STRUCTURE DEGREE JUMP

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The theory of the degrees of structures is a new area in the degree theory which has been studied for the last couple of years at the Faculty of Mathematics and Computer Science, University of Sofia, Bulgaria. The research was inspired by Prof. Soskov. Jojgov [5] studies minimal pairs of structure degrees, Bouchkova [3] considers 1-genericity, and Agelov [1] generalized the Coding Lemma of Slaman and Woodin. The obtained results have their analogues in the enumeration degree theory [4] which shows that the introduced terms keep the basic nice properties of the presently studied degrees.

Unlike the the other degree theories, the structure degrees consist of two-sorted algebraic structures, where one of the sorts is abstract, countable and infinite, and the other one is the set of the natural numbers. The main computational method is search computability (**SC**) [6]. Properties of the natural numbers, such as the information coding, the natural order etc., are excluded from the computations, which emphasizes the nature of the degree elements themselves without loading the model with external properties. The second sort is auxiliary and its elements are used only as counters in the course of the computations.

Consider the algebraic structure $\mathfrak{A} = (A, \omega, =_A, \neq_A, P_1, \dots, P_n)$, where

- A is an infinite countable set;
- ω is the set of the natural numbers;
- $P_i \subseteq A^{a_i} \times \omega^{b_i}$, $i = 1, \dots, n$ are partial predicates which take only value *true* (0) wherever defined.

Consider a countable, first-order language consisting of the binary predicate symbols $=, \neq$ (interpreted in $\mathfrak{A}$ as $=_A$ and $\neq_A$) and for each $i = 1, \dots, n$

- a_i-ary symbol T_i if $b_i = 0$, which is interpreted in $\mathfrak{A}$ as P_i;
- a_i-ary symbol $T_i^{\bar{y}}$ for each $\bar{y} = (y_1, \dots, y_{b_i}) \in \omega^{b_i}$ if $b_i \neq 0$ with the following interpretation: ${T_i^{\bar{y}}}_{\mathfrak{A}}(X_1|s_1, \dots, X_{a_i}|s_{a_i}) \simeq 0 \Leftrightarrow P_i(s_1, \dots, s_{a_i}, y_1, \dots, y_{b_i}) \simeq 0$.

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An elementary existential formula is a formula in prenex normal form with a finite number of quantifiers, which are only existential and the matrix is a finite conjunction of atomic predicates. These formulae are interpreted in $\mathfrak{A}$ in the usual way and the quantifiers are over the set A. If there are unevaluated free variables in the formula it is considered undefined. The elementary existential formulae can be effectively coded by natural numbers. If n is a code of certain formula, the formula itself is denoted by $\lceil n \rceil$.

Here we use the following definition of **SC** which is equivalent to the one in [6]. The set $B \subseteq A^k \times \omega^m$ is **SC** in $\mathfrak{A}$ iff there exist a recursive $m+1$-ary function γ and $t_1, \dots, t_q \in B$ such that for all $(\bar{s}, \bar{x}) \in A^k \times \omega^m$ the following equivalence holds:

$$(\bar{s}, \bar{x}) \in B \Leftrightarrow \exists n \in \omega(\lceil \gamma(n, \bar{x}) \rceil_{\mathfrak{A}}(\bar{W}|\bar{t}, \bar{X}|\bar{s}) \simeq 0).$$

For the structures $\mathfrak{A} = (A, \omega, =_A, \neq_A, P_1^{\mathfrak{A}}, \dots, P_m^{\mathfrak{A}})$ and $\mathfrak{B} = (A, \omega, =_A, \neq_A, P_1^{\mathfrak{B}}, \dots, P_n^{\mathfrak{B}})$ and the predicates $P_0, \dots, P_k$ we introduce the following definitions:

- $P_0 \leq_{\mathbf{SC}} \{P_1, \dots, P_k\} \leftrightharpoons P_0 \in \mathbf{SC}(P_1, \dots, P_k)$;
- $P_0 \leq_{\mathbf{SC}} \mathfrak{A} \leftrightharpoons P_0 \leq_{\mathbf{SC}} \{=_A, \neq_A, P_1^{\mathfrak{A}}, \dots, P_m^{\mathfrak{A}}\}$;
- $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B} \leftrightharpoons P_i^{\mathfrak{A}} \leq_{\mathbf{SC}} \mathfrak{B}$ for $i = 1, \dots m$;
- $\mathfrak{A} \equiv_{\mathbf{SC}} \mathfrak{B} \leftrightharpoons \mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B}$ and $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{A}$;
- $\mathfrak{A} \oplus \mathfrak{B} = (A, \omega, =_A, \neq_A, P_1^{\mathfrak{A}}, \dots, P_m^{\mathfrak{A}}, P_1^{\mathfrak{B}}, \dots, P_n^{\mathfrak{B}})$.

The relation $\leq_{\mathbf{SC}}$ is reflexive and transitive and induces an equivalence relation $\equiv_{\mathbf{SC}}$. The respective equivalence classes are called structure degrees and are noted by $d_s(\mathfrak{A}) = \{\mathfrak{B} | \mathfrak{B} \equiv_{\mathbf{SC}} \mathfrak{A}\}$.

Let $\mathfrak{D}$ be the set of all structure degrees. For $a, b \in \mathfrak{D}$ we define $a \leq_{\mathbf{SC}} b \leftrightharpoons \forall \mathfrak{A} \forall \mathfrak{B}(\mathfrak{A} \in a \,\&\, \mathfrak{B} \in b \Rightarrow \mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B})$. It is true that $a \leq_{\mathbf{SC}} b \Leftrightarrow \exists \mathfrak{A} \exists \mathfrak{B}(\mathfrak{A} \in a \,\&\, \mathfrak{B} \in b \,\&\, \mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B})$. Let $a \cup b = d_s(\mathfrak{A} \oplus \mathfrak{B})$. Then $\mathfrak{S} = \langle \mathfrak{D}, \leq_{\mathbf{SC}}, \cup, \mathfrak{O} \rangle$ is semi-lattice with the smallest element $\mathfrak{O} = (A, \omega, =_A, \neq_A)$.

For an arbitrary function f, let G_f be its graph. The structure $\alpha = (G_f, D)$, where $f : \omega \to N$ is a bijection and $D \subseteq \omega$, is called enumeration. If D is a total set (see [4] for a definition), we call the corresponding enumeration total.

We are going to define the structure degree jump similarly to the corresponding definition of enumeration jump [4]. For that purpose we need to code the information about all the search-computable predicates in the structure. Since in the abstract structures there are no coding methods, we introduce *-structures. Let $A_0 = A \cup \{0^*\}$, where $0^* \notin A$ and let $\langle , \rangle$ be a binary function in A which does not take values in A. Let A^* be the closure of A_0 with respect to $\langle , \rangle$ and let L and R be the decoding functions of $\langle , \rangle$. *-structure is a structure of the kind $(A^*, \omega, =_{A^*}, \neq_{A^*}, G_L, G_R, G_{\langle , \rangle}, P_1, \dots, P_n)$, where $P_i \subseteq {A^*}^{a_i} \times \omega^{b_i}$. The set of the natural numbers can be isomorphicly embedded in A^*. The element of A^* corresponding to to the natural number k is denoted by k^*. The *-structure of $\mathfrak{A} = (A, \omega, =_A, \neq_A, P_1^{\mathfrak{A}}, \dots, P_m^{\mathfrak{A}})$ is the structure $\mathfrak{A}^* = (A^*, \omega, =_{A^*}, \neq_{A^*}, G_L, G_R, G_{\langle , \rangle}, P_1, \dots, P_n)$. Each evaluation $X_1|s_1, \dots, X_n|s_n$ can be effectively coded (in the sense of **SC**) by some element of A^* denoted by $\lceil X_1|s_1, \dots, X_n|s_n \rceil$. The jump $\mathfrak{A}'$ of $\mathfrak{A}$ is defined by adding to $\mathfrak{A}^*$ two predicates $K^{\mathfrak{A}}$ and $\bar{K}^{\mathfrak{A}} \subseteq A^*$, where the first codes the information for the **SC** in $\mathfrak{A}$ predicates and the second one – for the predicates which are not **SC** in $\mathfrak{A}$. More exactly, the elements of $K^{\mathfrak{A}}$ are of the form

$$\langle k^*, \langle e^*, \langle \lceil X_1|s_1, \dots, X_k|s_k \rceil, \langle \lceil W_1|t_1, \dots, W_r|t_r \rceil, \langle x_1^*, \langle x_2^*, \dots, \langle x_{m-1}^*, x_m^* \rangle \dots \rangle\rangle\rangle\rangle\rangle$$

where e is the code of the partial recursive $m+1$-ary function φ_e (in the Gödel enumeration) and there exists a natural number n such that $\lceil \varphi_e(n, \bar{x}) \rceil_{\mathfrak{A}}(\bar{W}|\bar{t}, \bar{X}|\bar{s}) \simeq$

0. The elements of $\bar{K}^{\mathfrak{A}}$ have the same form, but for each natural number n, $\lceil\varphi_e(n,\bar{x})\rceil_{\mathfrak{A}}(\bar{W}|\bar{t},\bar{X}|\bar{s})$ is undefined. The relation $\leq_{\mathbf{SC}}$ naturally extends to $\leq^*_{\mathbf{SC}}$.

$\mathfrak{A}'$ is called *definable* iff there exists $\mathfrak{B} = (A, \omega, =_A, \neq_A, P_1, \dots, P_k)$, such that $\mathfrak{A}' \equiv^*_{\mathbf{SC}} \mathfrak{B}^*$.

The following properties are proved:

- if $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B}$ then $\mathfrak{A}' \leq^*_{\mathbf{SC}} \mathfrak{B}'$;
- $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{A}^*$;
- if α is an enumeration then α' is definable by $\beta = (f, D')$.

The main results are the following two theorems.

Theorem 1. $\mathfrak{B} \leq^*_{\mathbf{SC}} \mathfrak{A}'$ *iff for each total enumeration* α*,* $\mathfrak{A} \leq_{\mathbf{SC}} \alpha \Rightarrow \mathfrak{B} \leq_{\mathbf{SC}} \alpha'$.

Theorem 2. *Let* $\mathfrak{A}' \leq^*_{\mathbf{SC}} \alpha$*, where* α *is a total enumeration. Then there exists a total enumeration* β *such that* $\mathfrak{A} \leq_{\mathbf{SC}} \beta$ *and* $\beta' \equiv_{\mathbf{SC}} \alpha$.

To prove these results, forcing and enumeration methods are used. Theorem 1 is an analogue of Ash's theorem for enumeration degrees [2] and Theorem 2 is similar to the jump inversion theorem [7]. The above assertions characterize the relations $\mathfrak{A} \leq^*_{\mathbf{SC}} \mathfrak{B}'$ and $\mathfrak{B}' \leq^*_{\mathbf{SC}} \mathfrak{A}$ without using *-structures.

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