COSPECTRA OF JOINT SPECTRA OF A SEQUENCE OF STRUCTURES

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1. Introduction

1.1. Degree Spectrum and Cospectrum of a structure. Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$ be a partial structure over the set of all natural numbers $\mathbb{N}$,where each R_i is a subset of $\mathbb{N}^{r_i}$ and "$=$" and "$\neq$" are among $R_1, \dots, R_k$.

An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.

For every $A \subseteq \mathbb{N}^a$ define $f^{-1}(A) = \{\langle x_1 \dots x_a\rangle : (f(x_1), \dots, f(x_a)) \in A\}$.

Let $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k)$.

For any sets of natural numbers A and B the set A is enumeration reducible to B ($A \leq_e B$) if there is an enumeration operator Γ_z such that $A = \Gamma_z(B)$. By $deg_e(A)$ we denote the enumeration degree of the set A. The set A is total if $A \equiv_e A \oplus (\mathbb{N} \setminus A)$. A degree a is called total if a contains the e-degree of a total set. For every recursive ordinal α by $A^{(\alpha)}$ we shall denote the α-th enumeration jump of A [8].

The Degree Spectrum of $\mathfrak{A}$ is the set

$$Sp(\mathfrak{A}) = \{deg_e(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

The above notion is introduced by [5] for bijective enumerations and is used in [4, 2, 3, 7]. where some results about the degree spectra of structures are obtained.

If $a \in Sp(\mathfrak{A})$ and b is a total e-degree, $a \leq b$, then $b \in Sp(\mathfrak{A})$ [7]. So, the Degree Spectrum of $\mathfrak{A}$ is closed upwards.

Denote by $\mathcal{D}_e$ the set of all enumeration degrees. Let $\mathcal{A} \subseteq \mathcal{D}_e$. *The Co-set of* $\mathcal{A}$ is the set $Co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$Co(\mathcal{A}) = \{b : b \in \mathcal{D}_e \ \& \ (\forall a \in \mathcal{A})(b \leq a)\}.$$

2. Joint spectrum of a sequence of structures

Let ζ be a recursive ordinal and let $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ be a sequence of structures over the natural numbers.

The Joint Spectrum of the sequence $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the set

$$Sp(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{a : a \in \mathcal{D}_e \ \& \ (\forall \xi \leq \zeta)(a^{(\xi)} \in Sp(\mathfrak{A}_\xi))\}.$$

Let $\alpha \leq \zeta$. *The α - th Jump Spectrum of* $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the set

$$Sp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{a^{(\alpha)} : a \in Sp(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta})\}.$$

2.1. Cospectra of Joint Spectra of a sequence of structures. Let $\alpha \leq \zeta$. *The α - th Co-spectrum of* $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the Co-set of $Sp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta})$, i.e.

$$Cosp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{b : b \in \mathcal{D}_e \& (\forall a \in Sp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}))(b \leq a)\}.$$

In [10] we represented a characterization of the Cospectrum of the Joint Degree Spectrum of finitely many structures. Here we shall consider $Cosp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta})$.

2.1.1. Proposition. *For any $\alpha \leq \zeta$, $Cosp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = Cosp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\alpha})$.*

It is clear that $Cosp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\alpha}) \subseteq Cosp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta})$. The oposite follows from the Jump Inversion Theorem from [9] and the fact that fot each ξ the degree spectrum $Sp(\mathfrak{A}_\xi)$ is closed upwards.

2.2. The jump set of a sequence of sets. Let for each $\xi \leq \zeta$ f_ξ be an enumeration of $\mathfrak{A}_\xi$ and $f = \{f_\xi\}_{\xi\leq\zeta}$. For any recursive ordinal $\alpha \leq \zeta$ we define the *jump set $\mathcal{P}^f_\alpha$ of the sequence* $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ by means of transfinite recursion on α:

(i) $\mathcal{P}^f_0 = f_0^{-1}(\mathfrak{A}_0)$.

(ii) Let $\alpha = \beta + 1$. Then let $\mathcal{P}^f_\alpha = (\mathcal{P}^f_\beta)' \oplus f_\alpha^{-1}(\mathfrak{A}_\alpha)$.

(iii) Let $\alpha = \lim \alpha(p)$. Then set $\mathcal{P}^f_{<\alpha} = \{\langle p, x\rangle : x \in \mathcal{P}^f_{\alpha(p)}\}$ and let $\mathcal{P}^f_\alpha = \mathcal{P}^f_{<\alpha} \oplus f_\alpha^{-1}(\mathfrak{A}_\alpha)$.

2.2.1. Theorem. *Let $A \subseteq \mathbb{N}$. Then*

$$deg_e(A) \in Cosp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) \iff$$
$$(\text{ for every sequnce } f = \{f_\xi\}_{\xi\leq\zeta})(f_\xi \text{ enumeration of } \mathfrak{A}_\xi)(A \leq_e \mathcal{P}^f_\alpha).$$

It follows from the Jump Inversion Theorem from [9].

3. Generic enumerations and forcing

3.1. Satisfaction relation. Let $W_0, \dots, W_z, \dots$ be the Godel enumeration of the r.e. sets and D_v be the finite set having canonical code v.

For every $\alpha \leq \zeta$, e and x in $\mathbb{N}$ define the relations $f \models_\alpha F_e(x)$ and $f \models_\alpha \neg F_e(x)$ by transfite induction on α:

(i) $f \models_0 F_e(x)$ iff there exists a v such that $\langle v, x\rangle \in W_e$ and $D_v \subseteq f_0^{-1}(\mathfrak{A}_0)$;

(ii) $\alpha = \beta + 1$.
$$\begin{aligned} f \models_\alpha F_e(x) \iff & \exists v(\langle v, x\rangle \in W_e \ \& \ (\forall u \in D_v)(\\ & u = \langle 0, e_u, x_u\rangle \ \& \ f \models_\beta F_{e_u}(x_u) \ \vee \\ & u = \langle 1, e_u, x_u\rangle \ \& \ f \models_\beta \neg F_{e_u}(x_u) \ \vee \\ & u = \langle 2, x_u\rangle \ \& \ x_u \in f_\alpha^{-1}(\mathfrak{A}_\alpha))); \end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then
$$\begin{aligned} f \models_\alpha F_e(x) \iff & (\exists v)(\langle v, x\rangle \in W_e \ \& \ (\forall u \in D_v)(\\ & (u = \langle 0, p_u, e_u, x_u\rangle \ \& \ f \models_{\alpha(p_u)} F_{e_u}(x_u)) \vee \\ & (u = \langle 2, x_u\rangle \ \& \ x_u \in f_\alpha^{-1}(\mathfrak{A}_\alpha)))); \end{aligned}$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

3.1.1. Proposition. *For each* $A \subseteq \mathbb{N}$

$$A \leq_e \mathcal{P}^f_\alpha \iff \text{there is a number } e \text{ such that } A = \{x : f \models_\alpha F_e(x)\}.$$

3.2. Finite parts and forcing. The forcing conditions which we shall call *finite parts* are sequnces τ of finite mappings $\tau_\xi, \xi \leq \zeta$ from $\mathbb{N}$ to $\mathbb{N}$, so that $\bigcup_{\xi \leq \zeta} dom(\tau_\xi)$ is finite. If τ and ρ are finite parts, then $\tau \subseteq \rho$ if for each $\xi \leq \zeta$ $(\tau_\xi \subseteq \rho_\xi)$.

For every $\alpha \leq \zeta$, e and x in $\mathbb{N}$ and every finite part τ we define the forcing relations $\tau \Vdash_\alpha F_e(x)$ and $\tau \Vdash_\alpha \neg F_e(x)$ following the definition of "$\models_\alpha$".

(i) $\tau \Vdash_0 F_e(x) \iff$ there exists a v such that $\langle v, x\rangle \in W_e$ & $D_v \subseteq \tau_0^{-1}(\mathfrak{A}_0)$;

(ii) $\alpha = \beta + 1$.
$$\begin{aligned}\tau \Vdash_\alpha F_e(x) \iff \exists v(\langle v,x\rangle \in W_e \ \&\\ (\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta F_{e_u}(x_u) \vee\\ u = \langle 1, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta \neg F_{e_u}(x_u) \vee\\ u = \langle 2, x_u\rangle \ \&\ x_u \in \tau_\alpha^{-1}(\mathfrak{A}_\alpha)));\end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then
$$\begin{aligned}\tau \Vdash_\alpha F_e(x) \iff (\exists v)(\langle v,x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\ (u = \langle 0, p_u, e_u, x_u\rangle \ \&\ \tau \Vdash_{\alpha(p_u)} F_{e_u}(x_u))\vee\\ (u = \langle 2, x_u\rangle \ \&\ x_u \in \tau_\alpha^{-1}(\mathfrak{A}_\alpha))));\end{aligned}$$

(iv) $\tau \Vdash_\alpha \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash_\alpha F_e(x))$.

3.3. Forcing properties. Let $\alpha \leq \zeta, e, x \in \mathbb{N}$ and δ, τ be finite parts.

(1) If $\delta \subseteq \tau$, then
$\delta \Vdash_\alpha (\neg)F_e(x) \Longrightarrow \tau \Vdash_\alpha (\neg)F_e(x)$;

(2) If $(\forall \xi \leq \alpha)(\delta_\xi = \tau_\xi)$, then
$\delta \Vdash_\alpha (\neg)F_e(x) \Longrightarrow \tau \Vdash_\alpha (\neg)F_e(x)$.

Define $\delta \subseteq_\alpha \tau \iff (\forall \xi \leq \alpha)(\delta_\xi \subseteq \tau_\xi)$ & $(\forall \xi > \alpha)(\delta_\xi = \tau_\xi)$.

Let $\tau \Vdash^*_\alpha (\neg)F_e(x)$ be the same as $\tau \Vdash_\alpha (\neg)F_e(x)$ with the exception of

(iii) $\tau \Vdash_\alpha \neg F_e(x) \iff (\forall \rho \supseteq_\alpha \tau)(\rho \nVdash_\alpha F_e(x))$.

The next Lemma shows that actually the star forcing relation $\Vdash^*_\alpha$ coincides with the forcing relation $\Vdash_\alpha$.

3.3.1. Lemma. $\tau \Vdash_\alpha (\neg)F_e(x) \iff \tau \Vdash^*_\alpha (\neg)F_e(x)$.

3.4. Generic enumerations. For any $\alpha < \zeta, e, x \in \mathbb{N}$ denote by
$X^\alpha_{\langle e,x\rangle} = \{\rho : \rho \Vdash_\alpha F_e(x)\}$.

An enumeration f of $\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}$ is α-*generic* if for every $\beta < \alpha$, $e, x \in \mathbb{N}$

$$(\forall \tau \subseteq f)(\exists \rho \in X^\beta_{\langle e,x\rangle})(\tau \subseteq \rho) \Longrightarrow (\exists \tau \subseteq f)(\tau \in X^\beta_{\langle e,x\rangle}).$$

3.4.1. Lemma.

(1) *Let f be an α-generic enumeration, $\alpha < \zeta$. Then*

$$f \models_\alpha F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_\alpha F_e(x)).$$

(2) *Let f be an $\alpha+1$-generic enumeration. Then*

$$f \models_\alpha \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_\alpha \neg F_e(x)).$$

3.5. Forcing α - definable sets. The set $A \subseteq \mathbb{N}$ is *forcing α-definable* on $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ if there exist a finite part δ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \tau \supseteq \delta)(\tau \Vdash_\alpha F_e(x)).$$

3.5.1. Theorem. *Let $A \subseteq \mathbb{N}$.*

If $A \leq_e \mathcal{P}^f_\alpha$ for all f - enumerations of $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$, then A is forcing α-definable on $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$.

4. The Normal Form Theorem

In this section we shall give an explicit form of the forcing α-definable on $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ sets by means of *positive* recursive Σ^+_α formulae. These formulae can be considered as a modification of the Ash's formulae introduced in [1].

4.1. Recursive Σ^+_α formulae. Let, for each $\xi \leq \zeta$, $\mathcal{L}_\xi = \{T^\xi_1, \dots, T^\xi_{n_\xi}\}$ be the language of $\mathfrak{A}_\xi$. We suppose that the languages $\mathcal{L}_\xi$ are disjoint.

For each $\alpha \leq \zeta$, define the elementary Σ^+_α formulae and Σ^+_α formulae by transfinite induction on α, as follows.

(1) *An elementary Σ^+_0 formula* with free variables among $\bar{X}$ is an existential formula of the form:

$$\exists Y_1 \dots \exists Y_m \varphi(\bar{X}, Y_1, \dots, Y_m),$$

where φ is a finite conjunction of atomic formulae in $\mathcal{L}_0$;

(2) $\alpha = \beta + 1$. *An elementary Σ^+_α formula* is in the form

$$\exists Y_0 \dots \exists Y_m \varphi(\bar{X}, Y_0 \dots Y_m)$$

where φ is a finite conjunction of Σ^+_β formulae and negations of Σ^+_β formulae and atoms of $\mathcal{L}_\alpha$;

(3) Let $\alpha = \lim \alpha(p)$ be a limit ordinal and $\alpha \leq \zeta$. The elementary Σ^+_α formula are in the form

$$\exists Y_0 \dots \exists Y_m \varphi(\bar{X}, Y_0, \dots, Y_m),$$

where φ is a finite conjunction of atoms of $\mathcal{L}_\alpha$ and $\Sigma^+_{\alpha(p)}$ formulae.

(4) *A Σ^+_α formula* is an r.e. infinitary disjunction of elementary Σ^+_α formulae with free variables among $\bar{X}$.

Let Φ be a Σ^+_α formula $\alpha \leq \zeta$ with free variables among $X_0, \dots, X_i$ and let $t_0, \dots, t_i$ be elements of $\mathbb{N}$. Then by $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta} \models \Phi(X_0/t^0, \dots, X_i/t_i)$ we shall denote that Φ is true on $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ under the variable assignment v such that $v(X_0) = t_0, \dots, v(X_i) = t_i$.

4.2. The formally α-definable sets. The set $A \subseteq \mathbb{N}$ is *formally α-definable* on $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ if there exists an $e \in \mathbb{N}$ and a recursive sequence $\{\Phi\}^{\gamma(e,x)}$ of Σ^+_α formulae with free variables among $W_0, \dots, W_k$ and elements $t_0, \dots, t_k$ of $\mathbb{N}$ such that the following equivalence holds:

$$x \in A \iff \{\mathfrak{A}_\xi\}_{\xi\leq\zeta} \models \Phi^{\gamma(e,x)}(W_0/t_0 \dots W_k/t_k).$$

4.2.1. Theorem. *Let $A \subseteq \mathbb{N}$ be forcing α-definable on $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$. Then A is formally α-definable on $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$.*

4.2.2. Theorem. *Let $A \subseteq \mathbb{N}$. Then the following are equivalent:*

(1) $deg_e(A) \in Cosp^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta})$.
(2) *For every enumeration f of $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$, $A \leq_e \mathcal{P}^f_\alpha$.*
(3) *A is forcing α-definable on $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$.*
(4) *A is formally α-definable on $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$.*

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