}) \to P(P^\chi)$:

1) $\chi = \epsilon$

$M^\epsilon(A) \stackrel{def}{=} \{M + i^{\#e} | i^{\#e} \in A\}$

2) $\chi = t$

$M^t(A) \stackrel{def}{=} \{M + i^{\#e} | i^{\#e} \in A\} \bigcup \{M_t + i^{\#e} | i_t^{\#e} \in A\}$

3) $\chi = ms$

$M^{ms}(A) \stackrel{def}{=} \{M + i^{\#e} | i^{\#e} \in A\} \bigcup \{M_s + i^{\#e} | i_s^{\#e} \in A\}$

We define $rm : I^\chi_{states} \bigcup M^\chi_{states} \to I^\epsilon_s \bigcup M^\epsilon_s$:

$$rm(A) \stackrel{def}{=} \begin{cases} I + i^{\#e} & \text{if } A \in I^\chi_{states} \;\&\; e - i = \mu z[z = e' - i' \;\&\; I + i'^{\#e'} \in A] \\ M + i^{\#e} & \text{if } A \in M^\chi_{states} \;\&\; e - i = \mu z[z = e' - i' \;\&\; M + i'^{\#e'} \in A] \end{cases}$$

The element $rm(A)$ is called *right-most element of* A. To know whether we have to convert with the function m_n the candidate B that we receive from some state and input character $b_1 b_2 ... b_k$, it is suffice to check the value $f_n(rm(B), k)$ because for each nonfinal state A and each k it is true that $f_n(rm(A), k) = false \Leftrightarrow \forall S \in A(S = I + i^{\#e}$ for some i and $e \Rightarrow f_n(S,k) = false)$ and for each final state C it is true that $f_n(rm(C), k) = true \Leftrightarrow \forall S \in C(S = M + i^{\#e}$ for some i and $e \Rightarrow f_n(S,k) = true)$, i.e. it's no matter whether the type of the elemets of the candidate B is $I + i^{\#e}$ or it is $M + i^{\#e}$ - if $f_n(rm(B), k) = true$

then B is not good candidate and if $f_n(rm(B),k) = false$ then B is good.

We define *the function of the elementary transitions*
$\delta_e^{\forall,\chi} : (I_s^\chi \bigcup M_s^\chi) \times \Sigma_n^\forall \multimap I_{states}^\chi \bigcup M_{states}^\chi \bigcup \{\phi\}$:

1) Let $S = I + i^{\#e}$ or $S = I_t + i^{\#e}$ or $S = I_s + i^{\#e}$. We define $\delta_e^{\forall,\chi}(S,x)$:

1.1) $\neg ! r_n(S,x)$

$\neg ! \delta_e^{\forall,\chi}(S,x)$

1.2) $! r_n(S,x)$

$$\delta_e^{\forall,\chi}(S,x) \stackrel{def}{=} \begin{cases} I^\chi(\delta_e^{D,\chi}(i^{\#e}, r_n(S,x))) & \text{if } S = I + i^{\#e} \\ I^\chi(\delta_e^{D,\chi}(i_t^{\#e}, r_n(S,x))) & \text{if } S = I_t + i^{\#e} \\ I^\chi(\delta_e^{D,\chi}(i_s^{\#e}, r_n(S,x))) & \text{if } S = I_s + i^{\#e} \end{cases}$$

2) Let $S = M + i^{\#e}$ or $S = M_t + i^{\#e}$ or $S = M_s + i^{\#e}$. We define $\delta_e^{\forall,\chi}(S,x)$:

2.1) $\neg ! r_n(S,x)$

$\neg ! \delta_e^{\forall,\chi}(S,x)$

2.2) $! r_n(S,x)$

$$\delta_e^{\forall,\chi}(S,x) \stackrel{def}{=} \begin{cases} M^\chi(\delta_e^{D,\chi}(i^{\#e}, r_n(S,x))) & \text{if } S = M + i^{\#e} \\ M^\chi(\delta_e^{D,\chi}(i_t^{\#e}, r_n(S,x))) & \text{if } S = M_t + i^{\#e} \\ M^\chi(\delta_e^{D,\chi}(i_s^{\#e}, r_n(S,x))) & \text{if } S = M_s + i^{\#e} \end{cases}$$

We define $\bigsqcup : P(P(I_s^\chi)) \bigcup P(P(M_s^\chi)) \to P(I_s^\chi) \bigcup P(M_s^\chi)$:

$\bigsqcup A \stackrel{def}{=} \{\pi | \pi \in \bigcup A \;\&\; \neg \exists \pi' \in \bigcup A (\pi' <_s^\chi \pi)\}$

We difine $\bigtriangledown_a : I_{states}^\chi \bigcup M_{states}^\chi \to P(N)$. This function is used for checking whether the length of the word $b_1 b_2 ... b_k$ is suitable to define the transition function $\delta_n^{\forall,\chi}$. If $k \notin \bigtriangledown_a(Q)$ then $\neg ! \delta_n^{\forall,\chi}(Q, b_1 b_2 ... b_k)$.

1) Let $Q \in I_{states}^\chi$

1.1) $Q = \{I^{\#0}\}$

$\bigtriangledown_a(Q) \stackrel{def}{=} \{k | n \le k \le 2n+2\}$

1.2) $Q \ne \{I^{\#0}\}$

Let $rm(Q) = I + i^{\#e}$

$\bigtriangledown_a(Q) \stackrel{def}{=} \{k | 2n + i - e + 1 \le k \le 2n+2\}$

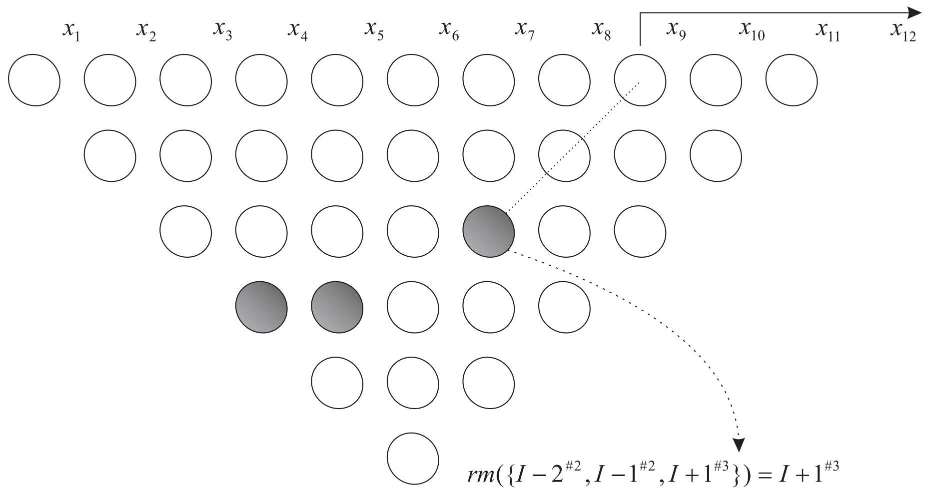

Fig. 16 $n = 5$, $\triangledown_a(\{I-2^{\#2}, I-1^{\#2}, I+1^{\#3}\}) = \{9, 10, 11, 12\}$
The length k of the input charcter $x_1x_2...x_k$ must be such that all elements of the type $I + i^{\#e}$ to be on the left of the diagonal $I + k - 2n + t^{\#t}$.

2) Let $Q \in M^{\chi}_{states}$
$\triangledown_a(Q) \stackrel{def}{=} \{k \in N | \forall \pi \in Q(if(k < n, M^{\#n-k}, M + n - k^{\#0}) \leq^{\chi}_{s} \pi)\} \backslash \{0\}$

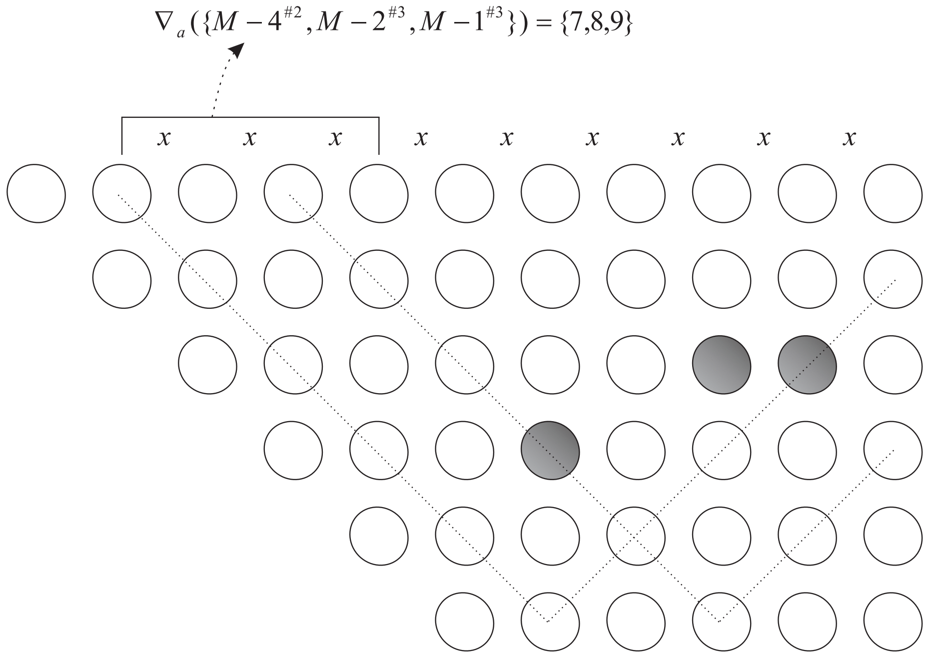

Fig. 17 $n = 5$, $\bigtriangledown_a(\{M - 4^{\#2}, M - 2^{\#3}, M - 1^{\#3}\}) = \{7, 8, 9\}$

We are ready to define the transition function $\delta_n^{\forall,\chi} : Q_n^{\forall,\chi} \times \Sigma_n^{\forall} \not\rightarrow Q_n^{\forall,\chi}$.
Let $Q \in Q_n^{\forall,\chi}$ and $x \in \Sigma_n^{\forall}$.
1) $|x| \notin \bigtriangledown_a(Q)$
$\neg!\delta_n^{\forall,\chi}(Q, x)$
2) $|x| \in \bigtriangledown_a(Q)$
2.1) $\bigcup_{q \in Q} \delta_e^{\forall,\chi}(q, x) = \phi$
$\neg!\delta_n^{\forall,\chi}(Q, x)$
2.2) $\bigcup_{q \in Q} \delta_e^{\forall,\chi}(q, x) \neq \phi$
Let $\Delta = \bigsqcup_{q \in Q} \delta_e^{\forall,\chi}(q, x)$.

$$\delta_n^{\forall,\chi}(Q, x) \stackrel{def}{=} \begin{cases} \Delta & \text{if } f_n(rm(\Delta), |x|) = false \\ m_n(\Delta, |x|) & \text{if } f_n(rm(\Delta), |x|) = true \end{cases}$$

In what follows we suppose that $I^{\chi}_{states} = \{A | \exists x \in \Sigma_n^{\forall^*}(\delta_n^{\forall,\chi^*}(\{I^{\#0}\}, x) = A)\ \&\ A \subseteq I^{\chi}_s\}$ and $M^{\chi}_{states} = \{A | \exists x \in \Sigma_n^{\forall^*}(\delta_n^{\forall,\chi^*}(\{I^{\#0}\}, x) = A)\ \&\ A \subseteq M^{\chi}_s\}$, i.e. we cosider that $A_n^{\forall,\chi}$ has no state that cannot be reached from $\{I^{\#0}\}$.

Definition of $A_n^{\forall,\epsilon}$ is given in [MSFASLD].

The three automata $A_1^{\forall,\epsilon}$, $A_1^{\forall,t}$ and $A_1^{\forall,ms}$ are depicted resp. on fig. 18, fig. 19 and fig. 20. In these figures x can be interpreted as 0 or 1 and the expressions in brackets are optional. For instance from $\{I^{\#1}, I + 1^{\#1}\}$ with $x10(x)$ we can reach $\{I^{\#1}\}$. This means that from $\{I^{\#1}, I + 1^{\#1}\}$ we can reach $\{I^{\#1}\}$ with 010, 110, 0100, 0101, 1100 and 1101.

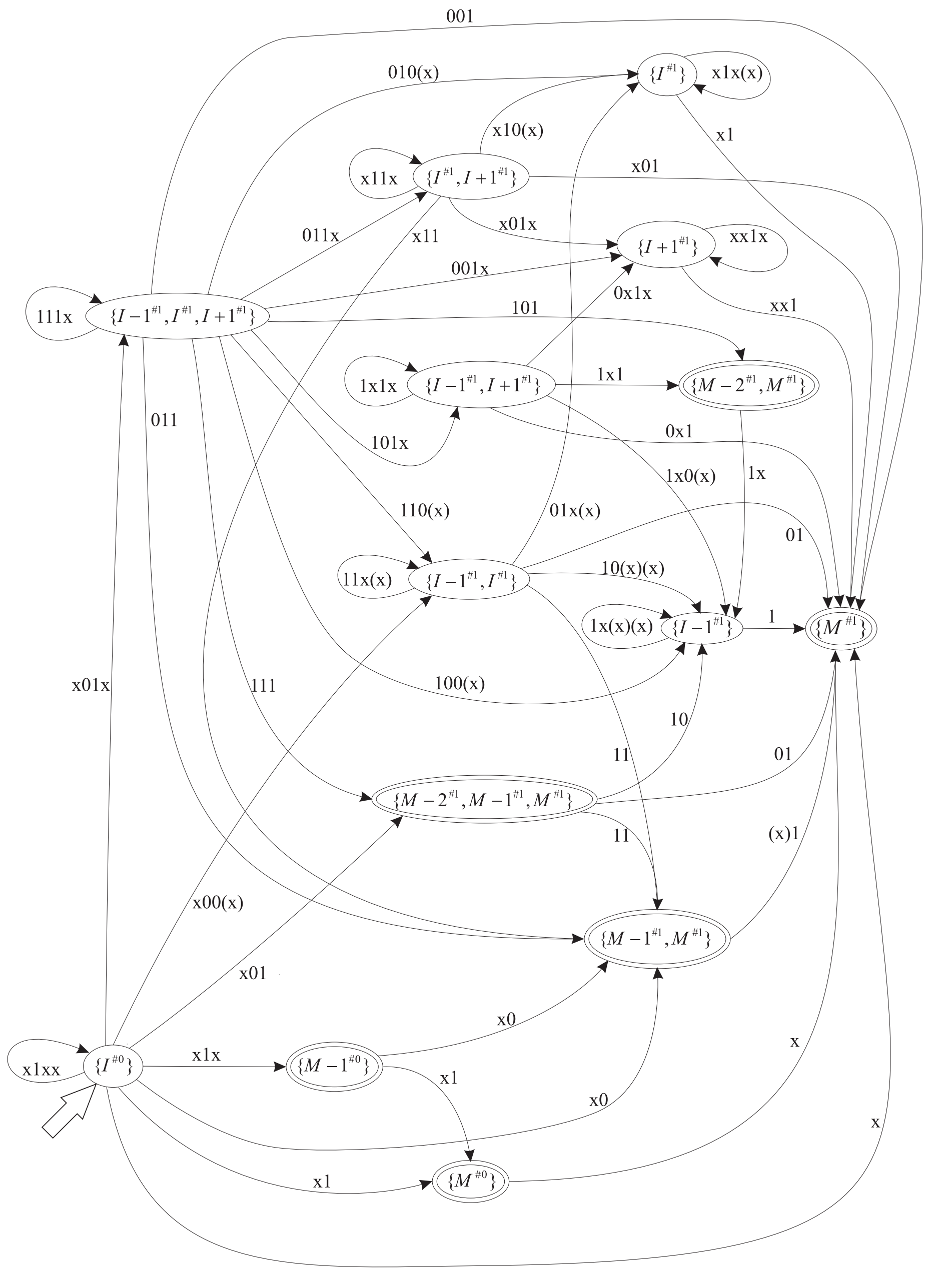

Fig. 18 $A_1^{\forall,\epsilon}$

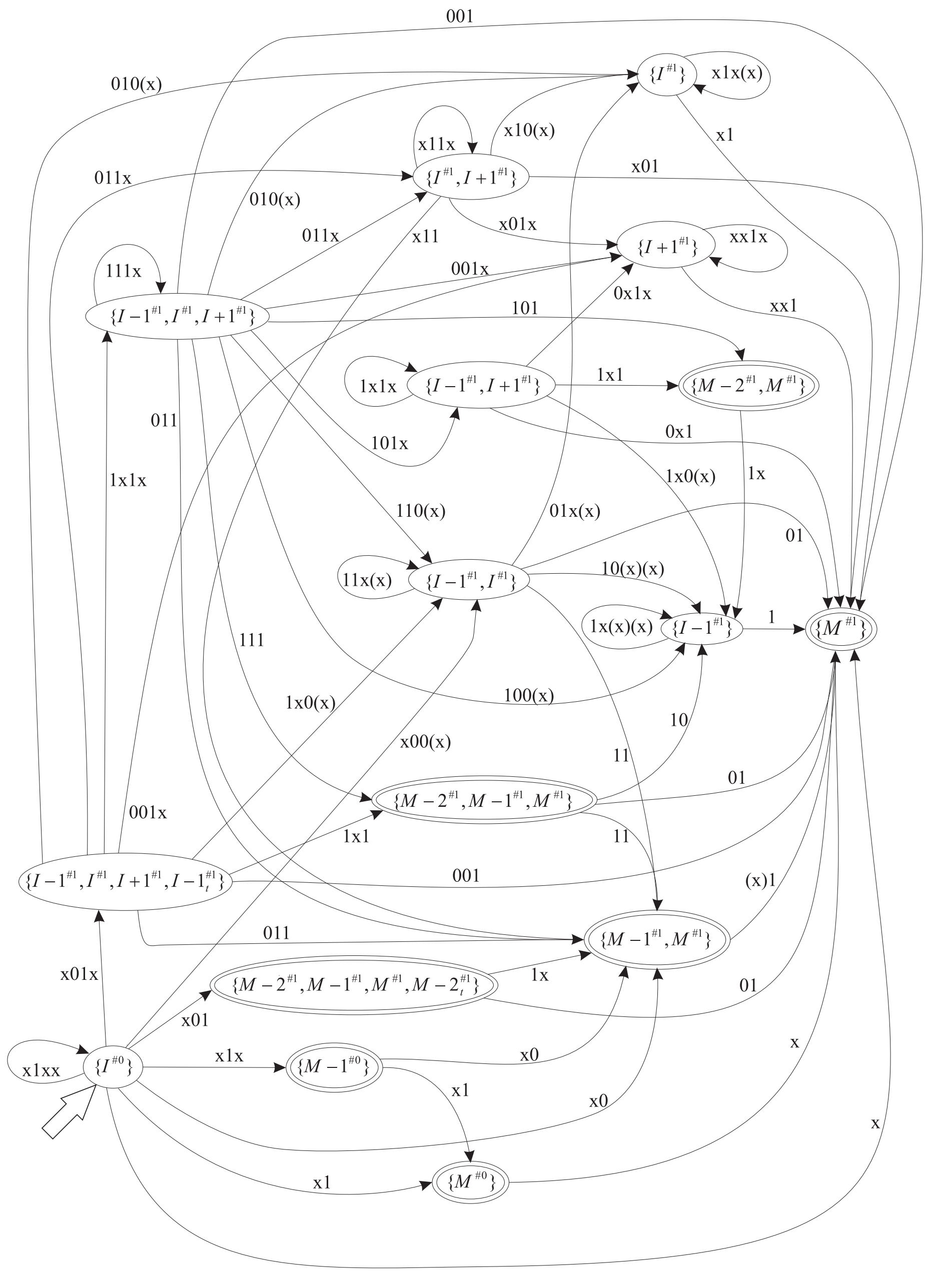

Fig. 19 $A_1^{\forall,t}$

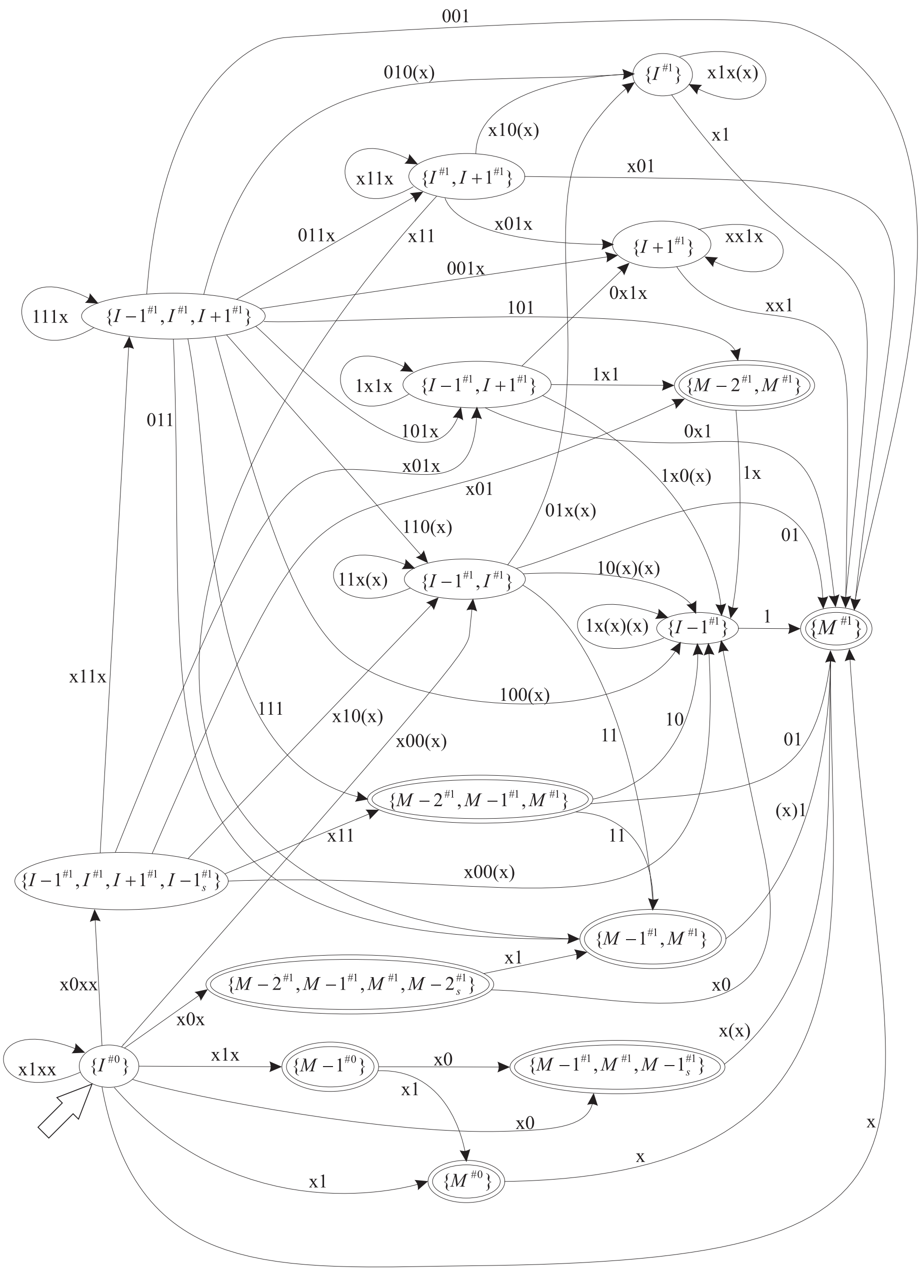

Fig. 20 $A_1^{\forall,ms}$

Definition 16 Let $n \in N$ and $\$ \notin \Sigma$.
$w_{-n+1} \stackrel{def}{=} w_{-n+2} \stackrel{def}{=} \ldots \stackrel{def}{=} w_0 \stackrel{def}{=} \$$

$s_n : \Sigma^* \times N^+ \not\longrightarrow (\Sigma \bigcup \{\$\})^*$

$$s_n(w,i) \stackrel{def}{=} \begin{cases} w_{i-n}w_{i-n+1}...w_v & \text{if } v \geq i-n \\ \neg! & \text{if } v < i-n \end{cases}$$

where $v = min(|w|, i+n+1)$.

$h_n : \Sigma^* \times \Sigma^+ \not\longrightarrow \Sigma_n^{\forall *}$

$$h_n(w, x_1x_2...x_t) \stackrel{def}{=} \begin{cases} \beta(x_1, s_n(w,1))\beta(x_2, s_n(w,2))...\beta(x_t, s_n(w,t)) & \text{if } t \leq |w|+n \\ \neg! & \text{if } t > |w|+n \end{cases}$$

We give an example how $A_n^{\forall,\chi}$ can be used. Let $w = abcabb$ and $x = dacab$. We want to know whether $x \in L_{Lev}^{\chi}(3, w)$. We find $b = h_3(w,x) = b_1b_2...b_5$. $b_1 = \beta(x_1, s_3(w,1)) = \beta(d, \$\$\$abcab) = 00000000$, $b_2 = \beta(x_2, s_3(w,2)) = \beta(a, \$\$abcabb) = 00100100$, $b_3 = \beta(x_3, s_3(w,3)) = \beta(c, \$abcabb) = 0001000$, $b_4 = \beta(x_4, s_3(w,4)) = \beta(a, abcabb) = 100100$ and $b_5 = \beta(x_5, s_3(w,5)) = \beta(b, bcabb) = 10011$. $x \in L_{Lev}^{\chi}(w,3) \Leftrightarrow b \in L(A_3^{\forall,\chi})$.

Proposition 19 Let $\chi \in \{\epsilon, t, ms\}$. Let $w \in \Sigma^*$, $x \in \Sigma^+$, $n \in N^+$, $!h_n(w,x)$, $b = h_n(w,x)$ and $|b| = |x| = t$. Let

$q_0^{\forall,\chi} = \{I^{\#0}\}$ and

$$q_{i+1}^{\forall,\chi} = \begin{cases} \delta_n^{\forall,\chi}(q_i^{\forall,\chi}, b_{i+1}) & \text{if } !q_i^{\forall,\chi} \ \& \ !\delta_n^{\forall,\chi}(q_i^{\forall,\chi}, b_{i+1}) \\ \neg! & \text{otherwise} \end{cases} \quad \text{for } 0 \leq i \leq t-1.$$

Let $|w| = p$. Let $s : [0,t] \to N$ and

$$s(i) \stackrel{def}{=} \begin{cases} p & \text{if } q_i^{\forall,\chi} \in F_n^{\forall,\chi} \\ i & \text{if } q_i^{\forall,\chi} \notin F_n^{\forall,\chi} \end{cases}$$

Let $A_n^{D,\chi}(w) =< \Sigma, Q_n^{D,\chi}, I^{D,\chi}, F_n^{D,\chi}, \delta_n^{D,\chi} >$. Let

$q_0^{D,\chi} = \{0^{\#0}\}$ and

$$q_{i+1}^{D,\chi} = \begin{cases} \delta_n^{D,\chi}(q_i^{D,\chi}, x_{i+1}) & \text{if } !q_i^{D,\chi} \ \& \ !\delta_n^{D,\chi}(q_i^{D,\chi}, x_{i+1}) \\ \neg! & \text{otherwise} \end{cases} \quad \text{for } 0 \leq i \leq t-1.$$

Let $d : (I_s^{\chi} \bigcup M_s^{\chi}) \times N \to Q^{ND,\chi}$ and

1) when $\chi = \epsilon$

$d(I + i^{\#e}, z) \stackrel{def}{=} z + i^{\#e}$ and

$d(M + i^{\#e}, z) \stackrel{def}{=} z + i^{\#e}$

2) when $\chi = t$

$d(I + i^{\#e}, z) \stackrel{def}{=} z + i^{\#e}$,

$d(M + i^{\#e}, z) \stackrel{def}{=} z + i^{\#e}$,

$d(I_t + i^{\#e}, z) \stackrel{def}{=} z + i_t^{\#e}$ and

$d(M_t + i^{\#e}, z) \stackrel{def}{=} z + i_t^{\#e}$

3) when $\chi = ms$

$d(I + i^{\#e}, z) \stackrel{def}{=} z + i^{\#e}$,

$d(M + i^{\#e}, z) \stackrel{def}{=} z + i^{\#e}$,

$d(I_s + i^{\#e}, z) \stackrel{def}{=} z + i_s^{\#e}$ and

$d(M_s + i^{\#e}, z) \stackrel{def}{=} z + i_s^{\#e}$.

Let $d : P(I_s^\chi \bigcup M_s^\chi) \times N \to P(Q^{ND,\chi})$ and

$d(A, z) \stackrel{def}{=} \{d(\pi, z) | \pi \in A\}$.

Then

I) $!q_i^{\forall,\chi} \Leftrightarrow !q_i^{D,\chi}$ and

II) $\forall i \in [0, t](!q_i^{\forall,\chi} \ \& \ !q_i^{D,\chi} \Rightarrow d(q_i^{\forall,\chi}, s(i)) = q_i^{D,\chi})$ and

III) $\forall i \in [1, t](!q_i^{\forall,\chi} \ \& \ !q_i^{D,\chi} \Rightarrow (q_i^{\forall,\chi} \in F_n^{\forall,\chi} \Leftrightarrow q_i^{D,\chi} \in F_n^{D,\chi}))$.

Proposition 19 is formulated in [MSFASLD] for $\chi = \epsilon$.

Proof

II) induction on i

1) $i = 0$

$s(0) = 0$. $d(q_0^{\forall,\chi}, s(0)) = \{0^{\#0}\} = q_0^{D,\chi}$.

2) Induction hypothesis $!q_i^{\forall,\chi} \ \& \ !q_i^{D,\chi} \Rightarrow d(q_i^{\forall,\chi}, s(i)) = q_i^{D,\chi}$. We have to prove that $!q_{i+1}^{\forall,\chi} \ \& \ !q_{i+1}^{D,\chi} \Rightarrow d(q_{i+1}^{\forall,\chi}, s(i+1)) = q_{i+1}^{D,\chi}$. Let $!q_{i+1}^{\forall,\chi}$ and $!q_{i+1}^{D,\chi}$. Therefore $!q_i^{\forall,\chi}$ and $!q_i^{D,\chi}$. Applying the induction hypothesis we receive that $d(q_i^{\forall,\chi}, s(i)) = q_i^{D,\chi}$.

Lemma If $q \in q_i^{\forall,\chi}$ and $\pi = d(q, s(i))$ then

- if $q_{i+1}^{\forall,\chi} \subseteq I_s^\chi \Leftrightarrow \delta_e^{\forall,\chi}(q, b_{i+1}) \subseteq I_s^\chi$ then

(1*) $d(\delta_e^{\forall,\chi}(q, b_{i+1}), s(i+1)) = \delta_e^{D,\chi}(\pi, x_{i+1})$

- if $\neg(q_{i+1}^{\forall,\chi} \subseteq I_s^\chi \Leftrightarrow \delta_e^{\forall,\chi}(q, b_{i+1}) \subseteq I_s^\chi)$ then

(2*) $d(m_n(\delta_e^{\forall,\chi}(q, b_{i+1}), |b_{i+1}|), s(i+1)) = \delta_e^{D,\chi}(\pi, x_{i+1})$

Proof Let $q \in q_i^{\forall,\chi}$ and $\pi = d(q, s(i))$. Let $\pi = j^{\#e}$ or $\pi = j_t^{\#e}$ or $\pi = j_s^{\#e}$.

We prove that $!r_n(q, b_{i+1})$ and $r_n(q, b_{i+1}) = \beta(x_{i+1}, w_{[\pi]})$:

1) $q_i^{\forall,\chi} \subseteq I_s^\chi$

In this case $s(i) = i$ and ($q = I + u^{\#e}$ or $q = I_t + u^{\#e}$ or $q = I_s + u^{\#e}$) for some u such that $j = i + u$. ($j = i + u$ because $d(q, s(i)) = \pi$.) Let $b_{i+1} = y_1 y_2 ... y_k$ ($k > 0$). Therefore

$$r_n(q, y_1 y_2 ... y_k) \stackrel{def}{=} \begin{cases} y_{n+u+1} y_{n+u+2} ... y_{n+u+h} & \text{if } h = min(n - e + 1, k - n - u) > 0 \\ \epsilon & \text{if } h = min(n - e + 1, k - n - u) = 0 \\ \neg! & \text{otherwise} \end{cases}$$

$$= \begin{cases} \beta(x_{i+1}, w_{i+1+u} w_{i+2+u} ... w_{i+h+u}) & \text{if } h > 0 \\ \epsilon & \text{if } h = 0 \\ \neg! & \text{otherwise} \end{cases}$$

$k = min(p, i+1+n+1) - (i+1-n) + 1 = min(p, i+n+2) + n - i$
$h = min(n-e+1, min(p, i+n+2) + n - i - n - u) =$
$min(n-e+1, min(p, i+n+2) - i - u) =$
$min(n-e+1, p-j, n-u+2)$
$q \in I_s^\chi$. Hence $|u| \leq e$ and $n-e+1 < n-u+2$.

Therefore $h = min(n-e+1, p-j) \geq 0$. Hence $!r_n(q, b_{i+1})$ and $r_n(q, b_{i+1}) = \beta(x_{i+1}, w_{[\pi]})$.

2) $q_i^{\forall,\chi} \subseteq M_s^\chi$

In this case $s(i) = p$ and ($q = M + u^{\#e}$ or $q = M_t + u^{\#e}$ or $q = M_s + u^{\#e}$) for some u such that $j = p + u$. Let $b_{i+1} = y_1 y_2 ... y_k$ ($k > 0$). Therefore

$$r_n(q, y_1 y_2 ... y_k) \stackrel{def}{=} \begin{cases} y_{k+u+1} y_{k+u+2} ... y_{k+u+h} & \text{if } h = min(n-e+1, -u) > 0 \\ \epsilon & \text{if } h = min(n-e+1, -u) = 0 \\ \neg! & \text{otherwise} \end{cases}$$

$$= \begin{cases} \beta(x_{i+1}, w_{i-n+k+u+1} w_{i-n+k+u+2} ... w_{i-n+k+u+h}) & \text{if } h > 0 \\ \epsilon & \text{if } h = 0 \\ \neg! & \text{otherwise} \end{cases}$$

$q_0^{\forall,\chi} \subseteq I_s^\chi$, $q_i^{\forall,\chi} \subseteq M_s^\chi$. Let i' be such that $i' \in [0, i-1] \,\&\, q_{i'}^{\forall,\chi} \subseteq I_s^\chi \,\&\, q_{i'+1}^{\forall,\chi} \subseteq M_s^\chi$. Therefore $f_n(q', |b_{i'+1}|) = true$ for some $q' \in I_s^\chi$. Hence $|b_{i'+1}| \leq 2n+1$ and $|b_{i'+1}| = n + p - i'$. Therefore $|b_{i+1}| = k = n + p - i$ and

$$r_n(q, y_1 y_2 ... y_k) = \begin{cases} \beta(x_{i+1}, w_{p+u+1} w_{p+u+2} ... w_{p+u+h}) & \text{if } h > 0 \\ \epsilon & \text{if } h = 0 \\ \neg! & \text{otherwise} \end{cases}$$

$h = min(n-e+1, -u) = min(n-e+1, p-j) \geq 0$. Hence $r_n(q, b_{i+1}) = \beta(x_{i+1}, w_{[\pi]})$.

We prove that
if $q_{i+1}^{\forall,\chi} \subseteq I_s^\chi \Leftrightarrow \delta_e^{\forall,\chi}(q, b_{i+1}) \subseteq I_s^\chi$ then 1^* and
if $\neg(q_{i+1}^{\forall,\chi} \subseteq I_s^\chi \Leftrightarrow \delta_e^{\forall,\chi}(q, b_{i+1}) \subseteq I_s^\chi)$ then 2^*:

1) $q_i^{\forall,\chi} \subseteq I_s^\chi$

In this case $s(i) = i$ and ($q = I + u^{\#e}$ or $q = I_t + u^{\#e}$ or $q = I_s + u^{\#e}$) for some u such that $j = i + u$.

Let

$$\Delta_I = \begin{cases} \delta_e^{D,\chi}(u^{\#e}, r_n(q, b_{i+1})) & \text{if } q = I + u^{\#e} \\ \delta_e^{D,\chi}(u_t^{\#e}, r_n(q, b_{i+1})) & \text{if } q = I_t + u^{\#e} \\ \delta_e^{D,\chi}(u_s^{\#e}, r_n(q, b_{i+1})) & \text{if } q = I_s + u^{\#e} \end{cases}$$

Let

$$\Delta'_I = \begin{cases} \delta_e^{D,\chi}(u + i^{\#e}, r_n(q, b_{i+1})) & \text{if } q = I + u^{\#e} \\ \delta_e^{D,\chi}(u + i_t^{\#e}, r_n(q, b_{i+1})) & \text{if } q = I_t + u^{\#e} \\ \delta_e^{D,\chi}(u + i_s^{\#e}, r_n(q, b_{i+1})) & \text{if } q = I_s + u^{\#e} \end{cases}$$

It follows form $i+u = j$ and $r_n(q, b_{i+1}) = \beta(x_{i+1}, w_{[\pi]})$ that $\Delta'_I = \delta_e^{D,\chi}(\pi, \beta(x_{i+1}, w_{[\pi]})) = \delta_e^{D,\chi}(\pi, x_{i+1})$.

$\delta_e^{\forall,\chi}(q, b_{i+1}) = I^{\chi}(\Delta_I) \subseteq I_s^{\chi}$.

1.1) $q_{i+1}^{\forall,\chi} \subseteq I_s^{\chi}$

In this case $s(i+1) = i+1$. We have to prove 1^*. $d(\delta_e^{\forall,\chi}(q, b_{i+1}), s(i+1)) = d(I^{\chi}(\Delta_I), i+1) = d(\{I_{(t|s)} + a - 1^{\#b} | a_{(t|s)}^{\#b} \in \Delta_I\}, i+1) = \{a + i_{(t|s)}^{\#b} | a_{(t|s)}^{\#b} \in \Delta_I\} =_{Proposition\ 18} \Delta'_I = \delta_e^{D,\chi}(\pi, x_{i+1})$.

1.2) $q_{i+1}^{\forall,\chi} \subseteq M_s^{\chi}$

In this case $s(i+1) = p$ and $|b_{i+1}| = p - i + n$. We have to prove 2^*.
$d(m_n(\delta_e^{\forall,\chi}(q, b_{i+1}), |b_{i+1}|), s(i+1)) =$
$d(m_n(I^{\chi}(\Delta_I), p - i + n), p) =$
$d(m_n(\{I_{(t|s)} - 1 + a^{\#b} | a_{(t|s)}^{\#b} \in \Delta_I\}, p - i + n), p) =$
$d(\{M_{(s|t)} + n + 1 - (p - i + n) + a - 1^{\#b} | a_{(t|s)}^{\#b} \in \Delta_I\}, p) = \{a + i_{(s|t)}^{\#b} | a_{(t|s)}^{\#b} \in \Delta_I\} = \Delta'_I =_{Proposition\ 18} \delta_e^{D,\chi}(\pi, x_{i+1})$.

2) $q_i^{\forall,\chi} \subseteq M_s^{\chi}$

In this case $s(i) = p$ and ($q = M + u^{\#e}$ or $q = M_t + u^{\#e}$ or $q = M_s + u^{\#e}$) for some u such that $j = p + u$.

Let

$$\Delta_M = \begin{cases} \delta_e^{D,\chi}(u^{\#e}, r_n(q, b_{i+1})) & \text{if } q = M + u^{\#e} \\ \delta_e^{D,\chi}(u_t^{\#e}, r_n(q, b_{i+1})) & \text{if } q = M_t + u^{\#e} \\ \delta_e^{D,\chi}(u_s^{\#e}, r_n(q, b_{i+1})) & \text{if } q = M_s + u^{\#e} \end{cases}$$

Let

$$\Delta'_M = \begin{cases} \delta_e^{D,\chi}(p + u^{\#e}, r_n(q, b_{i+1})) & \text{if } q = M + u^{\#e} \\ \delta_e^{D,\chi}(p + u_t^{\#e}, r_n(q, b_{i+1})) & \text{if } q = M_t + u^{\#e} \\ \delta_e^{D,\chi}(p + u_s^{\#e}, r_n(q, b_{i+1})) & \text{if } q = M_s + u^{\#e} \end{cases}$$

It follows from $p + u = j$ and $r_n(q, b_{i+1}) = \beta(x_{i+1}, w_{[\pi]})$ that $\Delta'_M = \delta_e^{D,\chi}(\pi, \beta(x_{i+1}, w_{[\pi]})) = \delta_e^{D,\chi}(\pi, x_{i+1})$.

$\delta_e^{\forall,\chi}(q, b_{i+1}) = M^{\chi}(\Delta_M) \subseteq M_s^{\chi}$.

2.1) $q_{i+1}^{\forall,\chi} \subseteq M_s^{\chi}$.

In this case $s(i+1) = p$. We have to prove 1^*. $d(\delta_e^{\forall,\chi}(q, b_{i+1}), s(i+1)) = d(M^{\chi}(\Delta_M), p) = d(\{M_{(t|s)} + a^{\#b} | a_{(t|s)}^{\#b} \in \Delta_M\}, p) = \{p + a_{(t|s)}^{\#b} | a_{(t|s)}^{\#b} \in \Delta_M\} =_{Proposition\ 18} \Delta'_M = \delta_e^{D,\chi}(\pi, x_{i+1})$.

2.2) $q_{i+1}^{\forall,\chi} \subseteq I_s^{\chi}$

It follows from $q_i^{\forall,\chi} \subseteq M_s^{\chi}$ that $|b_{i+1}| = p - i + n$. $s(i+1) = i+1$. We have to prove 2^*. $d(m_n(\delta_e^{\forall,\chi}(q, b_{i+1}), |b_{i+1}|), s(i+1)) = d(m_n(M^{\chi}(\Delta_M), |b_{i+1}|), s(i+1)) = d(m_n(\{M_{(t|s)} + a^{\#b} | a_{(t|s)}^{\#b} \in \Delta_M\}, p - i + n), i+1) = d(\{I_{(t|s)} + p - i + n - n - 1 + a^{\#b} | a_{(t|s)}^{\#b} \in \Delta_M\}, i+1) = \{p + a_{(t|s)}^{\#b} | a_{(t|s)}^{\#b} \in \Delta_M\} =_{Proposition\ 18} \Delta'_M = \delta_e^{D,\chi}(\pi, x_{i+1})$.

The *Lemma* is proved.

We prove that $d(q_{i+1}^{\forall,\chi}, s(i+1)) = q_{i+1}^{D,\chi}$. Let $\Delta = \bigsqcup_{q \in q_i} \delta_e^{\forall,\chi}(q, b_{i+1})$

1) $f_n(rm(\Delta), |b_{i+1}|) = false$

$d(q_{i+1}^{\forall,\chi}, s(i+1)) =$
$d(\bigsqcup_{q \in q_i^{\forall,\chi}} \delta_e^{\forall,\chi}(q, b_{i+1}), s(i+1)) =$
$\bigsqcup_{q \in q_i^{\forall,\chi}} d(\delta_e^{\forall,\chi}(q, b_{i+1}), s(i+1)) =_{1^* \,\&\, d(q_i^{\forall,\chi}, s(i)) = q_i^{D,\chi}}$
$\bigsqcup_{\pi \in q_i^{D,\chi}} \delta_e^{D,\chi}(\pi, x_{i+1}) = q_{i+1}^{D,\chi}$

2) $f_n(rm(\Delta), |b_{i+1}|) = true$

$d(q_{i+1}^{\forall,\chi}, s(i+1)) =$
$d(m_n(\bigsqcup_{q \in q_i^{\forall,\chi}} \delta_e^{\forall,\chi}(q, b_{i+1}), |b_{i+1}|), s(i+1)) =$
$\bigsqcup_{q \in q_i^{\forall,\chi}} d(m_n(\delta_e^{\forall,\chi}(q, b_{i+1}), |b_{i+1}|), s(i+1)) =_{2^* \,\&\, d(q_i^{\forall,\chi}, s(i)) = q_i^{D,\chi}}$
$\bigsqcup_{\pi \in q_i^{D,\chi}} \delta_e^{D,\chi}(\pi, x_{i+1}) = q_{i+1}^{D,\chi}$

II) is proved.

I) Induction on i

1) $i = 0$

$!q_i^{\forall,\chi}$ and $!q_i^{D,\chi}$

2) Induction hypothesis $!q_i^{\forall,\chi} \Leftrightarrow !q_i^{D,\chi}$

We have to prove that $!q_{i+1}^{\forall,\chi} \Leftrightarrow !q_{i+1}^{D,\chi}$.

2.1) $\neg !q_i^{\forall,\chi}$ and $\neg !q_i^{D,\chi}$

Therefore $\neg !q_{i+1}^{\forall,\chi}$ and $\neg !q_{i+1}^{D,\chi}$.

2.2) $!q_i^{\forall,\chi}$ and $!q_i^{D,\chi}$

We prove that $|b_{i+1}| \in \bigtriangledown_a(q_i^{\forall,\chi})$:

2.2.1) $q_i^{\forall,\chi} \subseteq I_s^\chi$

2.2.1.1) $q_i^{\forall,\chi} = \{I^{\#0}\}$

In the proof of the *lemma* we showed that if $q \in q_i^{\forall,\chi}$ then $!r_n(q, b_{i+1})$. Hence $!r_n(I^{\#0}, b_{i+1})$. Therefore $|b_{i+1}| \geq n$ and $|b_{i+1}| \in \bigtriangledown_a(q_i^{\forall,\chi})$.

2.2.1.2) $q_i^{\forall,\chi} \neq \{I^{\#0}\}$

Therefore $i > 0$ and $!q_{i-1}^{\forall,\chi}$. Let us suppose that $|b_{i+1}| \notin \bigtriangledown_a(q_i^{\forall,\chi})$. Hence $|b_{i+1}| < 2n + i - e + 1$ where $rm(q_i^{\forall,\chi}) = I + i^{\#e}$. $e \geq |i|$. Therefore $|b_{i+1}| \leq 2n$ and $|b_i| \leq 2n + 1$.

2.2.1.2.1) $q_{i-1}^{\forall,\chi} \subseteq I_s^\chi$

It follows from the definition of $\delta_n^{\forall,\chi}$ that $\neg f_n(rm(q_i^{\forall,\chi}), |b_i|)$ i.e. $\neg f_n(I + i^{\#e}, |b_i|)$. But $|b_i| \leq 2n + 1$ and $e \leq i + 2n + 1 - |b_i|$. Hence $f_n(I + i^{\#e}, |b_i|)$. Contradiction.

2.2.1.2.2) $q_{i-1}^{\forall,\chi} \subseteq M_s^\chi$

It follows from the definition of $\delta_n^{\forall,\chi}$ and $m_n(M + i + n + 1 - |b_i|^{\#e}, |b_i|) = I + i^{\#e}$ that $f_n(M + i + n + 1 - |b_i|^{\#e}, |b_i|)$. But it also follows from $|b_{i+1}| < 2n + i - e + 1$, $|b_{i+1}| \leq 2n$ and $|b_i| \leq 2n + 1$ that $e \leq i + n + 1 - |b_i| + n$. Hence $\neg f_n(M + i + n + 1 - |b_i|^{\#e}, |b_i|)$. Contradiction.

2.2.2) $q_i^{\forall,\chi} \subseteq M_s^\chi$

Therefore $s(i) = p$ and $|b_{i+1}| = p - i + n$. Let us suppose that $|b_{i+1}| \notin \bigtriangledown_a(q_i^{\forall,\chi})$. Let q be such that $q \in q_i^{\forall,\chi}$ and $if(|b_{i+1}| < n, M^{\#n-|b_{i+1}|}, M + n - |b_{i+1}|^{\#0}) \not\leq_s^\chi q$. (It follows from $|b_{i+1}| \notin \bigtriangledown_a(q_i^{\forall,\chi})$ that such q exists.) Let

$\pi = d(q, s(i))$. It follows from II) that $d(q_i^{\forall,\chi}, s(i)) = q_i^{D,\chi}$. Hence $\pi \in q_i^{D,\chi}$ and $if(p < i, p^{\#i-p}, i^{\#0}) \not\leq_s^\chi \pi$. It follows from the proof of correctness of $\delta_n^{D,\chi}$ that $\forall \pi \in q_i^{D,\chi}(if(p < i, p^{\#i-p}, i^{\#0}) \leq_s^\chi \pi)$. Contradiction.

We proved that $|b_{i+1}| \in \bigtriangledown_a(q_i^{\forall,\chi})$. It follows from $|b_{i+1}| \in \bigtriangledown_a(q_i^{\forall,\chi})$, II) and the definition of $\delta_n^{\forall,\chi}$ that if $\forall q \in q_i^{\forall,\chi}\forall\pi(\pi = d(q, s(i)) \Rightarrow (\delta_e^{\forall,\chi}(q, b_{i+1}) = \phi \Leftrightarrow \delta_e^{D,\chi}(\pi, x_{i+1}) = \phi))$ then $!q_{i+1}^{\forall,\chi} \Leftrightarrow !q_{i+1}^{D,\chi}$. We prove that $\forall q \in q_i^{\forall,\chi}\forall\pi(\pi = d(q, s(i)) \Rightarrow (\delta_e^{\forall,\chi}(q, b_{i+1}) = \phi \Leftrightarrow \delta_e^{D,\chi}(\pi, x_{i+1}) = \phi))$. Let $q = I_{(t|s)} + u^{\#e} \in q_i^{\forall,\chi}$ or $q = M_{(t|s)} + u^{\#e} \in q_i^{\forall,\chi}$ and $\pi = d(q, s(i))$. In the proof of the *lemma* we showed that $!r_n(q, b_{i+1})$ and $r_n(q, b_{i+1}) = \beta(x_{i+1}, w_{[\pi]})$. Therefore $\delta_e^{\forall,\chi}(q, b_{i+1}) = \phi \Leftrightarrow \delta_e^{D,\chi}(u_{(s|t)}^{\#e}, r_n(q, b_{i+1})) = \phi \Leftrightarrow \delta_e^{D,\chi}(u+s(i)_{(s|t)}^{\#e}, r_n(q, b_{i+1})) = \phi \Leftrightarrow \delta_e^{D,\chi}(\pi, x_{i+1}) = \phi$. I) is proved.

III) Let $i > 0$, $!q_i^{\forall,\chi}$ and $!q_i^{D,\chi}$

$\Rightarrow$) We prove that $q_i^{\forall,\chi} \in F_n^{\forall,\chi} \Rightarrow q_i^{D,\chi} \in F_n^{D,\chi}$

Let $q_i^{\forall,\chi} \in F_n^{\forall,\chi}$. Therefore $s(i) = p$ and $\exists q \in q_i^{\forall,\chi}(q \leq_s^\chi M^{\#n})$. Let $M + u^{\#f} \in q_i^{\forall,\chi}$ and $M + u^{\#f} \leq_s^\chi M^{\#n}$. It follows from II) that $p + u^{\#f} \leq_s^\chi p^{\#n}$ and $p + u^{\#f} \in q_i^{D,\chi}$. Therefore $q_i^{D,\chi} \in F_n^{D,\chi}$.

$\Leftarrow$) We prove that $q_i^{\forall,\chi} \notin F_n^{\forall,\chi} \Rightarrow q_i^{D,\chi} \notin F_n^{D,\chi}$

Let $q_i^{\forall,\chi} \notin F_n^{\forall,\chi}$. It follows from $i > 0$ that $!q_{i-1}^{\forall,\chi}$. And it follows from II) and the definition of $\delta_n^{D,\chi}$ that $\forall q \in q_{i-1}^{\forall,\chi}\exists\pi(\pi = d(q, s(i-1))\&\delta_e^{D,\chi}(\pi, x_i) \notin F_n^{D,\chi}) \Rightarrow q_i^{D,\chi} \notin F_n^{D,\chi}$. We prove that $\forall q \in q_{i-1}^{\forall,\chi}\exists\pi(\pi = d(q, s(i-1))\ \&\ \delta_e^{D,\chi}(\pi, x_i) \notin F_n^{D,\chi})$. Let $q \in q_{i-1}^{\forall,\chi}$ and $\pi = d(q, s(i-1))$.

1) $q = I + u^{\#e}$ or $q = I_t + u^{\#e}$ or $q = I_s + u^{\#e}$

In this case $q_{i-1}^{\forall,\chi} \subseteq I_s^\chi$.

1.1) $\delta_e^{\forall,\chi}(q, b_i) = \phi$

It follows from the *lemma* in the proof of II) that $\delta_e^{D,\chi}(\pi, x_i) = \phi \notin F_n^{D,\chi}$

1.2) $\delta_e^{\forall,\chi}(q, b_i) \neq \phi$

Let

$$\Delta_I = \begin{cases} \delta_e^{D,\chi}(u^{\#e}, r_n(I + u^{\#e}, x)) & \text{if } q = I + u^{\#e} \\ \delta_e^{D,\chi}(u_t^{\#e}, r_n(I_t + u^{\#e}, x)) & \text{if } q = I_t + u^{\#e} \\ \delta_e^{D,\chi}(u_s^{\#e}, r_n(I_s + u^{\#e}, x)) & \text{if } q = I_s + u^{\#e} \end{cases}$$

Therefore $\neg f_n(rm(I^\chi(\Delta_I)), |b_i|)$. (If $f_n(rm(I^\chi(\Delta_I)), |b_i|)$ then $f_n(rm(\bigsqcup_{\eta\in q_{i-1}^{\forall,\chi}} \delta_e^{\forall,\chi}(\eta, b_i)), |b_i|)$ and $q_i^{\forall,\chi} \in F_n^{\forall,\chi}$. Contradiction.)

1.2.1) $|b_i| = 2n + 2$

In this case $i + n + 1 \leq p$. It follows from II), $I^{\#0} \leq_s^\chi q$ and $s(i-1) = i - 1$ that $i - 1^{\#0} \leq_s^\chi \pi$. We have from the proof of correctness of $\delta_n^{D,\chi}$ that $\forall x \in \delta_e^{D,\chi}(\pi, x_i)(i^{\#0} \leq_s^\chi x)$. Therefore $\delta_e^{D,\chi}(\pi, x_i) \notin F_n^{D,\chi}$.

1.2.2) $|b_i| \leq 2n + 1$

In this case $|b_i| = p + n - i + 1$. Let $rm(I^\chi(\Delta_I)) = I + a^{\#b}$. Therefore $b > a+2n+1-(p+n-i+1)$ and $b > a+n+i-p$. Obviously if $d(I+a^{\#b}, s(i)) \notin F_n^{ND,\chi}$ then $\delta_e^{D,\chi}(\pi, x_i) \notin F_n^{D,\chi}$. We prove that $d(I + a^{\#b}, s(i)) \notin F_n^{ND,\chi}$. $s(i) = i$.

$d(I+a^{\#b}, s(i)) = i+a^{\#b}$. If follows from $b > a+n+i-p$ that $i+a^{\#b} \notin F_n^{ND,\chi}$. Therefore $\delta_e^{D,\chi}(\pi, x_i) \notin F_n^{D,\chi}$.

2) $q = M + u^{\#e}$ or $q = M_t + u^{\#e}$ or $q = M_s + u^{\#e}$

In this case $q_{i-1}^{\forall,\chi} \subseteq M_s^\chi$

2.1) $\delta_e^{\forall,\chi}(q, b_i) = \phi$

It follows from the *lemma* in the proof of II) that $\delta_e^{D,\chi}(\pi, x_i) = \phi \notin F_n^{D,\chi}$

2.2) $\delta_e^{\forall,\chi}(q, b_i) \neq \phi$

Let

$$\Delta_M = \begin{cases} \delta_e^{D,\chi}(i^{\#e}, r_n(M + i^{\#e}, x)) & \text{if } q = M + i^{\#e} \\ \delta_e^{D,\chi}(i_t^{\#e}, r_n(M_t + i^{\#e}, x)) & \text{if } q = M_t + i^{\#e} \\ \delta_e^{D,\chi}(i_s^{\#e}, r_n(M_s + i^{\#e}, x)) & \text{if } q = M_s + i^{\#e} \end{cases}$$

Therefore $f_n(rm(M^\chi(\Delta_M)), |b_i|)$. (If $\neg f_n(rm(M^\chi(\Delta_M)), |b_i|)$ then $\neg f_n(rm(\bigsqcup_{\eta \in q_{i-1}^{\forall,\chi}} \delta_e^{\forall,\chi}(\eta, b_i)), |b_i|)$ and $q_i^{\forall,\chi} \in F_n^{\forall,\chi}$. Contradiction.)

Let $rm(I^\chi(\Delta_I)) = M + a^{\#b}$. Therefore $b > a + n$. Obviously if $d(m_n(M + a^{\#b}, |b_i|), s(i)) \notin F_n^{ND,\chi}$ then $\delta_e^{D,\chi}(\pi, x_i) \notin F_n^{D,\chi}$. We prove that $d(m_n(M + a^{\#b}, |b_i|), s(i)) \notin F_n^{ND,\chi}$. $s(i) = i$. $|b_i| = n+p-i+1$. $d(m_n(M+a^{\#b}, |b_i|), s(i)) = d(I+a-n-1+|b_i|^{\#b}, s(i)) = d(I+a+p-i^{\#b}, s(i)) = a+p^{\#b}$. It follows from $b > a+n$ that $a+p^{\#b} \notin F_n^{ND,\chi}$. Therefore $\delta_e^{D,\chi}(\pi, x_i) \notin F_n^{D,\chi}$. III) is proved.

6 Building of $A_n^{\forall,\epsilon}$, $A_n^{\forall,t}$ and $A_n^{\forall,ms}$.

We are ready to present an algorithm that builds $A_n^{\forall,\chi}$. We use breadth first search strategy to generate all the states of $A_n^{\forall,\chi}$. $A_n^{\forall,\chi}$ is cyclic. That's why we use the function $HAS\_NEVER\_BEEN\_PUSHED(nextSt)$ to guarantee that the state *nextSt* is generated for the first time. This function returns true only if the state *nextSt* has not been pushed in the queue.

6.1 Summarized pseudo code

```
procedure Build_Automaton( n, χ );
begin
  PUSH_IN_QUEUE( {I^#0} );
  while( not EMPTY_QUEUE() ) do begin
    st := POP_FROM_QUEUE();
    for b in Σ_n^∀ do begin
      if( LENGTH(b) ∈ ▽_a( st ) ) then begin
        nextSt := δ_n^{∀,χ}( st, b );
        if( not EMPTY_STATE( nextSt ) ) then begin
          if( HAS_NEVER_BEEN_PUSHED( nextSt ) ) then begin
            PUSH_IN_QUEUE( nextSt )
          end
          ADD_TRANSITION( < st, b, nextSt > )
        end
```

```
      end
    end
  end
end;
```

6.2 Detailed pseudo code

I) Types

1) $STATE$ is the type of each finite set whose elements are $POSITION$s.

2) $POSITION$ is the type of each tuple $< parameter, type, X, Y >$ where $parameter \in \{I, M\}$, $type \in \{usual, t, s\}$, $X, Y \in Z$ ($I = 0, M = 1, usual = 0, t = 1, s = 2$).

3) $SETOFPOINTS$ is the type of each finite set whose elements are $POINT$s.

4) $POINT$ is the type of each tuple $< type, X, Y >$ where $type \in \{usual, t, s\}$, $X, Y \in Z$.

II) API

```
1) PROCEDURE PUSH_IN_QUEUE( st : STATE );
```

$PUSH\_IN\_QUEUE$ pushes st in $QUEUE$. ($QUEUE$ is the queue that we use.)

```
2) FUNCTION EMPTY_QUEUE() : BOOLEAN;
```

$EMPTY\_QUEUE$ returns $TRUE$ only if $QUEUE$ is empty.

```
3) FUNCTION POP_FROM_QUEUE() : STATE;
```

POP_FROM_QUEUE pops an element from $QUEUE$.

```
4) FUNCTION HAS_NEVER_BEEN_PUSHED( st : STATE ) : BOOLEAN;
```

$HAS\_NEVER\_BEEN\_PUSHED$ returns $TRUE$ only if st has not been pushed in $QUEUE$.

```
5) FUNCTION NEW_POSITION( parameter : { I, M };
                          type : { usual, t, s };
                          x,y : INTEGER ) : POSITION;
```

$NEW\_POSITION$ returns new $POSITION$ determined by $parameter$, $type$, x and y.

```
6) FUNCTION GET_POSITION_PARAM( pos : POSITION ) : { I, M };
```

$GET\_POSITION\_PARAM$ returns the $parameter$ of the $POSITION$ pos.

```
7) FUNCTION GET_POSITION_TYPE( pos : POSITION ) : { usual, t, s };
```

$GET\_POSITION\_TYPE$ returns the *type* of the $POSITION$ *pos*.

```
8) FUNCTION GET_POSITION_X( pos : POSITION ) : INTEGER;
```

$GET\_POSITION\_X$ returns the X of the $POSITION$ *pos*.

```
9) FUNCTION GET_POSITION_Y( pos : POSITION ) : INTEGER;
```

$GET\_POSITION\_Y$ returns the Y of the $POSITION$ *pos*.

```
10) FUNCTION EMPTY_STATE( st : STATE ) : BOOLEAN;
```

$EMPTY\_STATE$ returns $TRUE$ only if *st* is empty.

```
11) FUNCTION GET_FIRST_POSITION( st : STATE ) : POSITION;
```

$GET\_FIRST\_POSITION$ returns some of the elements that are in the $STATEst$. It doesn't matter which element is chosen.

```
12) PROCEDURE ADD_TRANSITION( st : STATE; b : STRING; nextSt : STATE );
```

$ADD\_TRANSITION$ adds the transition $< st, b, nextSt >$ in $AUTOMATON$. ($AUTOMATON$ is the output automaton.)

```
13) FUNCTION EMPTY_SET_OF_POINTS( set : SETOFPOINTS ) : BOOLEAN;
```

$EMPTY\_SET\_OF\_POINTS$ returns $TRUE$ only if *set* is empty.

```
14) FUNCTION NEW_POINT( type : { usual, t, s }; x,y : INTEGER ) : POINT;
```

NEW_POINT returns new $POINT$ determined by *type*, x and y.

```
15) FUNCTION GET_POINT_TYPE( pt : POINT ) : { usual, t, s };
```

GET_POINT_TYPE returns the *type* of *pt*.

```
16) FUNCTION GET_POINT_X( pt : POINT ) : INTEGER;
```

GET_POINT_X returns the X of *pt*.

```
17) FUNCTION GET_POINT_Y( pt : POINT ) : INTEGER;
```

GET_POINT_Y returns the Y of *pt*.

```
18) FUNCTION SUB_STRING( s : STRING; startPos : INTEGER; length : INTEGER ) :
                                                                      STRING;
```

SUB_STRING returns the string $s[startPos]s[startPos + 1]...s[startPos + length - 1]$.

```
19) VAR CHI : { epsilon, t, ms };
```

$epsilon = 0$, $t = 1$, $ms = 2$.

```
procedure Build_Automaton( n : INTEGER );
VAR st, nextSt : STATE;
    b            : STRING;
begin
    PUSH_IN_QUEUE( { NEW_POSITION( I, usual, 0, 0 ) } );
    while( not EMPTY_QUEUE() ) do begin
      st := POP_FROM_QUEUE();
      for b in { sym | sym : STRING and
                       1 <= LENGTH(sym) <= 2n+2 and
                       for all i( i in [1, LENGTH(sym)] =>
                                  ( sym[i] = 0 or sym[i] = 1 ) ) } do begin
        if( Length_Covers_All_The_Positions( n, LENGTH(b), st ) ) then begin
          nextSt := Delta( n, st, b );
          if( not EMPTY_STATE( nextSt ) ) then begin
            if( HAS_NEVER_BEEN_PUSHED( nextSt ) ) then begin
              PUSH_IN_QUEUE( nextSt )
            end
            ADD_TRANSITION( st, b , nextSt )
          end
        end
      end
    end
end;

function Length_Covers_All_The_Positions( n : INTEGER, k : INTEGER; st : STATE ) :
                                                                          BOOLEAN;
(* Length_Covers_All_The_Positions( n, k, st ) = true ⇔ k ∈ ▽_a( st ) *)
VAR pos, pi, q : POSITION;
begin
    pos = GET_FIRST_POSITION(st);
    if( GET_POSITION_PARAM( pos ) = I ) then begin
      if( st = { NEW_POSITION( I, usual, 0, 0 ) } ) then begin
        return( k >= GET_POSITION_X( pos ) + n ) end
      else begin
        for pi in st do begin
          if( k < 2*n + GET_POSITION_X( pi ) - GET_POSITION_Y( pi ) + 1 ) then begin
            return( false )
          end
        end
      end
    else begin
      if( k < n ) then begin
        q := NEW_POSITION( M, usual, 0, n - k ) end
      else begin
```

```
        q := NEW_POSITION( M, usual, n - k, 0 )
      end
      for pi in st do begin
        if( pi <> q and ( not Less_Than_Subsume( q, pi ) ) ) then begin
          return( false )
        end
      end
    end
    return( true )
end;

function Delta( n : INTEGER; st : STATE; b : STRING ) : STATE;
(* Delta( n, st, b ) corresponds to δ_n^{∀,χ}( st, b ) *)
VAR bAdd           : BOOLEAN;
    nextSt, deltaE : STATE;
    q, pi, p       : POSITION;
begin
  nextSt := {};
  for q in st do begin
    deltaE = Delta_E( n, q, b );
    if( not EMPTY_STATE( deltaE ) ) then begin
      for pi in deltaE do begin
        bAdd := true;
        for p in nextSt do begin
          if( Less_Than_Subsume( pi, p ) ) then begin
            nextSt := nextSt \ {p} end
          else begin
            if( p = pi or Less_Than_Subsume( p, pi ) ) then begin
              bAdd := false;
              goto LABEL1
            end
          end
        end
        LABEL1 :
        if( bAdd ) then begin
          nextSt := nextSt U {pi}
        end
      end
    end
  end
  if( F( n, RM(nextSt), LENGTH(b) ) ) then begin
    nextSt := M( n, nextSt, LENGTH(b) )
  end
  return( nextSt )
end;
```

```
function Less_Than_Subsume( q1 : POSITION; q2 : POSITION ) : BOOLEAN;
(* Less_Than_Subsume( q1, q2 ) = true $\Leftrightarrow$ q1 $<_s^{\chi}$ q2 *)
VAR m : INTEGER;
begin
  if( GET_POSITION_TYPE(q1) <> usual or GET_POSITION_Y(q2) <= GET_POSITION_Y(q1) )
  then begin
    return( false )
  end
  if( GET_POSITION_TYPE(q2) = t ) then begin
    m = GET_POSITION_X(q2) + 1 - GET_POSITION_X(q1) end
  else begin
    m = GET_POSITION_X(q2) - GET_POSITION_X(q1)
  end
  if( m < 0 ) then begin
    m = -m
  end
  return( m <= GET_POSITION_Y(q2) - GET_POSITION_Y(q1) )
end;

function Delta_E( n : INTEGER, q : POSITION, b : STRING ) : STATE;
(* Delta_E( n, q, b ) corresponds to $\delta_e^{\forall,\chi}$( q, b ) *)
var deltaED : SETOFPOINTS;
    st      : STATE;
    pi      : POINT;
begin
  deltaED := Delta_E_D( n,
                        NEW_POINT( GET_POSITION_TYPE(q),
                                   GET_POSITION_X(q),
                                   GET_POSITION_Y(q) ),
                        R(n, q, b) );
  if( EMPTY_SET_OF_POINTS( deltaED ) ) then begin
    return( {} )
  end
  st := {};
  if( GET_POSITION_PARAM( q ) = I ) then begin
    for pi in deltaED do begin
      st := st U { NEW_POSITION( I,
                                 GET_POINT_TYPE( pi ),
                                 GET_POINT_X( pi ) - 1,
                                 GET_POINT_Y( pi ) ) }
    end end
  else begin
    for pi in deltaED do begin
      st := st U { NEW_POSITION( M,
                                 GET_POINT_TYPE( pi ),
                                 GET_POINT_X( pi ),
```

```
                                 GET_POINT_Y( pi ) ) }
    end end
  end
  return( st )
end;

function M( n : INTEGER; st : STATE; k : INTEGER ) : STATE;
(* M( n, st, k ) corresponds to m_n( st, k ) *)
VAR m  : STATE;
    pi : POSITION;
begin
  m = {};
  for pi in st do begin
    if( GET_POSITION_PARAM(pi) = I ) then begin
      m := m U { NEW_POSITION( M,
                               GET_POSITION_TYPE(pi),
                               GET_POSITION_X(pi) + n + 1 - k,
                               GET_POSITION_Y(pi) ) } end
    else begin
      m := m U { NEW_POSITION( I,
                               GET_POSITION_TYPE(pi),
                               GET_POSITION_X(pi) - n - 1 + k,
                               GET_POSITION_Y(pi) ) } end
    end
  end
  return(m)
end;

function R( n : INTEGER; pos : POSITION; b : STRING ) : STRING;
(* R( n, pos, b ) corresponds to r_n( pos, b ) *)
VAR len : INTEGER;
begin
  if( GET_POSITION_PARAM( pos ) = I ) then begin
    if( n - GET_POSITION_Y(pos) + 1 < LENGTH(b) - n - GET_POSITION_X(pos) ) then begin
      len := n - GET_POSITION_Y(pos) + 1 end
    else begin
      len := LENGTH(b) - n - GET_POSITION_X(pos)
    end
    return( SUB_STRING( b, n + GET_POSITION_X(pos) + 1, len ) )
  end
  if( n - GET_POSITION_Y(pos) + 1 < -GET_POSITION_X(pos) ) then begin
    len := n - GET_POSITION_Y(pos) + 1 end
  else begin
    len := -GET_POSITION_X(pos)
  end
  return( SUB_STRING( b, LENGTH(b) + GET_POSITION_X(pos) + 1, len ) )
```

```
end;

function RM( st : STATE ) : POSITION;
(* RM( st ) corresponds to $rm$( st ) *)
VAR pi, rm : POSITION;
begin
  for pi in st do begin
    if( GET_POSITION_TYPE(pi) = usual ) then begin
      rm := pi
    end
  end
  for pi in st do begin
    if( GET_POSITION_TYPE(pi) = usual and
        GET_POSITION_X(pi) - GET_POSITION_Y(pi) >
        GET_POSITION_X(rm) - GET_POSITION_Y(rm) ) then begin
      rm := pi
    end
  end
  return( rm )
end;

function F( n : INTEGER; pos : POSITION; k : INTEGER ) : BOOLEAN;
(* F( n, pos, k ) corresponds to $f_n$( pos, k ) *)
begin
  if( GET_POSITION_PARAM(pos) = I ) then begin
    return( k <= 2*n + 1 and
            GET_POSITION_Y(pos) <= GET_POSITION_X(pos) + 2*n + 1 - k )
  end
  return( GET_POSITION_Y(pos) > GET_POSITION_X(pos) + n )
end;

function Delta_E_D( n : INTEGER; pt : POINT; h : STRING ) : SET_OF_POINTS;
(* Delta_E_D( n, pt, h ) corresponds to $\delta_e^{D,\chi}$( pt, h )
CHI corresponds to the metasymbol $\chi$ *)
VAR x,y,j,posOfFirst1 : INTEGER;
begin
  x := GET_POINT_X(pt)
  y := GET_POINT_Y(pt)
  if( CHI = epsilon ) then begin
    if( LENGTH(h) = 0 ) then begin
      if( y < n ) then begin
        return( { NEW_POINT(x, y+1, usual) } )
      end
      return( {} )
    end
    if( h[1] = 1 ) then begin
```

```
    return( { NEW_POINT(x+1, y, usual) } )
  end
  if( LENGTH(h) = 1 ) then begin
    if( y < n ) then begin
      return( { NEW_POINT(x, y+1, usual),
                NEW_POINT(x+1, y+1, usual) } )
    end
    return( {} )
  end
  posOfFirst1 := 0;
  for j := 2 to LENGTH(h) do begin
    if( h[j] = 1 ) then begin
      posOfFirst1 := j;
      goto LABEL2
    end
  end
  LABEL2 :
  if( posOfFirst1 = 0 ) then begin
    return( { NEW_POINT(x, y+1, usual),
              NEW_POINT(x+1, y+1, usual) } )
  end
  return( { NEW_POINT(x, y+1, usual),
            NEW_POINT(x+1, y+1, usual),
            NEW_POINT(x+j, y+j-1, usual) } ) end
else begin
  if( CHI = t ) then begin
    if( GET_POINT_TYPE(pt) = t ) then begin
      if( h[1] = 1 ) then begin
        return( { NEW_POINT(x+2, y) } )
      end
      return( {} )
    end
    if( LENGTH(h) = 0 ) then begin
      if( y < n ) then begin
        return( { NEW_POINT(x, y+1, usual) } )
      end
      return( {} )
    end
    if( h[1] = 1 ) then begin
      return( { NEW_POINT(x+1, y, usual) } )
    end
    if( LENGTH(h) = 1 ) then begin
      if( y < n ) then begin
        return( { NEW_POINT(x, y+1, usual),
                  NEW_POINT(x+1, y+1, usual) } )
      end
```

```
      return( {} )
    end
    if( h[2] = 1 ) then begin
      return( { NEW_POINT(x, y+1, usual),
                NEW_POINT(x+1, y+1, usual),
                NEW_POINT(x+2, y+1, usual),
                NEW_POINT(x, y+1, t) } )
    end
    posOfFirst1 := 0;
    for j := 3 to LENGTH(h) do begin
      if( h[j] = 1 ) then begin
        posOfFirst1 := j;
        goto LABEL3
      end
    end
    LABEL3 :
    if( posOfFirst1 = 0 ) then begin
      return( { NEW_POINT(x, y+1, usual),
                NEW_POINT(x+1, y+1, usual) } )
    end
    return( { NEW_POINT(x, y+1, usual),
              NEW_POINT(x+1, y+1, usual),
              NEW_POINT(x+j, y+j-1, usual) } )
  end
end
if( GET_POINT_TYPE(pt) = s ) then begin
  return( { NEW_POINT(x+1, y) } )
end
if( LENGTH(h) = 0 ) then begin
  if( y < n ) then begin
    return( { NEW_POINT(x, y+1, usual) } )
  end
  return( {} )
end
if( h[1] = 1 ) then begin
  return( { NEW_POINT(x+1, y, usual) } )
end
if( LENGTH(h) = 1 ) then begin
  if( y < n ) then begin
    return( { NEW_POINT(x, y+1, usual),
              NEW_POINT(x+1, y+1, usual),
              NEW_POINT(x, y+1, s) } )
  end
end
return( { NEW_POINT(x, y+1, usual),
          NEW_POINT(x+1, y+1, usual),
```

```
            NEW_POINT(x+2, y+1, usual),
            NEW_POINT(x, y+1, s) } )
end;
```

6.3 Complexity

In this section we give some rough evaluations for the number of the states of $A_n^{\forall,\chi}$ and the time and the space complexity of the algorithm that builds $A_n^{\forall,\chi}$.

1) $\chi = \epsilon$

1.1) We evaluate $|I^{\epsilon}_{states}|$

$f : I^{\epsilon}_{s} \to [1, 2n+1]$

$f(I + i^{\#e}) \stackrel{def}{=} i + e + 1$

We define $g : I^{\epsilon}_{states} \to \{0, 1, ..., 2n+1\}^*$. Let for each $A \in I^{\epsilon}_{states}$ and for each $j \in [1, 2n+1]$ it is true that $|g(A)| = 2n+1$ and

$$g(A)_j = \begin{cases} 0 & \text{if } A \bigcap A_j = \phi \\ f(\pi) & \text{if } \pi \in A \bigcap A_j \end{cases}$$

where $A_j = \{I - n + j - 1 - t^{\#n-t} | 0 \le t \le n\}$.

Obviously g is inective function and $\forall k \in [1, 2n+1] \forall A \in I_{states}(g(A)_k \neq 0 \,\&\, 1 \le r < k \Rightarrow g(A)_k > g(A)_r)$.

Therefore $|I^{\epsilon}_{states}| \le |W|$ where $W = \{w | w \in \{0, 1, ..., 2n+1\}^* \,\&\, |w| = 2n+1 \,\&\, \forall k \in [1, 2n+1] \forall r \in [1, k-1](w_k \neq 0 \Rightarrow w_k > w_r)\}$.

$$|W| = \sum_{k=1}^{2n+1} \left(\begin{array}{c} 2n+1 \\ k \end{array} \right)^2 = \left(\begin{array}{c} 2(2n+1) \\ 2n+1 \end{array} \right) - 1 < \frac{[2(2n+1)]!}{(2n+1)!(2n+1)!}$$

Applying the following extension of the Stirling's formula:

$$\sqrt{2\pi k}\, k^k e^{-k} e^{\frac{1}{12k+1}} < k! < \sqrt{2\pi k}\, k^k e^{-k} e^{\frac{1}{12k}} \text{(Robbins 1955, Feller 1968)}$$

we have

$$|W| < C \frac{\sqrt{(2n+1)}\,(4n+2)^{4n+2} e^{-(4n+2)}}{(2n+1)(2n+1)^{4n+2} e^{-(4n+2)}}$$

for some constant C. Therefore $|I^{\epsilon}_{states}| = O(2^{4n - \log_2 \sqrt{2n+1}})$

1.2) We evaluate $|M^{\epsilon}_{states}|$

We have

$$M^{\epsilon}_{states} = \bigcup_{k=0}^{n} \{A_k\}$$

where $A_k = \{A | A \in M^{\epsilon}_{states} \,\&\, \exists t (0 \le t \le k \,\&\, rm(A) = M - k + t^{\#t})\}$. Obviously $\forall k \in [0, n](|A_k| < |A_n|)$ and $|A_n| < |I^{\epsilon}_{states}|$. $|A_0| = 1$. Therefore $|M^{\epsilon}_{states}| = O(n 2^{4n - \log_2 \sqrt{2n+1}})$.

2) $\chi = t$ or $\chi = ms$

It follows from *Proposition* 21 that in this case

$$|I^{\chi}_{states}| \le \sum_{k=1}^{2n+1} \binom{2n+1}{k}^2 2^k \le 2^{2n+1} \sum_{k=1}^{2n+1} \binom{2n+1}{k}^2$$

Therefore $|I^{\chi}_{states}| = O(2^{6n-\log_2 \sqrt{2n+1}})$. Obviously $|M^{ms}_{states}| = O(n2^{6n-\log_2 \sqrt{2n+1}})$.

The space complexity of the procedure *Build_Automaton* is $O(|I^{\chi}_{states}| + |M^{\chi}_{states}|)$. (We don't take into account the memory that the algorithm uses to write the output.) The time complexity is $O(n^2(|I^{\chi}_{states}| + |M^{\chi}_{states}|))$.

6.4 Some final results

$\chi = \epsilon$			
n	$\vert I^{\epsilon}_{states}\vert$	$\vert M^{\epsilon}_{states}\vert$	$\vert\{< q_1, b, q_2 > \mid !\delta^{\forall,\epsilon}_n(q_1, b)\ \&\ q_2 = \delta^{\forall,\epsilon}_n(q_1, b)\}\vert$
1	8	6	163
2	50	40	5073
3	322	280	144133
4	2187	2025	4067325
5	15510	15026	116976045
6	113633	113841	3445035693

$\chi = t$			
n	$\vert I^{t}_{states}\vert$	$\vert M^{t}_{states}\vert$	$\vert\{< q_1, b, q_2 > \mid !\delta^{\forall,t}_n(q_1, b)\ \&\ q_2 = \delta^{\forall,t}_n(q_1, b)\}\vert$
1	9	7	187
2	66	54	6805
3	508	448	229025
4	4155	3884	7730973
5	35584	34711	267593313
6	315199	317409	9515031337

$\chi = ms$			
n	$\vert I^{ms}_{states}\vert$	$\vert M^{ms}_{states}\vert$	$\vert\{< q_1, b, q_2 > \mid !\delta^{\forall,ms}_n(q_1, b)\ \&\ q_2 = \delta^{\forall,ms}_n(q_1, b)\}\vert$
1	9	8	197
2	76	75	8307
3	676	725	317039
4	6339	7214	12126471
5	61914	73566	476227735

7 Minimality of $A^{\forall,\epsilon}_n$, $A^{\forall,t}_n$ and $A^{\forall,ms}_n$.

Proposition 20 Let $\chi \in \{\epsilon, t, ms\}$. Let $n \in N$ and $A \in I^{\chi}_{states}$. Then $\exists b \in \Sigma^{\forall}_n(!\delta^{\forall,\chi}_n(A, b)\ \&\ \delta^{\forall,\chi}_n(A, b) \in M^{\chi}_{states})$.

Proof Let k be the least element of $\bigtriangledown_a(A)$. Obviously $\delta_n^{\forall,\chi}(A, 1^k) \in M_{states}^{\chi}$.

Proposition 21
1) $A \in I_{states}^t \,\&\, I + i_t^{\#e} \in A \Rightarrow I + i + 1^{\#e} \in A$
2) $A \in M_{states}^t \,\&\, M + i_t^{\#e} \in A \Rightarrow M + i + 1^{\#e} \in A$
3) $A \in I_{states}^{ms} \,\&\, I + i_s^{\#e} \in A \Rightarrow I + i^{\#e} \in A$
4) $A \in M_{states}^{ms} \,\&\, M + i_s^{\#e} \in A \Rightarrow M + i^{\#e} \in A$

Proof
1) Let $A \in I_{states}^t$ and $I + i_t^{\#e}$. Therefore $\exists B \in I_{states}^t \bigcup M_{states}^t \exists x \in \Sigma_n^{\forall}(\delta_n^{\forall,t}(B, x) = A)$. Let $B \in I_{states}^t \bigcup M_{states}^t$ and $x \in \Sigma_n^{\forall}$ be such that $\delta_n^{\forall,t}(B, x) = A$. Therefore $\exists \pi \in B(I + i_t^{\#e} \in \delta_e^{\forall,t}(\pi, x) \vee I + i_t^{\#e} \in m_n(\delta_e^{\forall,t}(\pi, x), |x|))$. Let $\pi \in B$ be such that $I + i_t^{\#e} \in \delta_e^{\forall,t}(\pi, x)$ or $I + i_t^{\#e} \in m_n(\delta_e^{\forall,t}(\pi, x), |x|))$. Therefore $I + i + 1^{\#e} \in \delta_e^{\forall,t}(\pi, x)$ or $I + i + 1^{\#e} \in m_n(\delta_e^{\forall,t}(\pi, x), |x|))$. It follows from $I + i_t^{\#e} \in A$ that $\neg \exists \pi' \in A(\pi' <_s^t I + i + 1^{\#e})$ (if we suppose that $\pi' \in A$ and $\pi' <_s^t I + i + 1^{\#e}$ then $\pi' <_s^t I + i_t^{\#e}$ - contradiction). Therefore $I + i + 1^{\#e} \in A$.
2), 3) and 4) can be proved analogously.

Proposition 22 Let $\chi \in \{\epsilon, t, ms\}$. Let $n \in N$. Then $\forall A, B \in I_{states}^{\chi}(L(A) = L(B) \Rightarrow A = B)$.

Proof
We define $y : I_s^{\chi} \rightarrow N$:
$y(I + i^{\#e}) \stackrel{def}{=} e$
$y(I + i_t^{\#e}) \stackrel{def}{=} e$
$y(I + i_s^{\#e}) \stackrel{def}{=} e$
We define $min : P(I_s^{\chi}) \rightarrow P(I_s^{\chi})$:
$min(X) \stackrel{def}{=} \{\pi | \pi \in X \,\&\, \forall \pi' \in X(y(\pi') \geq y(\pi))\}$

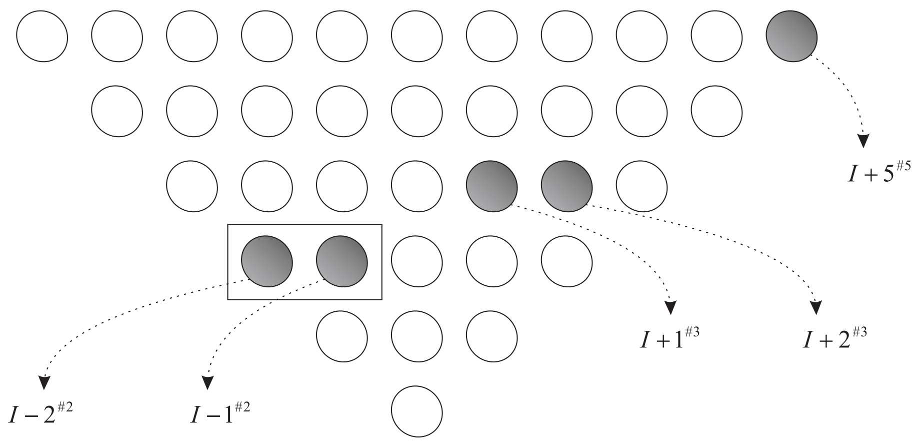

Fig. 21 $n = 5$,
$min(\{I - 2^{\#2}, I - 1^{\#2}, I + 1^{\#3}, I + 2^{\#3}, I + 5^{\#5}\}) = \{I - 2^{\#2}, I - 1^{\#2}\}$

Lemma $\forall A, B \in I^{\chi}_{states}(L(A) = L(B) \Rightarrow min(A) = min(B))$

Proof Let $A, B \in I^{\chi}_{states}$ and $L(A) = L(B)$. We define $MIN : I^{\chi}_{states} \to N$:
$MIN(X) = e \overset{def}{\Leftrightarrow} \exists\pi \in min(X)(y(\pi) = e)$
(The definition of MIN is correct because $X \in I^{\chi}_{states} \Rightarrow min(X) \neq \phi$ and $\pi_1 \in min(X)$ & $\pi_2 \in min(X) \Rightarrow y(\pi_1) = y(\pi_2)$.)
Let us suppose that $min(A) \neq min(B)$.
1) $MIN(A) < MIN(B)$
Using induction we build the rows $\{A_i\}^{\infty}_{i=0}$ and $\{B_i\}^{\infty}_{i=0}$ such that
$\forall i \in N($
$A_i \in I^{\chi}_{states}$ &
$B_i \in I^{\chi}_{states}$ &
$MIN(A_i) = MIN(A) + i$ &
$MIN(B_i) = MIN(B) + i$ &
$L(A_i) = L(B_i))$.
$A_0 = A$, $B_0 = B$. Let us suppose that A_i and B_i are built such that $A_i \in I^{\chi}_{states}$, $B_i \in I^{\chi}_{states}$, $MIN(A_i) = MIN(A) + i$, $MIN(B_i) = MIN(B) + i$ and $L(A_i) = L(B_i)$. We have to build A_{i+1} and B_{i+1}. Let $b_1 = 0^{2n+2} \in \Sigma^{\forall}_n$. We have $MIN(A_i) < MIN(B_i) \leq n$.
1.1) $\chi = \epsilon$
Therefore $!\delta^{\forall,\epsilon}_n(A_i, b_1)$ $(MIN(A_i) < n)$, $\delta^{\forall,\epsilon}_n(A_i, b_1) \in I^{\epsilon}_{states}$ and $MIN(\delta^{\forall,\epsilon}_n(A_i, b_1)) = MIN(A_i) + 1 = MIN(A) + i + 1$. Let $A_{i+1} = \delta^{\forall,\epsilon}_n(A_i, b_1)$. We prove that $!\delta^{\forall,\epsilon}_n(B_i, b_1)$. Let us suppose that $\neg!\delta^{\forall,\epsilon}_n(B_i, b_1)$. Let $b' \in \Sigma^{\forall}_n$ be such that $!\delta^{\forall,\epsilon}_n(A_{i+1}, b')$ and $\delta^{\forall,\epsilon}_n(A_{i+1}, b') \in M^{\epsilon}_{states}$. (It follows from *Proposition* 20 that such b' exists.) Therefore $b_1b' \in L(A_i)$ but $b_1b' \notin L(B_i)$. Hence $L(A_i) \neq L(B_i)$. Contradiction. Therefore $!\delta^{\forall,\epsilon}_n(B, b_1)$. Obviously $\delta^{\forall,\epsilon}_n(B, b_1) \in I^{\epsilon}_{states}$ and $MIN(\delta^{\forall,\epsilon}_n(B_i, b_1)) = MIN(B_i) + 1 = MIN(B) + i + 1$. Let $B_{i+1} = \delta^{\forall,\epsilon}_n(B_i, b_1)$. We have $L(A_{i+1}) = L(B_{i+1})$ (otherwise $L(A_i) \neq L(B_i)$).
1.2) $\chi = t$
It follows from *Proposition* 21 that $\exists j \exists e(I + j^{\#e} \in min(A_i))$. Therefore $!\delta^{\forall,t}_n(A_i, b_1)$, $\delta^{\forall,t}_n(A_i, b_1) \in I^{t}_{states}$ and $MIN(\delta^{\forall,t}_n(A_i, b_1)) = MIN(A_i) + 1 = MIN(A)+i+1$. Hence $!\delta^{\forall,t}_n(B_i, b_1)$, $\delta^{\forall,t}_n(B_i, b_1) \in I^{t}_{states}$ and $MIN(\delta^{\forall,t}_n(B_i, b_1)) = MIN(B_i)+1 = MIN(B)+i+1$. Let $A_{i+1} = \delta^{\forall,t}_n(A_i, b_1)$ and $B_{i+1} = \delta^{\forall,t}_n(B_i, b_1)$.
1.3) $\chi = ms$
Let $b' = 1^{2n+2}$. Obviously $!\delta^{\forall,ms}_n(A_i, b')$, $\delta^{\forall,ms}_n(A_i, b') \in I^{ms}_{states}$, $MIN(\delta^{\forall,ms}_n(A_i, b')) = MIN(A_i)$, $\neg\exists\pi \in \delta^{\forall,ms}_n(A_i, b')\exists j \exists e(\pi = I + j^{\#e}_s)$, $!\delta^{\forall,ms}_n(B_i, b')$, $\delta^{\forall,ms}_n(B_i, b') \in I^{ms}_{states}$, $MIN(\delta^{\forall,ms}_n(B_i, b')) = MIN(B_i)$ and $\neg\exists\pi \in \delta^{\forall,ms}_n(B_i, b')\exists j \exists e(\pi = I + j^{\#e}_s)$. Let $A' = \delta^{\forall,ms}_n(A_i, b')$ and $B' = \delta^{\forall,ms}_n(B_i, b')$. Therefore $!\delta^{\forall,ms}_n(A', b_1)$, $\delta^{\forall,ms}_n(A', b_1) \in I^{ms}_{states}$ and $MIN(\delta^{\forall,ms}_n(A', b_1)) = MIN(A') + 1 = MIN(A_i) + 1 = MIN(A)+i+1$. Obviously $L(A') = L(B')$. Hence $!\delta^{\forall,ms}_n(B', b_1)$ (otherwise $L(A') \neq L(B')$). It follows form $!\delta^{\forall,ms}_n(B', b_1)$ and $\neg\exists\pi \in \delta^{\forall,ms}_n(B_i, b')\exists j \exists e(\pi =$

$I+j_s^{\#e}$) that $\delta_n^{\forall,ms}(B',b_1) \in I_{states}^{ms}$ and $MIN(\delta_n^{\forall,ms}(B',b_1)) = MIN(B')+1 = MIN(B)+i+1$. Let $A_{i+1} = \delta_n^{\forall,ms}(A',b_1)$ and $B_{i+1} = \delta_n^{\forall,ms}(B',b_1)$.

The rows $\{A_i\}_{i=0}^{\infty}$ and $\{B_i\}_{i=0}^{\infty}$ are built. But such rows cannot exist. Contradiction.

2) $MIN(A) > MIN(B)$

Like in 1) we receive contradiction.

3) $MIN(A) = MIN(B) = m$

3.1) $\exists \pi \in min(A)(\pi \notin min(B))$

Let $\pi \in min(A)$ and $\pi \notin min(B)$.

3.1.1) $\chi = \epsilon$

Let $\pi = I + i^{\#m}$ and $c = 0^{n+i}10^{n+1-i}$.

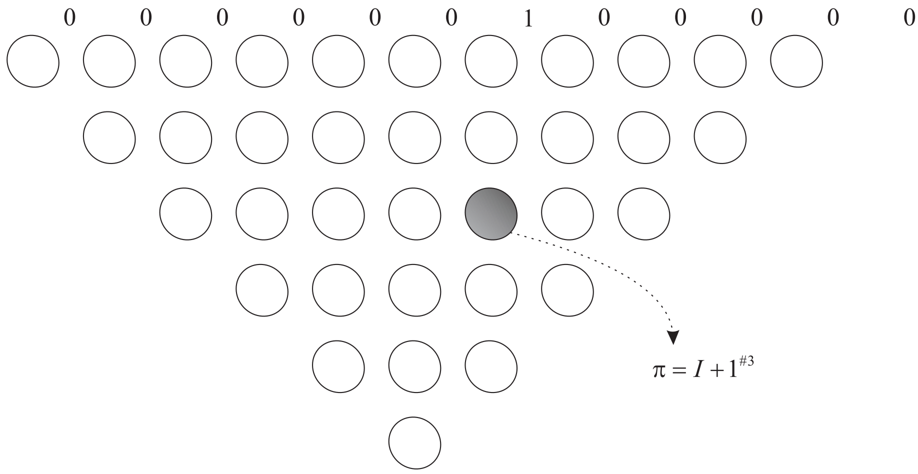

Fig. 22 $n = 5$, $\pi = I + 1^{\#3}$, $c = 0^6 10^5$

3.1.1.1) $m < n$

It follows from the definition of $\delta_n^{\forall,\epsilon}$ that $MIN(\delta_n^{\forall,\epsilon}(A,c)) = m$ and $MIN(\delta_n^{\forall,\epsilon}(B,c)) = m+1$. And it follows from 1) that $\exists c' \in \Sigma_n^{\forall *}(c' \in L(\delta_n^{\forall,\epsilon}(A,c))$ & $c' \notin L(\delta_n^{\forall,\epsilon}(B,c)))$. Hence we find cc' such that $cc' \in L(A)$ and $cc' \notin L(B)$ i.e. $L(A) \neq L(B)$. Contradiction.

3.1.1.2) $m = n$

It follows from the definition of $\delta_n^{\forall,\epsilon}$ that $!\delta_n^{\forall,\epsilon}(A,c)$ and $\neg!\delta_n^{\forall,\epsilon}(B,c)$, i.e. $L(A) \neq L(B)$. Contradiction.

3.1.2) $\chi = t$

3.1.2.1) $\pi = I + i^{\#m}$

Let $c = 0^{n+i}10^{n+1-i}$. Obviously $!\delta_n^{\forall,t}(A,c)$ and $\delta_n^{\forall,t}(A,c) \in I_{states}^t$. Let $A' = \delta_n^{\forall,t}(A,c)$. Obviously $MIN(A') = MIN(A) = m$, $\neg\exists\pi' \in min(A')\exists j(\pi' = I + j_t^m)$ and $\pi \in A'$. We have also $!\delta_n^{\forall,t}(B,c)$ (otherwise $L(A) \neq L(B)$). Let $B' = \delta_n^{\forall,t}(B,c)$. Therefore $L(A') = L(B')$. Hence $MIN(B') = m$ (if we suppose that $MIN(B') > m$ then it follows from 1) that $L(A') \neq L(B')$). Obviously $\pi \notin$

B' and $\neg\exists\pi' \in min(B')\exists j(\pi' = I + j_t^m)$. Since $\pi \in A'$ we have $!\delta_n^{\forall,t}(A', c)$ and $MIN(\delta_n^{\forall,t}(A', c)) = m$. Hence $!\delta_n^{\forall,t}(B', c)$ and $L(\delta_n^{\forall,t}(A', c)) = L(\delta_n^{\forall,t}(B', c))$. Since $\pi \notin B'$ and $\neg\exists\pi' \in min(B')\exists j(\pi' = I + j_t^m)$ we have $MIN(\delta_n^{\forall,t}(B', c)) = m + 1$. It follows from 1) that $L(\delta_n^{\forall,t}(A', c)) \neq L(\delta_n^{\forall,t}(B', c))$. Contradiction.

3.1.2.2) $\pi = I + i_t^{\#m}$

Let $c = 0^{n+i}10^{n+1-i}$. Obviously $!\delta_n^{\forall,t}(A, c)$, $\delta_n^{\forall,t}(A, c) \in I_{states}^t$, $MIN(\delta_n^{\forall,t}(A, c)) = m$, $I+i+1^{\#m} \in \delta_n^{\forall,t}(A, c)$, $!\delta_n^{\forall,t}(B, c)$, $\delta_n^{\forall,t}(B, c) \in I_{states}^t$, $MIN(\delta_n^{\forall,t}(B, c)) = m$, $I + i + 1^{\#m} \notin \delta_n^{\forall,t}(B, c)$ and $L(\delta_n^{\forall,t}(A, c)) = L(\delta_n^{\forall,t}(B, c))$. Like in 3.1.2.1) we receive contradiction.

3.1.3) $\chi = ms$

3.1.3.1) $\pi = I + i^{\#m}$.

It follows from *Proposition* 21 that $I+i_s^{\#m} \notin B$ (otherwise $\pi = I+i^{\#m} \in B$). Let $c = 0^{n+i}10^{n+1-i}$. Obviously $!\delta_n^{\forall,ms}(A, c)$ and $\delta_n^{\forall,ms}(A, c) \in I_{states}^{ms}$. Let $A' = \delta_n^{\forall,ms}(A, c)$. Obviously $MIN(A') = MIN(A) = m$, $\neg\exists\pi' \in min(A')\exists j(\pi' = I + j_s^m)$ and $\pi \in A'$. We have also $!\delta_n^{\forall,ms}(B, c)$ (otherwise $L(A) \neq L(B)$). Let $B' = \delta_n^{\forall,ms}(B, c)$. Therefore $L(A') = L(B')$. Hence $MIN(B') = m$ (if we suppose that $MIN(B') > m$ then it follows from 1) that $L(A') \neq L(B')$). Obviously $\pi \notin B'$ ($I + i_s^{\#m} \notin B$) and $\neg\exists\pi' \in min(B')\exists j(\pi' = I + j_s^m)$. Since $\pi \in A'$ we have $!\delta_n^{\forall,ms}(A', c)$ and $MIN(\delta_n^{\forall,ms}(A', c)) = m$. Hence $!\delta_n^{\forall,ms}(B', c)$ and $L(\delta_n^{\forall,ms}(A', c)) = L(\delta_n^{\forall,ms}(B', c))$. Since $\pi \notin B'$ and $\neg\exists\pi' \in min(B')\exists j(\pi' = I + j_s^m)$ we have $MIN(\delta_n^{\forall,ms}(B', c)) = m+1$. It follows from 1) that $L(\delta_n^{\forall,ms}(A', c)) \neq L(\delta_n^{\forall,ms}(B', c))$. Contradiction.

3.1.3.2) $\pi = I + i_s^{\#m}$.

Let $c = 0^{2n+2}$. Obviously $!\delta_n^{\forall,ms}(A, c)$, $\delta_n^{\forall,ms}(A, c) \in I_{states}^{ms}$, $MIN(\delta_n^{\forall,ms}(A, c)) = m$, $I+i^{\#m} \in \delta_n^{\forall,ms}(A, c)$, $!\delta_n^{\forall,ms}(B, c)$, $\delta_n^{\forall,ms}(B, c) \in I_{states}^{ms}$, $MIN(\delta_n^{\forall,ms}(B, c)) = m$, $I + i^{\#m} \notin \delta_n^{\forall,ms}(B, c)$ and $L(\delta_n^{\forall,ms}(A, c)) = L(\delta_n^{\forall,ms}(B, c))$. Like in 3.1.3.1) we receive contradiction.

3.2) $\exists\pi \in min(B)(\pi \notin min(A))$

Like in 3.1) we receive contradiction.

The *Lemma* is proved.

We define $floor : P(I_s^\chi) \times N^+ \to P(I_s^\chi)$:

$$floor(X, 1) \stackrel{def}{=} min(X)$$

$$floor(X, i + 1) \stackrel{def}{=} min(X \backslash (\bigcup_{1 \leq j \leq i} floor(X, j)))$$

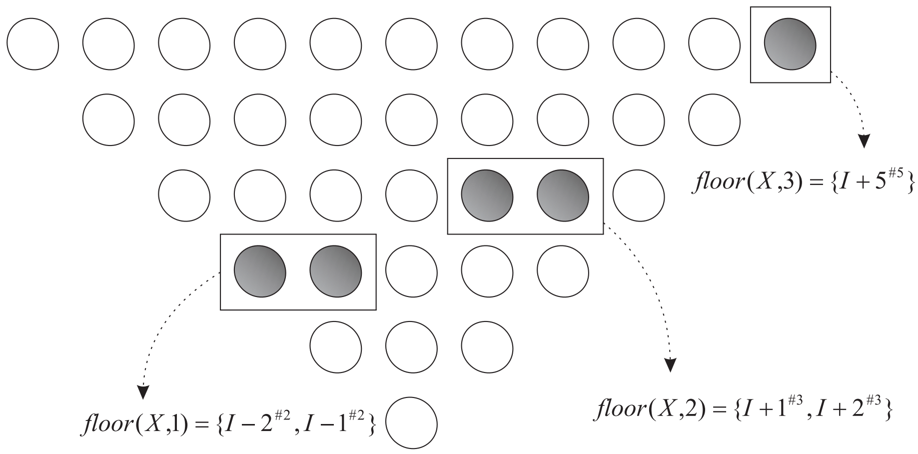

Fig. 23 $n = 5,\ X = \{I - 2^{\#2}, I - 1^{\#2}, I + 1^{\#3}, I + 2^{\#3}, I + 5^{\#5}\}$

We define $FLOOR : I^{\chi}_{states} \times N^+ \nrightarrow N$:

$$FLOOR(X, s) \stackrel{def}{=} \begin{cases} e & \text{if } \exists\pi \in floor(X, s)(y(\pi) = e) \\ \neg! & \text{if } floor(X, s) = \phi \end{cases}$$

We prove with induction on i that $\forall i \forall A, B \in I^{\chi}_{states}(L(A) = L(B) \Rightarrow floor(A, i) = floor(B, i))$.

1) $i = 1$

Let $A, B \in I^{\chi}_{states}$ and $L(A) = L(B)$. $floor(A, 1) = min(A) = min(B) = floor(B, 1)$

2) Induction hypothesis: $\forall j \leq i \forall A, B \in I^{\chi}_{states}(L(A) = L(B) \Rightarrow floor(A, j) = floor(B, j))$.

We have to prove that $\forall A, B \in I^{\chi}_{states}(L(A) = L(B) \Rightarrow floor(A, i + 1) = floor(B, i + 1))$. Let $A, B \in I^{\chi}_{states}$ and $L(A) = L(B)$. Let us suppose that $floor(A, i + 1) \neq floor(B, i + 1)$.

2.1) $\exists\pi \in floor(A, i + 1)(\pi \notin floor(B, i + 1))$

2.1.1) $\exists\pi \in floor(A, i + 1)\exists t\exists r(\pi \notin floor(B, i + 1)\ \&\ \pi = I + t^{\#r})$

Let $\pi = I+t^{\#r} \in floor(A, i+1)$ and $\pi \notin floor(B, i+1)$. Hence $!FLOOR(A, i+1)$ and $!FLOOR(A, 1)$. Let $f = FLOOR(A, i + 1) - FLOOR(A, 1)$ and $x = 0^{n+t}1^{n+1-t}$. We build rows $\{A_\alpha\}^f_{\alpha=0}$ and $\{B_\alpha\}^f_{\alpha=0}$ in such way that

$$\begin{array}{l} \forall\alpha \in [0, f](\\ \qquad A_\alpha, B_\alpha \in I^{\chi}_{states}\ \& \\ \qquad L(A_\alpha) = L(B_\alpha)\ \& \\ \qquad \pi \in A_\alpha\ \&\ \pi \notin B_\alpha\ \& \\ \qquad \neg\exists\pi'(\pi' <^{\chi}_{s} \pi\ \&\ \pi' \in A_\alpha \bigcup B_\alpha)\ \& \\ \qquad MIN(A_\alpha) = MIN(B_\alpha) = MIN(A) + \alpha). \end{array}$$

$A_0 = A$, $B_0 = B$. It follows from the induction hypothesis that $\neg\exists\pi'(\pi' <_s^\chi \pi \,\&\, \pi' \in A_0 \bigcup B_0)$. Let us suppose that A_α and B_α are built in such way that

$$\begin{array}{l} \alpha < f\,\& \\ A_\alpha, B_\alpha \in I^\chi_{states}\,\& \\ L(A_\alpha) = L(B_\alpha)\,\& \\ \pi \in A_\alpha \,\&\, \pi \notin B_\alpha\,\& \\ \neg\exists\pi'(\pi' <_s^\chi \pi \,\&\, \pi' \in A_\alpha \bigcup B_\alpha)\,\& \\ MIN(A_\alpha) = MIN(B_\alpha) = MIN(A) + \alpha. \end{array}$$

We have to build $A_{\alpha+1}$ and $B_{\alpha+1}$.

2.1.1.1) $\chi = \epsilon$

It follows from the definition of $\delta_n^{\forall,\epsilon}$ that $!\delta_n^{\forall,\epsilon}(A_\alpha, x)$. Let $A_{\alpha+1} = \delta_n^{\forall,\epsilon}(A_\alpha, x)$. Obviously $A_{\alpha+1} \in I^\epsilon_{states}$, $\pi \in A_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^\epsilon \pi \,\&\, \pi' \in A_{\alpha+1})$. We have also $!\delta_n^{\forall,\epsilon}(B_\alpha, x)$ (otherwise $L(A_\alpha) \neq L(B_\alpha)$). Let $B_{\alpha+1} = \delta_n^{\forall,\epsilon}(B_\alpha, x)$. Obviously $B_{\alpha+1} \in I^\alpha_{states}$ and $L(A_{\alpha+1}) = L(B_{\alpha+1})$. It follows from $\pi \notin B_\alpha$ and $\neg\exists\pi'(\pi' <_s^\epsilon \pi \,\&\, \pi' \in B_\alpha)$ that $\pi \notin B_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^\epsilon \pi \,\&\, \pi' \in B_{\alpha+1})$. Obviously $MIN(A_{\alpha+1}) = MIN(B_{\alpha+1}) = MIN(A_\alpha) + 1 = MIN(A) + \alpha + 1$.

2.1.1.2) $\chi = t$

It follows from the definition of $\delta_n^{\forall,t}$ that $!\delta_n^{\forall,t}(A_\alpha, x)$. Let $A_{\alpha+1} = \delta_n^{\forall,t}(A_\alpha, x)$. Obviously $A_{\alpha+1} \in I^t_{states}$. It follows from $\pi \in A_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^t \pi \,\&\, \pi' \in A_\alpha)$ that $\pi \in A_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^t \pi \,\&\, \pi' \in A_{\alpha+1})$. (If we suppose that $I + t_t^{\#b} \in A_\alpha$ and $b < r$ then it follows from *Proposition* 21 that $I + t + 1^{\#b} \in A_\alpha$. But $I + t + 1^{\#b} <_s^t \pi$. Contradiction.) We have $!\delta_n^{\forall,t}(B_\alpha, x)$ (otherwise $L(A_\alpha) \neq L(B_\alpha)$). Let $B_{\alpha+1} = \delta_n^{\forall,t}(B_\alpha, x)$. Obviously $B_{\alpha+1} \in I^t_{states}$ and $L(A_{\alpha+1}) = L(B_{\alpha+1})$. It follows from $\pi \notin B_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^t \pi \,\&\, \pi' \in B_\alpha)$ that $\pi \notin B_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^t \pi \,\&\, \pi' \in B_{\alpha+1})$. Obviously $MIN(A_{\alpha+1}) = MIN(B_{\alpha+1}) = MIN(A_\alpha) + 1 = MIN(A) + \alpha + 1$.

2.1.1.3) $\chi = ms$

It follows from the definition of $\delta_n^{\forall,ms}$ that $!\delta_n^{\forall,ms}(A_\alpha, x)$.

2.1.1.3.1) $\exists\pi' \in min(A_\alpha)\exists i\exists e(\pi' = I + i_s^{\#e})$

Obviously $!\delta_n^{\forall,ms^*}(A_\alpha, xx)$. Let $A_{\alpha+1} = \delta_n^{\forall,ms^*}(A_\alpha, xx)$. We have also $!\delta_n^{\forall,ms^*}(B_\alpha, xx)$. Let $B_{\alpha+1} = \delta_n^{\forall,ms^*}(B_\alpha, xx)$.

2.1.1.3.2) $\neg\exists\pi' \in min(A_\alpha)\exists i\exists e(\pi' = I + i_s^{\#e})$

Let $A_{\alpha+1} = \delta_n^{\forall,ms}(A_\alpha, x)$. Obviously $!\delta_n^{\forall,ms}(B_\alpha, x)$. Let $B_{\alpha+1} = \delta_n^{\forall,ms}(B_\alpha, x)$.

Obviously $A_{\alpha+1} \in I^{ms}_{states}$, $B_{\alpha+1} \in I^{ms}_{states}$, $L(A_{\alpha+1}) = L(B_{\alpha+1})$ and $MIN(A_{\alpha+1}) = MIN(B_{\alpha+1}) = MIN(A_\alpha) + 1 = MIN(A) + \alpha + 1$. It follows from $\pi \in A_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^{ms} \pi \,\&\, \pi' \in A_\alpha)$ that $\pi \in A_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^{ms} \pi \,\&\, \pi' \in A_{\alpha+1})$. (If we suppose that $I + a_s^{\#b} \in A_\alpha$ and $I + a^{\#b} <_s^{ms} \pi$ then it follows from *Proposition* 21 that $I + a^{\#b} \in A_\alpha$. Contradiction.) It follows from $\pi \notin B_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^{ms} \pi \,\&\, \pi' \in B_\alpha)$ that $\pi \notin B_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^{ms} \pi \,\&\, \pi' \in B_{\alpha+1})$. (If we suppose that $I + t_s^{\#r} \in B_\alpha$ then it follows from *Proposition* 21 that $\pi \in B_\alpha$. Contradiction.)

2.1.2) $\chi = t$ and $\exists\pi \in floor(A, i+1)\exists t\exists r(\pi \notin floor(B, i+1) \,\&\, \pi = I + t_t^{\#r})$

Let $\pi = I + t_t^{\#r}$, $\pi \in floor(A, i+1)$ and $\pi \notin floor(B, i+1)$. It follows from *Proposition* 21 that $I + t + 1^{\#r} \in A$. Hence $\neg\exists\pi'(\pi' <_s^t I + t +$

$1^{\#r}\,\&\,\pi' \in A)$. It follows from the induction hypothesis that $\neg\exists\pi'(\pi' <_s^t I+t+1^{\#r}\,\&\,\pi' \in B)$. Obviously $!\delta_n^{\forall,t}(A, 0^{n+t}1^{n+1-t})$ and $!\delta_n^{\forall,t}(B, 0^{n+t}1^{n+1-t})$. Let $A' = \delta_n^{\forall,t}(A, 0^{n+t}1^{n+1-t})$ and $B' = \delta_n^{\forall,t}(B, 0^{n+t}1^{n+1-t})$. Obviously $MIN(A') = MIN(B')$, $I+t+1^{\#r} \in A'$, $I+t+1^{\#r} \notin B'$ and $\neg\exists\pi'(\pi' <_s^t I+t+1^{\#r}\,\&\,\pi' \in A' \bigcup B')$. Let $f = r - FLOOR(A', 1)$ and $x = 0^{n+t+1}1^{n-t}$. Like in 2.1.1.2) we build rows $\{A_\alpha\}_{\alpha=0}^f$ and $\{B_\alpha\}_{\alpha=0}^f$ ($A_0 = A'$ and $B_0 = B'$).

2.1.3) $\chi = ms$ and $\exists\pi \in floor(A, i+1)\exists t\exists r(\pi \notin floor(B, i+1)\&\pi = I+t_s^{\#r})$

Let $\pi = I + t_s^{\#r}$, $\pi \in floor(A, i+1)$ and $\pi \notin floor(B, i+1)$. It follows from *Proposition* 21 that $I + t^{\#r} \in A$. Hence $\neg\exists\pi'(\pi' <_s^{ms} I + t^{\#r}\,\&\,\pi' \in A)$. It follows from the induction hypothesis that $\neg\exists\pi'(\pi' <_s^{ms} I + t^{\#r}\,\&\,\pi' \in B)$. Obviously $!\delta_n^{\forall,ms}(A, 0^{2n+2})$ and $!\delta_n^{\forall,ms}(B, 0^{2n+2})$. Let $A' = \delta_n^{\forall,ms}(A, 0^{2n+2})$ and $B' = \delta_n^{\forall,ms}(B, 0^{2n+2})$. Obviously $MIN(A') = MIN(B')$, $I+t^{\#r} \in A'$, $I+t^{\#r} \notin B'$ and $\neg\exists\pi'(\pi' <_s^{ms} I + t^{\#r}\,\&\,\pi' \in A' \bigcup B')$. Let $f = r - FLOOR(A', 1)$ and $x = 0^{n+t}1^{n+1-t}$. Like in 2.1.1.3) we build rows $\{A_\alpha\}_{\alpha=0}^f$ and $\{B_\alpha\}_{\alpha=0}^f$ ($A_0 = A'$ and $B_0 = B'$).

The rows $\{A_\alpha\}_{\alpha=0}^f$ and $\{B_\alpha\}_{\alpha=0}^f$ are built. Therefore $MIN(A_f) = MIN(B_f) = MIN(A_0) + f = MIN(A_0) + r - FLOOR(A_0, 1) = y(\pi)$. Hence $\pi \in min(A_f)$, $\pi \notin min(B_f)$ and $L(A_f) = L(B_f)$. Contradiction. Therefore $floor(A, i+1) = floor(B, i+1)$.

2.2) $\exists\pi \in floor(B, i+1)(\pi \neq floor(A, i+1))$

Like in 2.1) we recive contradiction.

We proved that $\forall i\forall A, B \in I_{states}^\chi(L(A) = L(B) \Rightarrow floor(A, i) = floor(B, i))$.

Therefore $\forall A, B \in I_{states}^\chi(L(A) = L(B) \Rightarrow A = B)$.

Proposition 23 Let $\chi \in \{\epsilon, t, ms\}$. Let $n \in N$. Then $\forall A, B \in M_{states}^\chi(L(A) = L(B) \Rightarrow A = B)$.

Proof We define $y : M_s^\chi \to N$:

$y(M + i^{\#e}) \stackrel{def}{=} e$

$y(M + i_t^{\#e}) \stackrel{def}{=} e$

$y(M + i_s^{\#e}) \stackrel{def}{=} e$

We define $min : P(M_s^\chi) \to P(M_s^\chi)$:

$min(X) \stackrel{def}{=} \{\pi | \pi \in X\,\&\,\forall\pi' \in X(y(\pi') \geq y(\pi))\}$

Lemma $\forall A, B \in M_{states}^\chi(L(A) = L(B) \Rightarrow min(A) = min(B))$

Proof Let $A, B \in M_{states}^\chi$ and $L(A) = L(B)$. We define $MIN : M_{states}^\chi \to N$:

$MIN(X) = e \stackrel{def}{\Leftrightarrow} \exists\pi \in min(X)(y(\pi) = e)$

Let us suppose that $min(A) \neq min(B)$.

1) $MIN(A) < MIN(B)$

Using induction we build rows $\{A_i\}_{i=0}^\infty$ and $\{B_i\}_{i=0}^\infty$ such that

$\forall i \in N($
$A_i \in M^{\chi}_{states}$ &
$B_i \in M^{\chi}_{states}$ &
$MIN(A_{i+1}) > MIN(A_i)$ &
$MIN(B_{i+1}) > MIN(B_i)$ &
$MIN(A_i) < MIN(B_i)$ &
$L(A_i) = L(B_i))$.

$A_0 = A$, $B_0 = B$. Let us suppose that A_i and B_i are built such that

$A_i \in M^{\chi}_{states}$, $B_i \in M^{\chi}_{states}$, $i > 0 \Rightarrow MIN(A_i) > MIN(A_{i-1})$, $i > 0 \Rightarrow MIN(B_i) > MIN(B_{i-1})$, $MIN(A_i) < MIN(B_i)$ and $L(A_i) = L(B_i)$.

We have to build A_{i+1} and B_{i+1}. Let $b_1 = 0^k$ where k is some element of $\bigtriangledown_a(A_i)$. We have $MIN(A_i) < MIN(B_i) \leq n$.

1.1) $\chi = \epsilon$

Therefore $!\delta^{\forall,\epsilon}_n(A_i, b_1)$ ($MIN(A_i) < n$) and $MIN(\delta^{\forall,\epsilon}_n(A_i, b_1)) = MIN(A_i)+1$. Let $A_{i+1} = \delta^{\forall,\epsilon}_n(A_i, b_1)$. We have $!\delta^{\forall,\epsilon}_n(B_i, b_1)$ (otherwise $L(A_i) \neq L(B_i)$). Therefore $MIN(\delta^{\forall,\epsilon}_n(B_i, b_1)) = MIN(B_i)+1$. Let $B_{i+1} = \delta^{\forall,\epsilon}_n(B_i, b_1)$. We have also $L(A_{i+1}) = L(B_{i+1})$ (otherwise $L(A_i) \neq L(B_i)$). Obviously $MIN(A_{i+1}) < MIN(B_{i+1})$. We prove that $A_{i+1} \in M^{\epsilon}_{states}$ and $B_{i+1} \in M^{\epsilon}_{states}$. We have $A_{i+1} \in M^{\epsilon}_{states} \Leftrightarrow B_{i+1} \in M^{\epsilon}_{states}$ (otherwise $L(A_i) \neq L(B_i)$). Let us suppose that $A_{i+1} \in I^{\epsilon}_{states}$ and $B_{i+1} \in I^{\epsilon}_{states}$. It follows from $MIN(A_{i+1}) < MIN(B_{i+1})$ that $A_{i+1} \neq B_{i+1}$. Applying *Proposition* 22 we receive that $L(A_{i+1}) \neq L(B_{i+1})$. Contradiction. Therefore $A_{i+1} \in M^{\epsilon}_{states}$ and $B_{i+1} \in M^{\epsilon}_{states}$.

1.2) $\chi = t$

It follows from *Proposition* 21 that $\exists j \exists e(M + j^{\#e} \in min(A_i))$ (if $M + j^{\#e}_t \in min(A_i)$ then $M + j + 1^{\#e} \in min(A_i)$). Therefore $!\delta^{\forall,t}_n(A_i, b_1)$ and $MIN(\delta^{\forall,t}_n(A_i, b_1)) = MIN(A_i) + 1$, $!\delta^{\forall,t}_n(B_i, b_1)$ (otherwise $L(A_i) \neq L(B_i)$) and $MIN(\delta^{\forall,t}_n(B_i, b_1)) = MIN(B_i) + 1$. Let $A_{i+1} = \delta^{\forall,t}_n(A_i, b_1)$ and $B_{i+1} = \delta^{\forall,t}_n(B_i, b_1)$. Obviously $L(A_{i+1}) = L(B_{i+1})$ and $MIN(A_{i+1}) < MIN(B_{i+1})$ Like in 1.1) we prove that $A_{i+1} \in M^{t}_{states}$ and $B_{i+1} \in M^{t}_{states}$.

1.3) $\chi = ms$

1.3.1) $min(A_i) \neq \{M^{\#MIN(A_i)}\}$

Let $b' = 1^{k_1}$ where k_1 is some element of $\bigtriangledown_a(A_i)$. Obviously $!\delta^{\forall,ms}_n(A_i, b')$. Let $A' = \delta^{\forall,ms}_n(A_i, b')$. Obviously $MIN(A') = MIN(A_i)$ and $\neg\exists\pi \in min(A')\exists j \exists e(\pi = M + j^{\#e}_s)$. We have $!\delta^{\forall,ms}_n(B_i, b')$ (otherwise $L(A_i) \neq L(B_i)$). Let $B' = \delta^{\forall,ms}_n(B_i, b')$. Obviously $MIN(B') = MIN(B_i) + 1$ or ($MIN(B') = MIN(B_i)$ and $\neg\exists\pi \in min(B')\exists j \exists e(\pi = M + j^{\#e}_s)$). Like in 1.1) we prove that $A' \in M^{ms}_{states}$ and $B' \in M^{ms}_{states}$. Let $b'' = 0^{k_2}$ where k_2 is some element of $\bigtriangledown_a(A')$. Therefore $!\delta^{\forall,ms}_n(A', b'')$ and $!\delta^{\forall,ms}_n(B', b'')$. Let $A_{i+1} = \delta^{\forall,ms}_n(A', b'')$ and $B_{i+1} = \delta^{\forall,ms}_n(A', b'')$. We have $MIN(A_{i+1}) = MIN(A_i)+1$, $MIN(B_{i+1}) \geq MIN(B_i)+1$, $A_{i+1} \in M^{ms}_{states}$ and $B_{i+1} \in M^{ms}_{states}$.

1.3.2) $min(A_i) = \{M^{\#MIN(A_i)}\}$

In this case $max(\bigtriangledown_a(A_i)) = MIN(A_i) + n < max(\bigtriangledown_a(B_i))$ (fig. 23).

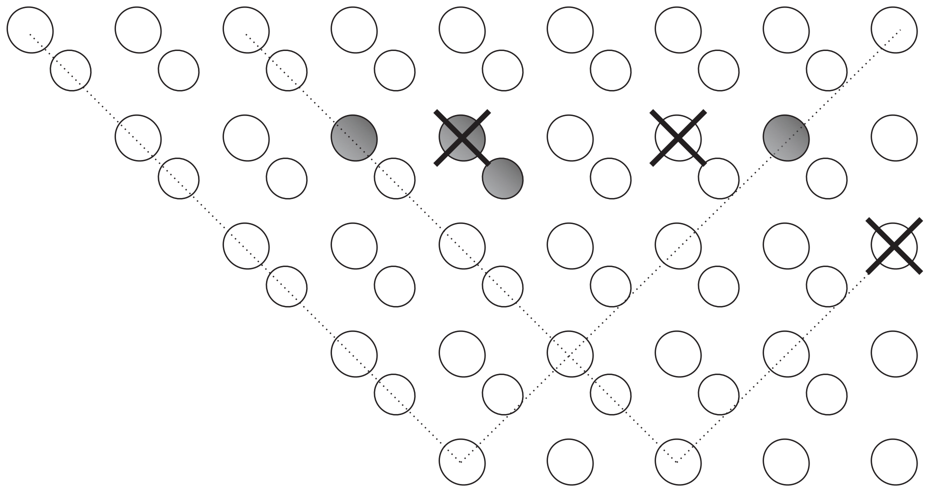

Fig. 24 $n = 4$, $A_i = \{M^{\#2}, M - 2^{\#3}, M - 4^{\#3}\}$,
$B_i = \{M - 1^{\#3}, M - 4^{\#3}, M - 5^{\#3}, M_s - 4^{\#3}\}$,
$max(\bigtriangledown_a(A_i)) = 6$, $max(\bigtriangledown_a(B_i)) = 8$

1.3.2.1) $B_i = \{M^{\#n}\}$

Obviously $L(A_i) \neq L(B_i)$. Contradiction.

1.3.2.2) $B_i \neq \{M^{\#n}\}$

Let $b = 1^{max(\bigtriangledown_a(B_i))}$. We have $!\delta_n^{\forall,ms}(B_i, b)$ and $\neg!\delta_n^{\forall,ms}(A_i, b)$. Hence $L(A_i) \neq L(B_i)$. Contradiction.

The rows $\{A_i\}_{i=0}^{\infty}$ and $\{B_i\}_{i=0}^{\infty}$ are built. But such rows cannot exist. Contradiction.

2) $MIN(B) < MIN(A)$

Like in 1) we receive contradiction.

3) $MIN(A) = MIN(B) = m$

3.1) $\exists\pi \in min(A)(\pi \notin min(B))$

Let $\pi \in min(A)$ and $\pi \notin min(B)$.

3.1.1) $\chi = \epsilon$

Let $\pi = M + i^{\#m}$.

3.1.1.1) $i < 0$

Let $c = 0^{k+i}10^{-i-1}$ where k is some element of $\bigtriangledown_a(A)$. Obviously $!\delta_n^{\forall,\epsilon}(A, c)$ and $MIN(\delta_n^{\forall,\epsilon}(A, c)) = m$. Therefore $!\delta_n^{\forall,\epsilon}(B, c)$ and $MIN(\delta_n^{\forall,\epsilon}(B, c)) = m+1$. We have $MIN(\delta_n^{\forall,\epsilon}(A, c)) < MIN(\delta_n^{\forall,\epsilon}(B, c))$. Obviously $\delta_n^{\forall,\epsilon}(A, c) \in M_{states}^{\epsilon}$, $\delta_n^{\forall,\epsilon}(B, c) \in M_{states}^{\epsilon}$ and $L(\delta_n^{\forall,\epsilon}(A, c)) = L(\delta_n^{\forall,\epsilon}(B, c))$. Like in 1) we receive contradiction.

3.1.1.2) $i = 0$

We prove that $m < n$. Let us suppose that $m = n$. Hence $M^{\#n} \notin min(B)$. Hence $\neg\exists\pi \in B\exists j\exists f(\pi = M + j^{\#f} \;\&\; f \leq j + n)$. Therefore $B \notin M_{states}^{\epsilon}$. Contradiction. So $m < n$. Obviously $max(\bigtriangledown_a(A)) = m + n < max(\bigtriangledown_a(B))$.

Let $b = 1^{max(\bigtriangledown_a(B))}$. Therefore $!\delta_n^{\forall,\epsilon}(B,b)$ and $\neg!\delta_n^{\forall,\epsilon}(A,b)$. Hence $L(A) \neq L(B)$. Contradiction.

3.1.2) $\chi = t$

3.1.2.1) $\pi = M + i^{\#m}$

3.1.1.1.1) $i < 0$

Let $c = 0^{k+i}10^{-i-1}$ where k is some element of $\bigtriangledown_a(A)$. Obviously $!\delta_n^{\forall,t}(A,c)$ and $MIN(\delta_n^{\forall,t}(A,c)) = m$. Therefore $!\delta_n^{\forall,t}(B,c)$ and $MIN(\delta_n^{\forall,t}(B,c)) = m$ (otherwise $L(A) \neq L(B)$). Therefore $M + i_t^{\#m} \in B$. Hence $i \leq -2$. Let $A' = \delta_n^{\forall,t}(A,c)$ and $B' = \delta_n^{\forall,t}(B,c)$. Obviously $A' \neq B'$ and $L(A') = L(B')$. Hence $A' \in M_{states}^t$ and $B' \in M_{states}^t$. Obviously $M + i + 1^{\#m} \in A'$, $M + i + 1^{\#m} \notin B'$ and $\neg\exists\pi'\exists j(\pi' = M + j_t^{\#m}$ & $\pi' \in min(A')\bigcup min(B'))$. Let $c' = 0^{k+i+1}10^{-i-2}$. Obviously $!\delta_n^{\forall,t}(A',c')$. Hence $!\delta_n^{\forall,t}(B',c')$ (otherwise $L(A') \neq L(B')$). Obviously $MIN(\delta_n^{\forall,t}(A',c')) = m$, $MIN(\delta_n^{\forall,t}(B',c')) = m+1$ and $L(\delta_n^{\forall,t}(A',c')) = L(\delta_n^{\forall,t}(B',c'))$. Like in 1) we receive contradiction.

3.1.1.1.2) $i = 0$

Like in 3.1.1.2) we receive contradiction.

3.1.2.2) $\pi = M + i_t^{\#m}$

Let $c = 0^{k+i}10^{-i-1}$ where k is some element of $\bigtriangledown_a(A)$. Obviously $!\delta_n^{\forall,t}(A,c)$, $MIN(\delta_n^{\forall,t}(A,c)) = m$, $!\delta_n^{\forall,t}(B,c)$, $MIN(\delta_n^{\forall,t}(B,c)) = m$, $\delta_n^{\forall,t}(A,c) \in M_{states}^t$, $\delta_n^{\forall,t}(B,c) \in M_{states}^t$, $M + i + 2^{\#m} \in \delta_n^{\forall,t}(A,c)$, $M + i + 2^{\#m} \notin \delta_n^{\forall,t}(B,c)$ and $L(\delta_n^{\forall,t}(A,c)) = L(\delta_n^{\forall,t}(B,c))$. Like in 3.1.2.1) we receive contradiction.

3.1.3) $\chi = ms$

3.1.3.1) $\pi = M + i^{\#m}$

3.1.3.1.1) $i < 0$

Let $c = 0^{k+i}10^{-i-1}$ where k is some element of $\bigtriangledown_a(A)$. Obviously $!\delta_n^{\forall,ms}(A,c)$ and $MIN(\delta_n^{\forall,ms}(A,c)) = m$. Therefore $!\delta_n^{\forall,ms}(B,c)$ and $MIN(\delta_n^{\forall,ms}(B,c)) = m$ (otherwise $L(A) \neq L(B)$). Let $A' = \delta_n^{\forall,ms}(A,c)$ and $B' = \delta_n^{\forall,ms}(B,c)$. Obviously $A' \neq B'$ and $L(A') = L(B')$. Hence $A' \in M_{states}^{ms}$ and $B' \in M_{states}^{ms}$. Obviously $M+i+1^{\#m} \in A'$, $M+i+1^{\#m} \notin B'$ (if we suppose that $M+i_s^{\#m} \in B$ then it follows from *Proposition* 21 that $M + i^{\#m} = \pi \in B$ - contradiction) and $\neg\exists\pi'\exists j(\pi' = M + j_s^{\#m}$ & $\pi' \in min(A')\bigcup min(B'))$. We have $i + 1 < 0$ (if we suppose that $i + 1 = 0$ then like in 3.1.1.2) we receive contradiction). Let $c' = 0^{k+i+1}10^{-i-2}$. Obviously $!\delta_n^{\forall,ms}(A',c')$, $!\delta_n^{\forall,ms}(B',c')$, $MIN(\delta_n^{\forall,ms}(A',c')) = m$, $MIN(\delta_n^{\forall,ms}(B',c')) = m+1$ and $L(\delta_n^{\forall,ms}(A',c')) = L(\delta_n^{\forall,ms}(B',c'))$. Like in 1) we receive contradiction.

3.1.3.1.2) $i = 0$

Like in 3.1.1.2) we receive contradiction.

3.1.3.2) $\pi = M + i_s^{\#m}$

Let $c = 0^k$ where k is some element of $\bigtriangledown_a(A)$. Obviously $!\delta_n^{\forall,ms}(A,c)$, $MIN(\delta_n^{\forall,ms}(A,c)) = m$, $!\delta_n^{\forall,ms}(B,c)$, $MIN(\delta_n^{\forall,ms}(B,c)) = m$, $\delta_n^{\forall,ms}(A,c) \in M_{states}^{ms}$, $\delta_n^{\forall,ms}(B,c) \in M_{states}^{ms}$, $M + i + 1^{\#m} \in \delta_n^{\forall,ms}(A,c)$, $M + i + 1^{\#m} \notin \delta_n^{\forall,ms}(B,c)$ and $L(\delta_n^{\forall,ms}(A,c)) = L(\delta_n^{\forall,ms}(B,c))$. Like in 3.1.3.1) we receive contradiction.

3.2) $\exists\pi \in min(B)(\pi \notin min(A))$

Like in 3.1) we receive contradiction.

The *Lemma* is proved.

We define $floor : P(M_s^\chi) \times N^+ \to P(M_s^\chi)$:

$$floor(X, 1) \stackrel{def}{=} min(X)$$

$$floor(X, i+1) \stackrel{def}{=} min(X \backslash (\bigcup_{1 \leq j \leq i} floor(X, j)))$$

We define $FLOOR : M_{states}^\chi \times N^+ \not\to N$.

$$FLOOR(X, s) \stackrel{def}{=} \begin{cases} e, & \text{if } \exists \pi \in floor(X, s)(y(\pi) = e) \\ \neg!, & \text{if } floor(X, s) = \phi \end{cases}$$

We prove with induction on i that $\forall i \forall A, B \in M_{states}^\chi(L(A) = L(B) \Rightarrow floor(A, i) = floor(B, i))$.

1) $i = 1$

Let $A, B \in M_{states}^\chi$ and $L(A) = L(B)$. $floor(A, 1) = min(A) = min(B) = floor(B, 1)$

2) Induction hypothesis: $\forall j \leq i \forall A, B \in M_{states}^\chi(L(A) = L(B) \Rightarrow floor(A, j) = floor(B, j))$.

We have to prove that $\forall A, B \in M_{states}^\chi(L(A) = L(B) \Rightarrow floor(A, i+1) = floor(B, i+1))$. Let $A, B \in M_{states}^\chi$ and $L(A) = L(B)$. Let us suppose that $floor(A, i+1) \neq floor(B, i+1)$.

2.1) $\exists \pi \in floor(A, i+1)(\pi \notin floor(B, i+1))$

2.1.1) $\exists \pi \in floor(A, i+1) \exists t \exists r(\pi \notin floor(B, i+1) \,\&\, \pi = M + t^{\#r})$

Let $\pi = M + t^{\#r} \in floor(A, i+1)$ and $\pi \notin floor(B, i+1)$. We build rows $\{A_\alpha\}_{\alpha=0}^{-t}$ and $\{B_\alpha\}_{\alpha=0}^{-t}$ in such way that

$$\begin{aligned} \forall \alpha \in [0, -t](& \\ & A_\alpha, B_\alpha \in M_{states}^\chi \,\& \\ & L(A_\alpha) = L(B_\alpha) \,\& \\ & M + t + \alpha^{\#r} \in A_\alpha \,\&\, M + t + \alpha^{\#r} \notin B_\alpha \,\& \\ & \neg\exists \pi'(\pi' <_s^\chi M + t + \alpha^{\#r} \,\&\, \pi' \in A_\alpha \bigcup B_\alpha)). \end{aligned}$$

$A_0 = A$, $B_0 = B$. It follows from the induction hypothesis that $\neg\exists\pi'(\pi' <_s^\chi M + t^{\#r} \,\&\, \pi' \in A_0 \bigcup B_0)$. Let us suppose that A_α and B_α are build in such way that

$$\begin{aligned} & \alpha < -t \,\& \\ & A_\alpha, B_\alpha \in M_{states}^\chi \,\& \\ & L(A_\alpha) = L(B_\alpha) \,\& \\ & M + t + \alpha^{\#r} \in A_\alpha \,\&\, M + t + \alpha^{\#r} \notin B_\alpha \,\& \\ & \neg\exists \pi'(\pi' <_s^\chi M + t + \alpha^{\#r} \,\&\, \pi' \in A_\alpha \bigcup B_\alpha). \end{aligned}$$

We have to build $A_{\alpha+1}$ and $B_{\alpha+1}$. Let $x = 0^{k+t}10^{-t-1}$ where k is some element of $\bigtriangledown_a(A_\alpha)$.

2.1.1.1) $\chi = \epsilon$

It follows from the definition of $\delta_n^{\forall,\epsilon}$ that $!\delta_n^{\forall,\epsilon}(A_\alpha, x)$. Hence $!\delta_n^{\forall,\epsilon}(B_\alpha, x)$. Let $A_{\alpha+1} = \delta_n^{\forall,\epsilon}(A_\alpha, x)$ and $B_{\alpha+1} = \delta_n^{\forall,\epsilon}(B_\alpha, x)$. It follows from $M + t + \alpha^{\#r} \in A_\alpha$, $M + t + \alpha^{\#r} \notin B_\alpha$ and $\neg\exists\pi'(\pi' <_s^\chi M + t + \alpha^{\#r} \,\&\, \pi' \in A_\alpha \bigcup B_\alpha)$ that $A_{\alpha+1} \neq B_{\alpha+1}$. Hence $A_{\alpha+1} \in M_{states}^\epsilon$ and $B_{\alpha+1} \in M_{states}^\epsilon$. Since $M + t + \alpha^{\#r} \in A_\alpha$ we have $M + t + \alpha + 1^{\#r} \in A_{\alpha+1}$. Hence $\neg\exists\pi'(\pi' <_s^\epsilon M + t + \alpha + 1^{\#r} \,\&\, \pi' \in A_{\alpha+1})$. It follows from $\neg\exists\pi'(\pi' <_s^\epsilon M + t + \alpha^{\#r} \,\&\, \pi' \in B_\alpha)$ and $M + t + \alpha^{\#r} \notin B_\alpha$ that $M + t + \alpha + 1^{\#r} \notin B_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^\epsilon M + t + \alpha + 1^{\#r} \,\&\, \pi' \in B_{\alpha+1})$. Obviously $L(A_{\alpha+1}) = L(B_{\alpha+1})$.

2.1.1.2) $\chi = t$

It follows from the definition of $\delta_n^{\forall,t}$ that $!\delta_n^{\forall,t}(A_\alpha, x)$. Hence $!\delta_n^{\forall,t}(B_\alpha, x)$. Let $A_{\alpha+1} = \delta_n^{\forall,t}(A_\alpha, x)$ and $B_{\alpha+1} = \delta_n^{\forall,t}(B_\alpha, x)$. Obviously $A_{\alpha+1} \neq B_{\alpha+1}$. Hence $A_{\alpha+1} \in M_{states}^t$ and $B_{\alpha+1} \in M_{states}^t$. Since $\neg\exists\pi'(\pi' <_s^t M + t + \alpha^{\#r} \,\&\, \pi' \in A_\alpha)$ and $M + t + \alpha^{\#r} \in A_\alpha$ we have $M + t + \alpha + 1^{\#r} \in A_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^t M + t + \alpha + 1^{\#r} \,\&\, \pi' \in A_{\alpha+1})$. (If we suppose that $M + t + \alpha_t^{\#b} \in A_\alpha$ and $b < r$ then it follows from *Proposition* 21 that $M + t + \alpha + 1^{\#b} \in A_\alpha$. But $M + t + \alpha + 1^{\#b} <_s^t M + t + \alpha^{\#r}$. Contradiction.) It follows from $\neg\exists\pi'(\pi' <_s^t M + t + \alpha^{\#r} \,\&\, \pi' \in B_\alpha)$ and $M + t + \alpha^{\#r} \notin B_\alpha$ that $M + t + \alpha + 1^{\#r} \notin B_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^t M + t + \alpha + 1^{\#r} \,\&\, \pi' \in B_{\alpha+1})$. Obviously $L(A_{\alpha+1}) = L(B_{\alpha+1})$.

2.1.1.3) $\chi = ms$

It follows from the definition of $\delta_n^{\forall,ms}$ that $!\delta_n^{\forall,ms}(A_\alpha, x)$. Hence $!\delta_n^{\forall,ms}(B_\alpha, x)$. Let $A_{\alpha+1} = \delta_n^{\forall,ms}(A_\alpha, x)$ and $B_{\alpha+1} = \delta_n^{\forall,ms}(B_\alpha, x)$. Obviously $A_{\alpha+1} \neq B_{\alpha+1}$. Hence $A_{\alpha+1} \in M_{states}^{ms}$ and $B_{\alpha+1} \in M_{states}^{ms}$. Since $\neg\exists\pi'(\pi' <_s^{ms} M + t + \alpha^{\#r} \,\&\, \pi' \in A_\alpha)$ and $M + t + \alpha^{\#r} \in A_\alpha$ we have $M + t + \alpha + 1^{\#r} \in A_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^{ms} M+t+\alpha+1^{\#r} \& \pi' \in A_{\alpha+1})$. (If we suppose that $M+t+\alpha_s^{\#b} \in A_\alpha$ and $b < r$ then it follows from *Proposition* 21 that $M + t + \alpha^{\#b} \in A_\alpha$. But $M + t + \alpha^{\#b} <_s^{ms} M + t + \alpha^{\#r}$. Contradiction.) It follows from $\neg\exists\pi'(\pi' <_s^{ms} M+t+\alpha^{\#r} \& \pi' \in B_\alpha)$ and $M+t+\alpha^{\#r} \notin B_\alpha$ that $M+t+\alpha+1^{\#r} \notin B_{\alpha+1}$ and $\neg\exists\pi'(\pi' <_s^{ms} M+t+\alpha+1^{\#r} \& \pi' \in B_{\alpha+1})$. (If we suppose that $M+t+\alpha_s^{\#r} \in B_\alpha$ then it follows from *Proposition* 21 that $M + t + \alpha^{\#r} \in B_\alpha$. Contradiction.) Obviously $L(A_{\alpha+1}) = L(B_{\alpha+1})$.

2.1.2) $\chi = t$ and $\exists\pi \in floor(A, i+1)\exists t\exists r(\pi \notin floor(B, i+1) \,\&\, \pi = M + t_t^{\#r})$

Let $\pi = M + t_t^{\#r}$, $\pi \in floor(A, i+1)$ and $\pi \notin floor(B, i+1)$. It follows from *Proposition* 21 that $M + t + 1^{\#r} \in A$. Hence $\neg\exists\pi'(\pi' <_s^t M + t + 1^{\#r} \,\&\, \pi' \in A)$. It follows from the induction hypothesis that $\neg\exists\pi'(\pi' <_s^t M + t + 1^{\#r} \,\&\, \pi' \in B)$. Let k_1 be some element of $\bigtriangledown_a(A)$. Obviously $!\delta_n^{\forall,t}(A, 0^{k_1+t+1}10^{-t-2})$. Hence $!\delta_n^{\forall,t}(B, 0^{k_1+t+1}10^{-t-2})$. Let $A' = \delta_n^{\forall,t}(A, 0^{k_1+t+1}10^{-t-2})$ and $B' = \delta_n^{\forall,t}(B, 0^{k_1+t+1}10^{-t-2})$. Obviously $M + t + 2^{\#r} \in A'$, $M + t + 2^{\#r} \notin B'$ and $\neg\exists\pi'(\pi' <_s^t M+t+2^{\#r} \& \pi' \in A' \bigcup B')$. Like in 2.1.1.2) we build rows $\{A_\alpha\}_{\alpha=0}^{-t-2}$ and $\{B_\alpha\}_{\alpha=0}^{-t-2}$ ($A_0 = A'$ and $B_0 = B'$).

2.1.3) $\chi = ms$ and $\exists\pi \in floor(A, i+1)\exists t\exists r(\pi \notin floor(B, i+1) \& \pi = M + t_s^{\#r})$

Let $\pi = M + t_s^{\#r}$, $\pi \in floor(A, i+1)$ and $\pi \notin floor(B, i+1)$. It follows from *Proposition* 21 that $M + t^{\#r} \in A$. Hence $\neg\exists\pi'(\pi' <_s^{ms} M + t^{\#r} \,\&\, \pi' \in A)$. It follows from the induction hypothesis that $\neg\exists\pi'(\pi' <_s^{ms} M + t^{\#r} \,\&\, \pi' \in B)$. Let k_1 be some element of $\bigtriangledown_a(A)$. Obviously $!\delta_n^{\forall,ms}(A, 0^{k_1})$. Hence $!\delta_n^{\forall,ms}(B, 0^{k_1})$. Let $A' = \delta_n^{\forall,ms}(A, 0^{k_1})$ and $B' = \delta_n^{\forall,ms}(B, 0^{k_1})$. Obviously $M + t + 1^{\#r} \in A'$,

$M+t+1^{\#r} \notin B'$ and $\neg\exists\pi'(\pi' <_s^{ms} M+t+1^{\#r} \,\&\, \pi' \in A' \bigcup B')$. Like in 2.1.1.3) we build rows $\{A_\alpha\}_{\alpha=0}^{-t-1}$ and $\{B_\alpha\}_{\alpha=0}^{-t-1}$ ($A_0 = A'$ and $B_0 = B'$).

The rows $\{A_\alpha\}_{\alpha=0}^{-t}$ and $\{B_\alpha\}_{\alpha=0}^{-t}$ are built. Therefore $M^{\#r} \in A_{-t}$, $M^{\#r} \notin B_{-t}$ and $\neg\exists\pi'(\pi' <_s^\chi M^{\#r} \,\&\, \pi' \in A_\alpha \bigcup B_\alpha)$. We have $r < n$ (otherwise $M^{\#n} \notin min(B_{-t})$ and $B_{-t} \notin M_{states}^\chi$). Obviously $max(\bigtriangledown_a(A_{-t})) = r + n < max(\bigtriangledown_a(B_{-t}))$. Let $b = 1^{max(\bigtriangledown_a(B_{-t}))}$. Therefore $!\delta_n^{\forall,\epsilon}(B_{-t}, b)$ and $\neg!\delta_n^{\forall,\epsilon}(A_{-t}, b)$. Hence $L(A) \neq L(B)$. Contradiction. Therefore $floor(A, i+1) = floor(B, i+1)$.

2.2) $\exists\pi \in floor(B, i+1)(\pi \neq floor(A, i+1))$

Like in 2.1) we receive contradiction.

We proved that $\forall i \forall A, B \in M_{states}^\chi(L(A) = L(B) \Rightarrow floor(A, i) = floor(B, i))$.

Therefore $\forall A, B \in M_{states}^\chi(L(A) = L(B) \Rightarrow A = B)$.

Corollary Let $\chi \in \{\epsilon, t, ms\}$. Let $n \in N$. It follows from *Proposition* 22 and *Proposition* 23 that $A_n^{\forall,\chi}$ is minimal.

8 Some properties of $A_n^{\forall,\epsilon}$.

Proposition 24 Let $n \in N$, $Q \subseteq I_s^\epsilon$, $Q \neq \phi$ and $\forall q_1, q_2 \in Q(q_1 \not<_s^\epsilon q_2)$. Then $\exists b \in \Sigma_n^{\forall *}(\delta_n^{\forall,\epsilon *}(\{I^{\#0}\}, b) = Q)$.

Proof

1) $Q = \{I^{\#0}\}$

$b = \epsilon$

2) $Q \neq \{I^{\#0}\}$

We build rows $\{b_i\}_{i=1}^{n+1}$ and $\{q_i\}_{i=1}^{n+1}$:

2.1) Let $b_1 = 0^n010^n$ and $q_1 = \delta_n^{\forall,\epsilon}(\{I^{\#0}\}, b_1)$

2.2) Let us supose that b_i and q_i are build

Let $b_{i+1} = x_1 x_2 ... x_{2n+2}$ where

$$x_j = \begin{cases} 1 & \text{if } I + j - n - 1^{\#e} \in Q \text{ for some } e \leq i \text{ or } j = n + 2 + i \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq j \leq 2n + 2$. Let $q_{i+1} = \delta_n^{\forall,\epsilon}(q_i, b_{i+1})$.

Obviously $\delta_n^{\forall,\epsilon *}(\{I^{\#0}\}, b) = Q$.

Proposition 25 Let $n \in N$, $Q \subseteq M_s^\epsilon$, $\forall q_1, q_2 \in Q(q_1 \not<_s^\epsilon q_2)$, $\exists q \in Q(q \leq_s^\epsilon M^{\#n})$ and $\exists i \in [-n, 0] \forall q \in Q(M + i^{\#0} \leq_s^\epsilon q)$. Then $\exists b \in \Sigma_n^{\forall *}(\delta_n^{\forall,\epsilon *}(\{I^{\#0}\}, b) = Q)$.

Proof

1) $Q = \{M^{\#0}\}$

$\delta_n^{\forall,\epsilon}(\{I^{\#0}\}, 0^n1) = Q$

2) $Q \neq \{M^{\#0}\}$

Let $i \in [-n, 0]$ be such that $\forall q \in Q(M + i^{\#0} \leq_s^\epsilon q)$. Let $Q_0 = \{I + j - i^{\#f} | M + j^{\#f} \in Q\}$. We define $s : I_{states}^\epsilon \to \Sigma_n^\forall$:

$s(A) \overset{def}{=} 1^m$ where $m = n + max\{x - y | I + x^{\#y} \in A\} + n + 1$.

Let $k = |s(Q_0)|$. We define $M_0 = \delta_n^{\forall,\epsilon}(Q_0, s(Q_0))$ and $M_{i+1} = \delta_n^{\forall,\epsilon}(M_i, 1^{k-(i+1)})$. Obviously $\exists p \in N(M_p = Q)$. Therefore $\exists b \in \Sigma_n^{\forall *}(\delta_n^{\forall,\epsilon *}(\{I^{\#0}\}, b) = Q)$.

Corollary Let $n \in N$. It follows from *Proposition* 24 and *Proposition* 25 that

I) $Q \in I_{states}^\epsilon \Leftrightarrow Q \subseteq I_s^\epsilon \;\&\; Q \neq \phi \;\&\; \forall q_1, q_2 \in Q(q_1 \not<_s^\epsilon q_2)$

II) $Q \in M_{states}^\epsilon \Leftrightarrow Q \subseteq M_s^\epsilon \;\&\; Q \neq \phi \;\&\; \forall q_1, q_2 \in Q(q_1 \not<_s^\epsilon q_2) \;\&\; max\{x - y | M + x^{\#y} \in Q\} \leq min\{x + y | M + x^{\#y} \in Q\} \;\&\; max\{x - y | M + x^{\#y} \in Q\} \geq -n$

Corollary $A_n^{\forall,\epsilon}$ can be computed in space $O(n^2)$. (We don't take into account the memory needed to write the output automaton.)

Instead of keeping all the states in a queue like in the algorithm in 6 we may use an enumeration of the states to pass through all the states. In this way the time needed to compute the automaton depends on the enumeration. In the following pseudo code we use one naive enumeration to illustrate the idea.

```
procedure Build_Automaton( n : INTEGER );
VAR st, nextSt : STATE;
    b          : STRING;
begin
  for st in P( { pos | pos : POSITION and
                       GET_POSITION_PARAM(pos) = I and
                       GET_POSITION_TYPE(pos) = usual and
                       GET_POSITION_Y(pos) >= GET_POSITION_X(pos) and
                       GET_POSITION_Y(pos) >= -GET_POSITION_X(pos) and
                       0 <= GET_POSITION_Y(pos) <= n } ) do begin
    if( Belongs_To_I_States(st) ) then begin
      for b in { sym | sym : STRING and
                       1 <= LENGTH(sym) <= 2n+2 and
                       for all i( i in [1, LENGTH(sym)] =>
                                  ( sym[i] = 0 or sym[i] = 1 ) ) } do begin
        if( Length_Covers_All_The_Positions( n, LENGTH(b), st ) ) then begin
          nextSt := Delta( n, st, b );
          if( not EMPTY_STATE( nextState ) ) then begin
            if( HAS_NEVER_BEEN_PUSHED( nextSt ) ) then begin
              PUSH_IN_QUEUE( nextState )
            end
            ADD_TRANSITION( st, b , nextSt )
          end
        end
      end
```

```
    end
  end

  for st in P( { pos | pos : POSITION and
                       GET_POSITION_PARAM(pos) = M and
                       GET_POSITION_TYPE(pos) = usual and
                       GET_POSITION_Y(pos) >= -GET_POSITION_X(pos) - n and
                       0 <= GET_POSITION_Y(pos) <= n and
                       GET_POSITION_X(pos) <= 0 } ) do begin
    if( Belongs_To_M_States(st) ) then begin
      for b in { sym | sym : STRING and
                       1 <= LENGTH(sym) <= 2n+2 and
                       for all i( i in [1, LENGTH(sym)] =>
                                  ( sym[i] = 0 or sym[i] = 1 ) ) } do begin
        if( Length_Covers_All_The_Positions( n, LENGTH(b), st ) ) then begin
          nextSt := Delta( n, st, b );
          if( not EMPTY_STATE( nextState ) ) then begin
            if( HAS_NEVER_BEEN_PUSHED( nextSt ) ) then begin
              PUSH_IN_QUEUE( nextState )
            end
            ADD_TRANSITION( st, b , nextSt )
          end
        end
      end
    end
  end
end;

function Belongs_To_I_States( st : STATE ) : BOOLEAN;
VAR q1, q2 : POSITION;
begin
  if( EMPTY_STATE( st : STATE ) ) then begin
    return( false )
  end
  for q1 in st do begin
    for q2 in st do begin
      if( Less_Than_Subsume( q1, q2 ) ) then begin
        return( false )
      end
    end
  end
  return( true )
end;

function Belongs_To_M_States( st : STATE ) : BOOLEAN;
VAR q1, q2, leftMost, rightMost : POSITION;
```

```
begin
  if( EMPTY_STATE( st : STATE ) ) then begin
    return( false )
  end
  leftMost := GET_FIRST_POSITION(st);
  rightMost := GET_FIRST_POSITION(st);
  for q1 in st do begin
    if( GET_POSITION_X(q1) + GET_POSITION_Y(q1) <
        GET_POSITION_X(leftMost) + GET_POSITION_Y(leftMost) ) then begin
      leftMost := q1
    end
    if( GET_POSITION_X(q1) - GET_POSITION_Y(q1) >
        GET_POSITION_X(rightMost) - GET_POSITION_Y(rightMost) ) then begin
      rightMost := q1
    end
    for q2 in st do begin
      if( Less_Than_Subsume( q1, q2 ) ) then begin
        return( false )
      end
    end
  end
  if( GET_POSITION_X(leftMost) + GET_POSITION_Y(leftMost) <
      GET_POSITION_X(rightMost) - GET_POSITION_Y(rightMost) ) then begin
    return( false )
  end
  return( true )
end;
```

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