REGULAR ENUMERATIONS FOR ABSTRACT STRUCTURES

VESSELA BALEVA

ABSTRACT. Using the method of regular enumerations in the context of definability, we obtain a normal form for the sets which are Σ_{n+1} admissible in some partial structure.

1. INTRODUCTION

In the present paper we are using the method of regular enumerations [9] in the context of definability on abstract structures.

For the sake of simplicity we consider only unary sets. All the definitions and results can be easily generalized for sets of arbitrary finite arity.

Given two sets of natural numbers A and B, we say that A is enumeration-reducible to B ($A \leq_e B$) if $A = \Gamma_z(B)$ for some enumeration operator Γ_z [7, 1, 3, 5, 8]. In other words, if D_v denotes the finite set with a canonical code v and $W_0, \ldots, W_z, \ldots$ is the Gödel enumeration of the r. e. sets, we have

$$A \leq_e B \Leftrightarrow \exists z \forall x (x \in A \Leftrightarrow \exists v (\langle v, x\rangle \in W_z \,\&\, D_v \subseteq B))$$

Given a set A, denote by A^+ the set $A \oplus (\omega \backslash A)$. The set A is called *total* iff $A \equiv_e A^+$. Note that the graph G_f of each total function f is a total set.

Given a set A, let $K_A^0 = \{\langle x, z\rangle | x \in \Gamma_z(A)\}$. We define the *e-jump* A'_e of A to be the set $(K_A^0)^+$.

Several properties of the e-jump are proved in [6, 9, 5]. Since we are going to consider only the e-jump here, we omit the subscript e in the notation of the e-jump. For each set B, $B^{(0)} = B$ and $B^{(n+1)}$ is the e-jump of $B^{(n)}$.

1991 *Mathematics Subject Classification.* 03D30.

Key words and phrases. enumerations, enumeration reducibility, enumeration jump, forcing, abstract structures, admissible sets, definability.

This work was partially supported by the Ministry of education and science, Contract I604/96.

Let N be an infinite countable set and ω be the set of the natural numbers. We assume that we have an equality ($=_N$) and an inequality ($\neq_N$) in N. Consider $n+1$ sets $B_0, \dots, B_n$, such that $B_i \subseteq N$ for each $i \in [0, n]$. The algebraic structure $\alpha = (N, \omega, =_N, \neq_N, G_f, D)$, where:

- $f : \omega \to N$ is a bijection
- $D \subseteq \omega$ is a total set

is called an *enumeration*. From now on we write $\alpha = (f, D)$ to denote the enumeration α and if $D = G_g$ for some total g, then we write $\alpha = (f, g)$.

The set $A \subseteq N$, is called *admissible* relatively $B_0, \dots, B_n$ iff for each enumeration $\alpha = (f, D)$ it is true that:

$$f^{-1}(B_0) \leq_e D \ \& \ \dots \ \& \ f^{-1}(B_n) \leq_e D^{(n)} \Rightarrow f^{-1}(A) \leq_e D^{(n)}.$$

The aim of the present paper is to obtain a normal form of the admissible sets.

Consider a countable first-order language consisting of the binary predicate symbols $=$, $\neq$ (interpreted as $=_N$ and $\neq_N$) and unary predicate symbols T_i for each $i \in [0, n]$ (interpreted as B_i and taking only value true (0), whenever defined).

An *elementary existential* formula is a formula in a prenex normal form with a finite number of quantifiers which are only existential, and a matrix which is a finite conjunction of atomic predicates of $=$, $\neq$ and T_0. These formulae are interpreted in the usual way and the quantifiers are over the set N. The elementary existential formulae can be effectively coded by natural numbers. If n is the code of a certain formula, the formula itself is denoted by $\lceil n \rceil$. We use the notation $\varphi(Z_1, \dots, Z_a)$ for the formula φ with free variables among $Z_1, \dots, Z_a$.

Bellow we define Σ_i^+-formulae and Π_i^+-formulae for each $i \leq n$.

A Σ_0^+-formula is a formula of the form $\bigvee_{\gamma(n)} \lceil \gamma(n) \rceil (Z_1, \dots, Z_a)$, where γ is a recursive function and $\lceil \gamma(n) \rceil (Z_1, \dots, Z_a)$ is an elementary existential formula. These formulae are interpreted in the usual way. Π_0^+-formula $\Psi(Z_1, \dots, Z_a)$ is a formula of the form $\neg\Phi(Z_1, \dots, Z_a)$, where $\Phi(Z_1, \dots, Z_a)$ is a Σ_0^+-formula. If $\overline{s} \in N^a$ then:

$$\Psi(\overline{Z}|\overline{s}) \simeq 0 \Leftrightarrow \Phi(\overline{Z}|\overline{s}) \text{ is undefined.}$$

Proceeding by induction, suppose that $i < n$ and for each $j \in [0, i]$ we have defined Σ_j^+- and Π_j^+-formulae which can be effectively coded by natural numbers. An elementary Σ_{i+1}^+-formula is a formula in a prenex normal form with a finite number of existential quantifiers and a matrix which is a finite conjunction of atomic predicates of T_{i+1}, $=$,

$\neq$ and Σ_i^+- and Π_i^+-formulae. These formulae are interpreted in the usual way and they can be effectively coded by natural numbers.

A Σ_{i+1}^+-formula is a formula of the form $\bigvee_{\gamma(n)}\lceil\gamma(n)\rceil(Z_1,\dots,Z_a)$, where γ is a recursive function and $\lceil\gamma(n)\rceil(Z_1,\dots,Z_a)$ is an elementary Σ_i^+-formula. A Π_{i+1}^+-formula $\Psi(Z_1,\dots,Z_a)$ is a formula of the form $\neg\Phi(Z_1,\dots,Z_a)$, where $\Phi(Z_1,\dots,Z_a)$ is a Σ_{i+1}^+ formula. These formulae are interpreted similarly to the Σ_0^+- and Π_0^+-formulae.

The set $A \subseteq N$ is called *definable* iff there exist a Σ_n^+-formula $\Phi(W_1,\dots,W_r,Z)$ and $t_1,\dots,t_r \in N$, such that for all $s \in N$,

$$s \in A \Leftrightarrow \Phi(\overline{W}|\bar{t}, Z|s) \simeq 0.$$

We are going to prove following result, which gives a normal form for the admissible sets.

Theorem 1. *Let $A \subseteq N$. Then A is admissible iff A is definable.*

The "only if" part of the theorem is obvious, so we must only prove that if A is admissible then A is definable.

2. Regular Enumerations

The method of regular enumerations is introduced and studied in [9]. In this paper we adapt it for abstract structures.

Let us fix $n \geq 0$ and subsets $B_0,\dots,B_n$ of N. Since for every bijective mapping f of ω into N $f^{-1}(B_i) \equiv_e f^{-1}(B_i) \oplus \omega$ we may suppose that $f^{-1}(B_i)$ and hence B_i is not empty. We use the term *finite part* to denote an ordered pair $\tau = (f_\tau, g_\tau)$, such that:

- f_τ is a finite injective mapping of ω into N;
- g_τ is a finite mapping of ω into ω defined on a finite segment $[0, q-1]$ of ω.

The finite parts will be denoted by the letters τ, δ, ρ and Δ. If $\mathrm{dom}(g_\tau) = [0, q-1]$, then let $\mathrm{lh}(g_\tau) = q$. We assume that an effective coding of all sequences and all finite mappings of ω into ω defined on finite segment, is fixed. Let $\tau = (f_\tau, g_\tau)$ and $\rho = (f_\rho, g_\rho)$. If $f_\tau \subseteq f_\rho$ and $g_\tau \subseteq g_\rho$ we write $\tau \subseteq \rho$.

Bellow we define *i-regular* finite parts for each $i \leq n$.

A 0-regular finite part is a finite part $\tau = (f_\tau, g_\tau)$, such that $\mathrm{dom}(g_\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(g_\tau)$, $g_\tau(z) \in f_\tau^{-1}(B_0)$.

If $\mathrm{dom}(g_\tau) = [0, 2q+1]$, then the 0-rank, $|\tau|_0$, of τ is equal to $q+1$, the number of all odd elements of $\mathrm{dom}(g_\tau)$. For each 0-regular finite part τ, let $B_0^{g_\tau}$ be the set of the odd elements of $\mathrm{dom}(g_\tau)$.

Given a 0-regular finite part $\tau = (f_\tau, g_\tau)$, let

$$\begin{aligned} g_\tau \Vdash_0 F_e(x) &\Leftrightarrow \exists v(\langle v, x\rangle \in W_e \ \& \ \forall u \in D_v(g_\tau((u)_0) \simeq (u)_1)), \\ g_\tau \Vdash_0 \neg F_e(x) &\Leftrightarrow \forall(\text{0-regular } \rho = (f_\rho, g_\rho))(\tau \subseteq \rho \Rightarrow g_\rho \nVdash_0 F_e(x)). \end{aligned}$$

Proceeding by induction, suppose that we have defined the i-regular finite parts for some $i < n$ and for each i-regular finite part $\tau = (f_\tau, g_\tau)$ we have defined its i-rank $|\tau|_i$, the set $B_i^{g_\tau}$ and the relations $g_\tau \Vdash_i F_e(x)$ and $g_\tau \nVdash_i F_e(x)$.

Let f_τ be a finite mapping of ω into N and g'_τ be a finite mapping of ω into ω, such that $\mathrm{dom}(g'_\tau) = [0, q' - 1]$ and $\tau' = (f_\tau, g'_\tau)$ is i-regular. Let

$$G = \{g_\rho | \rho = (f_\rho, g_\rho) \text{ is } i\text{-regular} \ \& \ \tau' \subseteq \rho \ \& \ g'_\tau \subset g_\rho \ \& \ g_\rho \Vdash_i F_e(x)\}.$$

We say that g''_τ is *appropriate* for f_τ, g'_τ, e and x (we denote this by $\mathrm{app}(g''_\tau, f_\tau, g'_\tau, e, x)$), iff one of the following is true:

- $G \neq \emptyset$, $g''_\tau \Vdash_i F_e(x)$, (f_τ, g''_τ) is i-regular, $g'_\tau \subset g''_\tau$ and $\mathrm{lh}(g''_\tau) = \min\{\mathrm{lh}(g) | g \in G\}$.
- $G = \emptyset$, (f_τ, g''_τ) is i-regular, $|(f_\tau, g''_\tau)|_i = |(f_\tau, g'_\tau)|_i + 1$ and $g'_\tau \subset g''_\tau$.

Let τ be a finite part, g_τ be defined on $[0, q-1]$ and $r \geq 0$. Then τ is $(i+1)$-regular with $(i+1)$-rank $r+1$, iff there exist natural numbers

$$0 < n_0 < b_0 < n_1 < b_1 \cdots < n_r < b_r < n_{r+1} = q,$$

such that $(f_\tau, g_\tau \restriction n_0)$ is an i-regular finite part with i-rank 1 and for all j, $0 \leq j \leq r$ it is true that:

1. $\mathrm{app}(g_\tau \restriction b_j, f_\tau, g_\tau \restriction (n_j + 1), (j)_0, (j)_1)$
2. $g_\tau(b_j) \in f_\tau^{-1}(B_{i+1})$
3. $(g_\tau \restriction n_{j+1}, f_\tau)$ is an i-regular extension of $(g_\tau \restriction (b_j + 1), f_\tau)$ with i-rank $|(g_\tau \restriction (b_j + 1), f_\tau)|_i + 1$.

Let $B_{i+1}^{g_\tau} = \{b_0, \ldots, b_r\}$. The following lemma shows that the $(i+1)$-rank is well defined. Its proof follows easily from the definition of $(i+1)$-regular finite parts.

Lemma 1. *Let τ be an $(i+1)$-regular finite part. Then*

(i) Let $m_0, a_0, \ldots, m_p, a_p, m_{p+1}$ and $n_0, b_0, \ldots, n_r, b_r, n_{r+1}$ be two sequences of natural numbers satisfying 1.-3. Then $r = p$, $n_{p+1} = m_{p+1}$ and for all $j \leq p$, $n_j = m_j$ and $b_j = a_j$.

(ii) If ρ is $(i+1)$-regular, $\tau \subseteq \rho$ and $|\tau|_{i+1} = |\rho|_{i+1}$ then $g_\rho = g_\tau$.

(iii) τ is i-regular and $|\tau|_i > |\tau|_{i+1}$.

To complete the definition of the regular finite parts, let for each $(i+1)$-regular finite part τ

$$\begin{aligned} g_\tau \Vdash_i F_e(x) \quad &\Leftrightarrow \quad \exists v(\langle v,x\rangle \in W_e \ \&\ \forall u \in D_v(u = \langle e_u, x_u, \varepsilon\rangle \ \& \\ &\qquad \& \ \varepsilon \in \{0,1\} \ \&\ g_\tau \Vdash_i (\neg)^\varepsilon F_{e_u}(x_u))), \\ g_\tau \Vdash_{i+1} \neg F_e(x) \quad &\Leftrightarrow \quad \forall(i+1\text{-regular } \rho = (f_\rho, g_\rho)) \\ &\qquad (\tau \subseteq \rho \Rightarrow g_\rho \nVdash_{i+1} F_e(x)). \end{aligned}$$

Lemma 2.

(i) There exists an $i+1$-regular finite part with $(i+1)$-rank 1.
(ii) If τ is an $(i+1)$-regular finite part then there exists an $(i+1)$-regular finite part ρ, such that $\tau \subseteq \rho$ and $|\rho|_{i+1} = |\tau|_{i+1} + 1$.

The proof of this lemma also follows immediately from the definitions.

The enumeration $\alpha = (f,g)$ is called *regular* iff the following two conditions hold:

- For each finite part $\delta \subseteq \alpha$, there exists a n-regular extension τ of δ, such that $\tau \subseteq \alpha$.
- If $i \le n$ and $z \in B_i$ then there exists an i-regular $\tau \subseteq \alpha$, such that $f_\tau^{-1}(z) \in B_i^{g_\tau}$.

Given a regular enumeration $\alpha = (f,g)$ and $i \le k$, let

$$B_i^g = \{b | \exists(\tau = (f_\tau, g_\tau) \subseteq \alpha)(\tau \text{ is } i\text{-regular} \ \&\ b \in B_i^{g_\tau})\}.$$

Clearly $f^{-1}(B_i) = g(B_i^g)$. Similarly to the analogous proposition 2.8 in [9] one can prove the following lemma:

Lemma 3. *Suppose that $\alpha = (f,g)$ is a regular enumeration. If $i \le n$ then $f^{-1}(B_i) \le_e g^{(i)}$.*

Let g be a total mapping of ω into ω. For each $i < n$, e and x we define the relation $g \models_i F_e(x)$ by induction on i:

$$\begin{aligned} g \models_0 F_e(x) \quad &\Leftrightarrow \quad \exists v(\langle v,x\rangle \in W_e \ \&\ \forall u \in D_v(g((u)_0) \simeq (u)_1)), \\ g \models_i F_e(x) \quad &\Leftrightarrow \quad \exists v(\langle v,x\rangle \in W_e \ \&\ \forall u \in D_v((u = \langle e_u, x_u, 0\rangle \ \& \\ &\qquad \& \ g \models_i F_e(x)) \vee (u = \langle e_u, x_u, 1\rangle \ \&\ g \not\models_i F_e(x)))). \end{aligned}$$

Let for each $i \in [0,n]$

$$g \models_i \neg F_e(x) \Leftrightarrow g \not\models_i F_e(x).$$

The following lemma can be proved by induction on i.

Lemma 4. *Let g be a total mapping on ω into ω, $A \subseteq \omega$ and $i \le k$. Then $A \le_e g^{(i)}$ iff there exists e such that for all x, $x \in A \Leftrightarrow g \models_i F_e(x)$.*

Lemma 5 (Truth Lemma)**.** *Let $\alpha = (f, g)$ be a regular enumeration. Then for all $i \leq n$,*

$$g \models_i F_e(x) \Leftrightarrow \exists \tau \subseteq \alpha(\tau \textit{ is } i\textit{-regular} \,\&\, g_\tau \Vdash_i F_e(x)).$$

Proof. We will use an induction on i. The lemma is obviously true for $i = 0$. Suppose that $i < n$ and it is true for i. First, we are going to show that

$$g \models_i \neg F_e(x) \Leftrightarrow \exists \tau \subseteq \alpha(\tau \text{ is } i\text{-regular} \,\&\, g_\tau \Vdash_i \neg F_e(x)).$$

Suppose that $g \models_i \neg F_e(x)$ and for each i-regular $\tau \subseteq \alpha$, $g_\tau \nVdash \neg F_e(x)$. Then for each i-regular finite part τ of α there exists an i-regular ρ, such that $\tau \subseteq \rho$ and $g_\rho \Vdash F_e(x)$. Let δ be an $(i+1)$-regular finite part of α, such that $|\delta|_{i+1} > \langle e, x \rangle$. By the definition of the $(i+1)$-regular finite parts, there exists an i-regular $\rho' \subseteq \delta$, such that $g_{\rho'} \Vdash_i F_e(x)$. By induction $g \models_i F_e(x)$. A contradiction.

Suppose now that $\tau \subseteq \alpha$ is i-regular, $g_\tau \Vdash_i \neg F_e(x)$ and $g \models_i F_e(x)$. By induction, there exists an i-regular $\rho \subseteq \alpha$, such that $g_\rho \Vdash_i F_e(x)$. Using the monotonicity of $\Vdash_i$, we can assume that $\tau \subseteq \rho$ and get a contradiction. Now the lemma easily follows from the definitions and monotonicity. □

3. Normal Form of the Admissible Sets

Now we are ready to prove that if a set is admissible relatively $B_0, \ldots, B_n$ then it is definable. First of all, we need to prove that each admissible set has a normal form based on forcing relation and regular finite parts [10]. After that we can "translate" this normal form into a Σ_n^+-formula.

We say that $A \subseteq N$ has a *forcing normal form* iff there exist a natural number e and a n-regular finite part δ, such that for each $s \in N$ the following equivalence is true:

$$\begin{aligned} s \in A \quad \Leftrightarrow \quad & \exists x \exists \tau \supseteq \delta(\tau \text{ is } n\text{-regular finite part } \& \\ & \& \; f_\tau(x) \simeq s \;\&\; g_\tau \Vdash_n F_e(x)). \end{aligned}$$

Theorem 2 (Forcing normal form)**.** *Let $A \subseteq N$. If A is admissible then A has a forcing normal form.*

Proof. Suppose that A has not a forcing normal form. We are going to construct by steps a regular enumeration $\alpha = (f, g)$, such that for each $i \in [0, n]$ $f^{-1}(B_i) \leq_e g^i$, but $\neg(A \leq_e g^{(n)})$. At each step q we shall define a n-regular finite part δ_q, such that $\delta_q \subseteq \delta_{q+1}$.

Let $s_0, s_1, \dots$ be an arbitrary enumeration of N and δ_0 be an arbitrary n-regular finite part with n-rank 1. Let $q > 0$ and let δ_r be defined for all $r < q$.

I. $(q)_0 = 3n$. Let s be the first element of the sequence $s_0, s_1, \dots$ which does not belong to range$(f_{\delta_{q-1}})$ and z be the smallest natural number, which does not belong to dom$(f_{\delta_{q-1}})$. We define $f_{\delta_q}(z) \simeq s$ and $f_{\delta_q}(x) \simeq f_{\delta_{q-1}}(x)$ for $x \neq z$ and $g_{\delta_q} = g_{\delta_{q-1}}$.

II. $(q)_0 = 3n + 1$. Let δ_q be an arbitrary n-regular finite part, such that $\delta_q \supseteq \delta_{q-1}$ and $|\delta_q|_n = |\delta_{q-1}|_n + 1$.

III. $(q)_0 = 3n + 2$ and $(q)_1 = e$. Since A has not a forcing normal form, for δ_{q-1} and e there exists $s \in N$, such that the following equivalence is not true:

$$\begin{aligned} s \in A \iff & \exists x \exists \tau \supseteq \delta_{q-1}(\tau \text{ is } n\text{-regular finite part } \& \\ & \& \ f_\tau(x) \simeq s \ \& \ g_\tau \Vdash_n F_e(x)). \end{aligned}$$

1. Let $s \in A$ and $\forall x \forall \tau \supseteq \delta_{q-1}(\tau$ is n-regular finite part & $f_\tau(x) \simeq s \Rightarrow g_\tau \nVdash_n F_e(x))$. Let $\alpha = (f, g)$ be a regular enumeration, such that $\alpha \supseteq \delta_{q-1}$. We shall prove that $f^{-1}(A) \neq \{x | g \models_n F_e(x)\}$. Let $x = f^{-1}(s)$. Suppose that $x \in \{y | g \models_n F_e(y)\}$. Using the truth lemma and the monotonicity of the forcing we obtain a finite part τ, such that $f_\tau(x) = s$, $\delta_{q-1} \subseteq \tau$, and $g_\tau \Vdash F_e(x)$. A contradiction.
2. Let $s \notin A$ and $\exists x \exists \tau \supseteq \delta_{q-1}(\tau$ is n-regular finite part & $f_\tau(x) \simeq s$ & $g_\tau \Vdash_n F_e(x))$. Let us fix τ with the above properties and let $\alpha = (f, g)$ be a regular enumeration, such that $\alpha \supseteq \tau$. Using the monotonicity of the forcing we have that $g \models_n F_e(x)$ and $f(x) = s$, but $s \notin A$. Hence $f^{-1}(A) \neq \{x | g \models_n F_e(x)\}$. So in this case we define $\delta_q = \tau$.

Let $\alpha = (f, g)$ be a regular enumeration defined as follows: $f = \bigcup_{q \in \omega} f_{\delta_q}$ and $g = \bigcup_{q \in \omega} g_{\delta_q}$. Using Lemma 3 and Lemma 4 we obtain that A is not admissible which proves the theorem. □

Let us fix a variable Z. Denote by Var the set of all remaining variables. Let us fix a recursive bijective mapping var of the natural numbers onto Var.

We use the sign "$*$" to denote the concatenation operation on sequences of natural numbers, and "$\subseteq$" to denote the relation "is a subsequence of".

Bellow we define *i-patterns* of the i-regular finite parts for each $i \leq n$ and when an i-regular finite part τ is *coordinated* with the i-pattern σ.

Let $i = 0$. Then σ is a 0-pattern iff it is the code of a sequence of natural numbers of the form $\langle r_0, \dots, r_{2q+1} \rangle$. The 0-rank of σ, $|\sigma|_0$, is

$q+1$. The 0-regular finite part τ is coordinated with the 0-pattern σ iff $\{r_1, r_3, \dots, r_{2q+1}\} \subseteq \operatorname{dom}(f_\tau)$, $g_\tau(j) \simeq r_j$ for $j \in [0, 2q+1]$ and $\operatorname{lh}(g_\tau) = 2q+2$. We denote $\overline{\sigma} = (\operatorname{var}(r_1), \operatorname{var}(r_3), \dots, \operatorname{var}(r_{2q+1}))$ and $f_\tau(\overline{\sigma}) = (f_\tau(r_1), f_\tau(r_3), \dots, f_\tau(r_{2q+1}))$.

Let $i > 0$. Then σ is an i-pattern iff it is the code of a sequence of natural numbers of the form:

$$\langle \eta_0, n_0, \langle \xi_0, \varepsilon_0 \rangle, b_0, \eta_1, \dots, \eta_r, n_r, \langle \xi_r, \varepsilon_r \rangle, b_r, \eta_{r+1} \rangle,$$

where η_0 is an $(i-1)$-pattern with $(i-1)$-rank 1 and for each $j \in [0, r]$ the following conditions are satisfied:

1. $\varepsilon_j \in \{0, 1\}$ and ξ_j is an $(i-1)$-pattern such that $\xi_j \supseteq \eta_j * \langle n_0 \rangle$ and if $\varepsilon = 1$ then $|\eta_j|_{i-1} = |\eta_j|_{i-1} + 1$.
2. η_{j+1} is an $(i-1)$-pattern, such that $\eta_{j+1} \supseteq \xi_j * \langle b_0 \rangle$ and $|\eta_{j+1}|_{i-1} = |\xi_j|_{i-1} + 1$.

The i-rank of σ, $|\sigma|_i$, is $r+1$. The i-regular finite part τ is coordinated with the i-pattern σ, if the following conditions are satisfied:

- $\{b_0, \dots, b_r\} \subseteq \operatorname{dom}(f_\tau)$.
- If $m_0, a_0, \dots, m_r, a_r, m_{r+1}$ is a sequence of natural numbers satisfying 1. - 3. of the definition of i-regular finite part for τ then $(f_\tau, g_\tau \restriction m_0)$ is coordinated with $(i-1)$-pattern η_0 and for each $j \in [0, r]$ we have:
  1. $g_\tau(m_j) = n_j$;
  2. $(f_\tau, g_\tau \restriction a_j)$ is coordinated with ξ_j;
  3. if $\varepsilon_j = 0$ then $g_\tau \Vdash_{i-1} F_{(j)_0}((j)_1)$ else $g_\tau \nVdash_{i-1} F_{(j)_0}((j)_1)$;
  4. $g_\tau(a_j) = b_j$;
  5. $(f_\tau, g_\tau \restriction m_{j+1})$ is coordinated with the $(i-1)$-pattern η_{j+1}.

Let $\overline{\sigma} = (\overline{\eta}_{r+1}, \operatorname{var}(b_0), \dots, \operatorname{var}(b_r))$ and $f_\tau(\overline{\sigma}) = (f_\tau(\overline{\eta}_{r+1}), f_\tau(b_0), \dots, f_\tau(b_r))$. Let for $i \in [0, n]$

$$\Re_i(\delta, x) = \{s | s \in N \;\&\; \exists \tau \supseteq \delta (f_\tau(x) \simeq s \;\&\; \tau \text{ is } i\text{-regular})\}.$$

Lemma 6. *There exists an uniform effective way given g_δ, $y_1, \dots, y_r$, such that $\delta = (f_\delta, g_\delta)$ is i-regular and $\operatorname{dom}(f_\delta) = \{y_1, \dots, y_r\}$, and given natural numbers e and x to define a Σ_i^+-formula $\Phi^{\delta,e,x}$ with free variables among $\operatorname{var}(y_1) = Y_1, \dots, \operatorname{var}(y_r) = Y_r, Z$, such that for all $s \in \Re_i(\delta, x)$,*

$$\Phi^{\delta,e,x}(\overline{Y} | \overline{f_\delta(y)}, Z | s) \simeq 0 \Leftrightarrow \exists \tau \supseteq \delta (\tau \textit{ is } i\textit{-regular} \;\&\; g_\tau \Vdash_i F_e(x)).$$

Proof. We prove the lemma by induction on i. For $i = 0$ it immediately follows from the definitions.

Let $i > 0$ and let assume that for each $j \in [0, i-1]$ the lemma is true. Using the inductive assumption one can easily prove the following lemmas.

Lemma 7. *There exists an uniform effective way given $(i-1)$-pattern σ, natural numbers e and x and a finite set $D = \{y_1, \dots, y_r\}$, to define a Σ_i^+-formula $\Phi^{\sigma,e,x,D}$ with free variables among* $\mathrm{var}(y_1) = Y_1, \dots, \mathrm{var}(y_r) = Y_r$, $\overline{\sigma}$, *such that for each $(i-1)$-regular finite part τ coordinated with σ and such that $D \subseteq \mathrm{dom}(f_\tau)$ it is true that:*

$$\Phi^{\delta,e,x,D}(\overline{Y}|\overline{f_\delta(y)}, \overline{\sigma}|f_\tau(\overline{\sigma})) \simeq 0 \Leftrightarrow$$
$$\exists \Delta \supseteq \tau(\Delta \textit{ is } (i-1)\textit{-regular} \ \& \ g_\Delta \Vdash_{i-1} F_e(x)).$$

Lemma 8. *There exists an uniform effective way given an $(i-1)$-pattern σ, natural numbers e and x and a finite set $D = \{y_1, \dots, y_r\}$ to define a Σ_i^+-formula $\Phi^{\sigma,e,x,D}$ with free variables among* $\mathrm{var}(y_1) = Y_1, \dots, \mathrm{var}(y_r) = Y_r$, $\overline{\sigma}$ *such that for each $(i-1)$-regular finite part τ coordinated with σ and such that $D \subseteq \mathrm{dom}(f_\tau)$ it is true that:*

$$\Phi^{\delta,e,x,D}(\overline{Y}|\overline{f_\delta(y)}, \overline{\sigma}|f_\tau(\overline{\sigma})) \simeq 0 \Leftrightarrow g_\tau \Vdash_{i-1} F_e(x).$$

Lemma 9. *There exists an uniform effective way given an $(i-1)$-pattern σ, natural numbers l, e and x and a finite set $D = \{y_1, \dots, y_r\}$ to define a Σ_i^+-formula $\Phi^{\sigma,l,e,x,D}$ with free variables among* $\mathrm{var}(y_1) = Y_1, \dots, \mathrm{var}(y_r) = Y_r$, $\overline{\sigma}$ *such that for each $(i-1)$-regular finite part τ coordinated with σ and such that $D \subseteq \mathrm{dom}(f_\tau)$ it is true that:*

$$\Phi^{\delta,l,e,x,D}(\overline{Y}|\overline{f_\delta(y)}, \overline{\sigma}|f_\tau(\overline{\sigma})) \simeq 0 \Leftrightarrow$$
$$\exists \Delta \supseteq \tau(\Delta \textit{ is } (i-1)\textit{-regular} \ \& \ \mathrm{lh}(g_\Delta) < l \ \& \ g_\Delta \Vdash_{i-1} F_e(x)).$$

Lemma 10. *There exists an uniform effective way given an $(i-1)$-pattern σ and a finite set $D = \{y_1, \dots, y_r\}$ to define a Σ_i^+-formula $\Phi^{\sigma,D}$ with free variables $U_1, \dots, U_k$ among* $\mathrm{var}(y_1) = Y_1, \dots, \mathrm{var}(y_r) = Y_r$, $\overline{\sigma}$ *such that for all $\overline{s} \in N^k$ it is true that:*

$$\Phi^{\delta,D}(\overline{U}|\overline{s}) \simeq 0 \Leftrightarrow$$
$$\exists \tau(\tau \textit{ is } (i-1)\textit{-regular finite part coordinated with } \sigma \ \&$$
$$\& \ \{y_1, \dots, y_r\} \subseteq \mathrm{dom}(f_\tau) \ \&$$
$$\& \ f_\tau(\mathrm{var}^{-1}(U_1)) = s_1, \ \& \ \dots \ \& \ f_\tau(\mathrm{var}^{-1}(U_k)) = s_k).$$

Let us fix g_δ, $y_1, \dots, y_r$, e and x. Let D be a finite set of natural numbers. We say that D is compatible with g_δ iff the following conditions are true:

- Each $u \in D$ is of the form $\langle e_u, x_u, \varepsilon_u \rangle$, where $\varepsilon_u \in \{0, 1\}$.
- There are not elements u and w of D, such that $u = \langle e, x, 0 \rangle$ and $w = \langle e, x, 1 \rangle$.
- If $\langle e, x, \varepsilon \rangle \in D$ and $\langle e, y \rangle \leq |\delta|_i$, then if $\varepsilon = 0$ then $g_\delta \Vdash_{i-1} F_e(y)$ else $g_\delta \nVdash_{i-1} F_e(y)$.

Let σ be an i-pattern. We say that D is compatible with $\sigma = \langle \eta_0, n_0, \langle \xi_0, \varepsilon_0 \rangle, b_0, \eta_1, \ldots, \eta_r, n_r, \langle \xi_r, \varepsilon_r \rangle, b_r, \eta_{r+1} \rangle$ iff $\langle e, y, \varepsilon \rangle \in D$ implies $\varepsilon_{\langle e,y \rangle} = \varepsilon$. We call σ compatible with g_δ iff for each τ coordinated with σ, $g_\delta \subseteq g_\tau$. We define $\mathrm{lh}(\sigma)$ as $\mathrm{lh}(g_\tau)$, where τ is $(i-1)$-regular and coordinated with σ.

Consider the r. e. set

$$\begin{aligned} W \;=\; & \{\langle \sigma v \rangle | \langle v, x \rangle \in W_e \;\&\; \sigma \text{ is an } i\text{-pattern } \& \\ & \;\&\; D_v \text{ is compatible with } \sigma \text{ and } g_\delta \;\&\; \sigma \text{ is compatible with } g_\tau\}. \end{aligned}$$

Let $\langle \sigma, v \rangle \in W$, $\sigma = \langle \eta_0, n_0, \langle \xi_0, \varepsilon_0 \rangle, b_0, \eta_1, \ldots, \eta_r, n_r, \langle \xi_r, \varepsilon_r \rangle, b_r, \eta_{r+1} \rangle$, $j > |\delta|_i$ and $D = \{y_1, \ldots, y_r, x\}$.

I. $\varepsilon_j = 0$. We define $\Phi^j = \Phi^{\xi_j, D} \wedge \Phi^{\eta_{j+1}, D} \wedge \Phi^{\xi_j, (j)_0, (j)_1, D} \wedge \Phi_1 \wedge T_i(\mathrm{var}(b_j))$, where $\Phi^{\xi_j, D}$ and $\Phi^{\eta_{j+1}, D}$ are the formulae from Lemma 10 and $\Phi^{\xi_j, (j)_0, (j)_1, D}$ is the formula from Lemma 8 and if $|\xi_j|_{i-1} > |\eta_j| + 1$ then $\Phi_1 = \neg \Phi^{\eta_j * \langle n_0 \rangle, \mathrm{lh}(\xi_j)(j)_0, (j)_1, D}$ is the formula from Lemma 9 else $\Phi = (Z = Z)$.

II. $\varepsilon_j = 1$. We define $\Phi^j = \Phi^{\xi_j, D} \wedge \Phi^{\eta_{j+1}, D} \wedge \neg \Phi^{\xi_j, (j)_0, (j)_1, D} \wedge T_i(\mathrm{var}(b_j))$, where the first two formulae are the same as above and $\Phi^{\xi_j, (j)_0, (j)_1, D}$ is the formula from Lemma 7.

We denote by E the set of all variables in $\overline{\sigma}$ and $\{Y_1, \ldots, Y_r\}$. Let $\{W_1, \ldots, W_p\}$ be the set $E \setminus \{Y_1, \ldots, Y_r\}$. If $\mathrm{var}(x) \in E$ we define $\varphi^\sigma = \bigwedge_{U, W \in E, U \neq W} U \neq W \wedge \mathrm{var}(x) = Z$ else $\varphi^\sigma = \bigwedge_{U, W \in E, U \neg W} U \neq W$. Let $\Phi^{\langle \sigma, v \rangle}$ be the following formula:

$$\exists W_1, \ldots, \exists W_p (\bigwedge_{j > |\delta|_i} \Phi^j \wedge \varphi^\sigma).$$

Note that the above is an elementary Σ_i^+-formula. Now we are ready to define our Σ_i^+-formula:

$$\Phi^{\delta, e, x} = \bigvee_{\langle \sigma, v \rangle \in W} \Phi^{\langle \sigma, v \rangle}.$$

This completes the proof of Lemma 6. □

Using the previous lemma and Theorem 2 one can easily obtain our main result, Theorem 1.

4. Conclusions

In the papers [2] and [4] is obtained a normal form of the Σ_n-admissible sets in total structures. In the particular case where $B_1 = \cdots = B_n = N$ we find a normal form for the sets which are Σ_{n+1}-admissible in some partial structure. It would be interesting to extend the method of regular enumerations for the constructive ordinals and to prove a similar theorem.

The author would like to thank a lot Ivan Soskov for his help and patience.

References

1. C. J. Ash, *Generalizations of enumeration reducibility using recursive infinitary propositional senetences*, Ann. Pure Appl. Logic **58** (1992), 173–184.
2. C. J. Ash, J. F. Knight, M. Manasse, and T. Slaman, *Generic copies of countable structures*, Ann. Pure Appl. Logic **42** (1989), 195–205.
3. J. Case, *Maximal arithmetical reducibilities*, Z. Math. Logik Grundlag. Math. **20** (1974), 261–270.
4. J. Chisholm, *Effective model theory vs. recursive model theory*, J. Symbolic Logic **55** (1990), 1168–1191.
5. S. B. Cooper, *Enumeration reducibilty, nondeterministic computations and relative computability of partial functions*, Recursion theory week, Oberwolfach 1989, Lecture notes in mathematics (Heidelberg) (G. E. Sacks K. Ambos-Spies, G. Muler, ed.), vol. 1432, Springer-Verlag, 1990, pp. 57–110.
6. K. McEvoy, *Jumps of quasi-minimal enumeration degrees*, J. Symbolic Logic **50** (1985), 839–848.
7. H. Rogers Jr., *Theory of recursive functions and effective computability*, McGraw-Hill Book Company, New York, 1967.
8. A. L. Selman, *Arithmetical reducibilities i*, Z. Math. Logik Grundlag. Math. **17** (1971), 335–350.
9. I. N. Soskov, *A jump inversion theorem for the enumeration jump*, Arch. Math. Logic, to appear.
10. ———, *Intrinsically Π^1_1 relations*, Mathematical Logic Quarterly **42** (1996), 109–126.

Faculty of Mathematics and Comp. Science, Sofia University, Blvd. James Bourchier 5, 1164 Sofia, Bulgaria,

E-mail address: `vbaleva@fmi.uni-sofia.bg`