ω-Degree Spectra

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Abstract. We present a notion of a degree spectrum of a structure with respect to countably many sets, based on the notion of ω-enumeration reducibility. We prove that some properties of the degree spectrum such as the minimal pair theorem and the existence of quasi-minimal degree are true for the ω-degree spectrum.

1 Introduction

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_s)$ be a structure, where $\mathbb{N}$ is the set of all natural numbers, each R_i is a subset of $\mathbb{N}^{r_i}$ and the equality $=$ and the inequality $\neq$ are among $R_1, \dots, R_s$. *An enumeration* f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.

Given an enumeration f of $\mathfrak{A}$ and a subset A of $\mathbb{N}^a$ let $f^{-1}(A) = \{\langle x_1, \dots, x_a\rangle \mid (f(x_1), \dots, f(x_a)) \in A\}$. Denote by $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_s)$.

Given a set X of natural numbers denote by $d_{\mathrm{e}}(X)$ the enumeration degree of X and by $d_{\mathrm{T}}(X)$ the Turing degree of X.

The notion of *Turing degree spectrum* of $\mathfrak{A}$ is introduced by Richter [6]: $\mathrm{DS}_{\mathrm{T}}(\mathfrak{A}) = \{d_{\mathrm{T}}(f^{-1}(\mathfrak{A})) \mid f$ is an injective enumeration of $\mathfrak{A}\}$. Soskov [8] initiated the study of the properties of the degree spectra as sets of enumeration degrees. *The enumeration degree spectrum of* $\mathfrak{A}$ (called shortly degree spectrum of $\mathfrak{A}$) is the set: $\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f$ is an enumeration of $\mathfrak{A}\}$.

The benefit of considering all enumerations of the structure instead of only the injective ones is that every degree spectrum is upwards closed with respect to total enumeration degrees [8]. Soskov considered the notion of *co-spectrum* $\mathrm{CS}(\mathfrak{A})$ of $\mathfrak{A}$ as the set of all lower bounds of the elements of the degree spectrum of $\mathfrak{A}$ and proved several properties which show that the degree spectra behave with respect to their co-spectra very much like the cones of the enumeration degrees $\{\mathbf{x} \mid \mathbf{x} \geq \mathbf{a}\}$ behave with respect to the ideals $\{\mathbf{x} \mid \mathbf{x} \leq \mathbf{a}\}$. Further properties true of the degree spectra but not necessarily true of all upwards closed sets are: the minimal pair theorem for the degree spectrum and the existence of quasi-minimal degree for the degree spectrum.

In this paper we shall relativize Soskov's approach to degree spectra by considering multi-component spectra, i.e. a degree spectrum with respect to a given sequence of sets of natural numbers, considering the ω-enumeration reducibility introduced and studied in [10–12]. It is a uniform reducibility between sequences

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of sets of natural numbers. We shall prove that the so defined ω-degree spectrum preserves almost all properties of the degree spectrum and generalizes the notion of relative spectrum, i.e multi-component spectrum of a structure with respect to finitely many structures, introduced in [13].

2 ω-Enumeration Degrees

We assume the reader is familiar with enumeration reducibility and refer to [3] for further background. Denote by $\mathcal{D}_{\mathrm{e}}$ the set of all enumeration degrees. Recall that an enumeration degree $\mathbf{a}$ is total if $\mathbf{a}$ contains a total set, i.e. a set A such that $A \equiv_{\mathrm{e}} A^+$, where $A^+ = A \oplus (\mathbb{N} \setminus A)$. If X is a total set then $A \leq_{\mathrm{e}} X \iff A$ is c.e. in X. Cooper [2] introduced the jump operation "$'$" for enumeration degrees. By A' we denote the enumeration jump of the set A and $d_{\mathrm{e}}(A)' = d_{\mathrm{e}}(A')$.

Denote by $\mathcal{S}$ the set of all sequences of sets of natural numbers. For each element $\mathcal{B} = \{B_n\}_{n<\omega}$ of $\mathcal{S}$ call *the jump class of* $\mathcal{B}$ the set

$$J_{\mathcal{B}} = \{d_{\mathrm{T}}(X) \mid (\forall n)(B_n \text{ is c.e. in } X^{(n)} \text{ uniformly in } n)\} \ .$$

For every two sequences $\mathcal{A}$ and $\mathcal{B}$ let $\mathcal{A} \leq_\omega \mathcal{B}$ ($\mathcal{A}$ is *ω-enumeration reducible to* $\mathcal{B}$) if $J_{\mathcal{B}} \subseteq J_{\mathcal{A}}$ and let $\mathcal{A} \equiv_\omega \mathcal{B}$ if $J_{\mathcal{A}} = J_{\mathcal{B}}$. The relation $\equiv_\omega$ is an equivalence relation on $\mathcal{S}$. Let the *ω-enumeration degree of* $\mathcal{B}$ be $d_\omega(\mathcal{B}) = \{\mathcal{A} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$ and $\mathcal{D}_\omega = \{d_\omega(\mathcal{B}) \mid \mathcal{B} \in \mathcal{S}\}$. If $\mathbf{a} = d_\omega(\mathcal{A})$ and $\mathbf{b} = d_\omega(\mathcal{B})$ then $\mathbf{a} \leq_\omega \mathbf{b}$ if $\mathcal{A} \leq_\omega \mathcal{B}$. Denote by $\mathbf{0}_\omega = d_\omega(\emptyset_\omega)$, where $\emptyset_\omega$ is the sequence with all members equal to $\emptyset$. There is a natural embedding of the enumeration degrees into the ω-enumeration degrees. Given a set A denote by $A \uparrow \omega$ the sequence $\{A_n\}_{n<\omega}$, where $A_0 = A$ and for all $n > 0$ $A_n = \emptyset$. For every $A, B \subseteq \mathbb{N}$ we have that $A \leq_{\mathrm{e}} B \iff A \uparrow \omega \leq_\omega B \uparrow \omega$. So the mapping $\kappa(d_{\mathrm{e}}(A)) = d_\omega(A \uparrow \omega)$ gives an isomorphic embedding of $\mathcal{D}_{\mathrm{e}}$ to $\mathcal{D}_\omega$. We shall identify the enumeration degree $d_{\mathrm{e}}(A)$ with its representation $d_\omega(A \uparrow \omega)$ in $\mathcal{D}_\omega$. So when $\mathbf{a} = d_{\mathrm{e}}(A)$ and $\mathbf{b} \in \mathcal{D}_\omega$ then by writing $\mathbf{a} \leq_\omega \mathbf{b}$ ($\mathbf{b} \leq_\omega \mathbf{a}$) we mean $d_\omega(A \uparrow \omega) \leq_\omega \mathbf{b}$ ($\mathbf{b} \leq_\omega d_\omega(A \uparrow \omega)$).

Given a sequence of sets of natural numbers $\mathcal{B} = \{B_n\}_{n<\omega}$ we define the respective *jump sequence* $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_n(\mathcal{B})\}_{n<\omega}$ by induction on n:

(1) $\mathcal{P}_0(\mathcal{B}) = B_0$;

(2) $\mathcal{P}_{n+1}(\mathcal{B}) = (\mathcal{P}_n(\mathcal{B}))' \oplus B_{n+1}$.

Note that if $X \subseteq \mathbb{N}$, then $\mathcal{P}_n(X \uparrow \omega) \equiv_{\mathrm{e}} X^{(n)}$ uniformly in n.

The following theorem of Soskov and Kovachev [10] gives an explicit characterization of the uniform reducibility.

Theorem 1. *Let $\mathcal{A} = \{A_n\}_{n<\omega}$ and $\mathcal{B} = \{B_n\}_{n<\omega}$ be elements of $\mathcal{S}$. The following conditions are equivalent:*

(1) *$\mathcal{A} \leq_\omega \mathcal{B}$, i.e. for every total set X, if $B_n \leq_{\mathrm{e}} X^{(n)}$ uniformly in n then $A_n \leq_{\mathrm{e}} X^{(n)}$ uniformly in n.*

(2) *$A_n \leq_{\mathrm{e}} \mathcal{P}_n(\mathcal{B})$ uniformly in n, i.e. there is a computable function g such that $A_n = \mathit{\Gamma}_{g(n)}(\mathcal{P}_n(\mathcal{B}))$ for every n.*

It follows that if $X \subseteq \mathbb{N}$ then for every sequence $\mathcal{A} = \{A_n\}_{n<\omega}$ we have: $A_n \leq_{\mathrm{e}} X^{(n)}$ uniformly in n if and only if $\mathcal{A} \leq_\omega \{X^{(n)}\}_{n<\omega}$ if and only if $\mathcal{A} \leq_\omega X \uparrow \omega$. It is clear also that $\mathcal{A} \equiv_\omega \mathcal{P}(\mathcal{A})$.

With a slight modification of the proof of Theorem 1 we have the following:

Corollary 2. *Let $\mathcal{A}_0, \ldots, \mathcal{A}_r, \ldots$ be sequences of sets such that for every r, $\mathcal{A}_r \not\leq_\omega \mathcal{B}$. There is a total set X such that $\mathcal{B} \leq_\omega \{X^{(n)}\}_{n<\omega}$ and $\mathcal{A}_r \not\leq_\omega \{X^{(n)}\}_{n<\omega}$ for each r.*

The jump operator on the ω-enumeration degrees is defined by Soskov [11]. For every $\mathcal{A} \in \mathcal{S}$ *the ω-enumeration jump of $\mathcal{A}$* is $\mathcal{A}' = \{\mathcal{P}_{n+1}(\mathcal{A})\}_{n<\omega}$ and $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$. Furthermore $\mathcal{A}^{(k+1)} = (\mathcal{A}^{(k)})'$ and $d_\omega(\mathcal{A})^{(k+1)} = d_\omega(\mathcal{A}^{(k+1)})$. Then $\mathcal{A}^{(k)} = \{\mathcal{P}_{n+k}(\mathcal{A})\}_{n<\omega}$ for each k.

3 The ω-Degree Spectra of Structures

In this section we shall generalize the notion of degree spectrum of $\mathfrak{A}$ by considering a multi-component spectrum. The first step in this direction was the notion of relative spectrum of the structure $\mathfrak{A}$ with respect to finitely many given structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ studied in [13]. *The relative spectrum* $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set
$\{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f$ is an enumeration of $\mathfrak{A}$ s. t. $(\forall k \leq n)(f^{-1}(\mathfrak{A}_k) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(k)})\}$. It turns out that all properties of the degree spectra obtained by Soskov [8] remain true for the relative spectra.

We shall define here the notion of a spectrum of the structure $\mathfrak{A}$ with respect to a given infinite sequence of sets using the ω-enumeration reducibility.

Let $\mathcal{B} = \{B_n\}_{n<\omega}$ be a sequence of sets of natural numbers. An enumeration f of $\mathfrak{A}$ is called *acceptable with respect to the sequence* $\mathcal{B}$ if for every n, $f^{-1}(B_n) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(n)}$ uniformly in n. Denote by $\mathcal{E}(\mathfrak{A}, \mathcal{B})$ the class of all acceptable enumerations of $\mathfrak{A}$ with respect to the sequence $\mathcal{B}$.

Definition 3. *The ω-degree spectrum* of the structure $\mathfrak{A}$ with respect to the sequence $\mathcal{B}$ is the set $\mathrm{DS}(\mathfrak{A}, \mathcal{B}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})\}$.

The notion of the ω-degree spectrum is a generalization of the relative spectrum since $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{DS}(\mathfrak{A}, \mathcal{B})$, where $\mathcal{B} = \{B_k\}_{k<\omega}$, $B_0 = \emptyset$, B_k is the positive diagram of the structure $\mathfrak{A}_k$ for $0 < k \leq n$ and $B_k = \emptyset$ for all $k > n$.

Given an enumeration f of $\mathfrak{A}$ denote by $\mathcal{P}^f = \{\mathcal{P}^f_n\}_{n<\omega}$ the respective jump sequence of the sequence $\{f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0), f^{-1}(B_1), \ldots, f^{-1}(B_n), \ldots\}$ where $\mathcal{P}^f_n = \mathcal{P}_n(\{f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0), f^{-1}(B_1), \ldots, f^{-1}(B_n), \ldots\})$. Note that if f is an acceptable enumeration of $\mathfrak{A}$ with respect to $\mathcal{B}$ then $\mathcal{P}^f \equiv_\omega \{f^{-1}(\mathfrak{A})^{(n)}\}_{n<\omega} \equiv_\omega f^{-1}(\mathfrak{A}) \uparrow \omega$. So $f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ if and only if $\mathcal{P}^f \leq_\omega f^{-1}(\mathfrak{A}) \uparrow \omega$.

First we shall see that the ω-degree spectrum of the structure $\mathfrak{A}$ with respect to $\mathcal{B}$ is upwards closed with respect to total enumeration degrees.

Lemma 4. *Let f be an enumeration of $\mathfrak{A}$ and F be a total set such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$ and $f^{-1}(B_n) \leq_{\mathrm{e}} F^{(n)}$ uniformly in n. Then there exists an acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathcal{B}$ such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$.*

Proof. The construction of g is the following. Let $s \neq t \in \mathbb{N}$.

$$g(x) \simeq \begin{cases} f(x/2) & \text{if } x \text{ is even,} \\ s & \text{if } x = 2z+1 \text{ and } z \in F, \\ t & \text{if } x = 2z+1 \text{ and } z \notin F. \end{cases}$$

It is easy to see that $F \oplus f^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} g^{-1}(\mathfrak{A})$ and hence $F \equiv_{\mathrm{e}} g^{-1}(\mathfrak{A})$. Moreover for every set $B \subseteq \mathbb{N}$ we have that $g^{-1}(B) \leq_e F \oplus f^{-1}(B)$.

Then $g^{-1}(B_n) \leq_{\mathrm{e}} F \oplus f^{-1}(B_n) \leq_{\mathrm{e}} F \oplus F^{(n)} \equiv_{\mathrm{e}} F^{(n)} \equiv_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(n)}$ uniformly in n. And thus g is an acceptable enumeration of $\mathfrak{A}$ with respect to $\mathcal{B}$. □

Using Theorem 1 and the previous lemma one can find an acceptable enumeration $f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ such that $f^{-1}(\mathfrak{A})$ is a total set.

Another corollary of Lemma 4 is the following:

Proposition 5. *The ω-degree spectrum is upwards closed with respect to total e-degrees, i.e. if* $\mathbf{b}$ *is a total e-degree and for some* $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$, $\mathbf{b} \geq_{\mathrm{e}} \mathbf{a}$ *then* $\mathbf{b} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$.

It is obvious that $\mathrm{DS}(\mathfrak{A}) \supseteq \mathrm{DS}(\mathfrak{A}, \mathcal{B})$ for every structure $\mathfrak{A}$ and every $\mathcal{B} \in \mathcal{S}$. It is easy to find a structure $\mathfrak{A}$ and a sequence of sets $\mathcal{B}$ so that $\mathrm{DS}(\mathfrak{A}) \neq \mathrm{DS}(\mathfrak{A}, \mathcal{B})$. For example consider the structure $\mathfrak{A} = \{\mathbb{N}, S, =, \neq\}$, where $S \subseteq \mathbb{N}^2$ is defined as $S = \{(n, n+1) \mid n \in \mathbb{N}\}$. It is clear that the structure $\mathfrak{A}$ admits an effective enumeration f, i.e. $f^{-1}(\mathfrak{A})$ is c.e. Thus $\mathbf{0}_{\mathrm{e}} \in \mathrm{DS}(\mathfrak{A})$. By Proposition 5 all total enumeration degrees are elements of $\mathrm{DS}(\mathfrak{A})$. Consider now an arbitrary acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathcal{B} = \{B_n\}_{n<\omega}$. Fix a number x_0 such that $f(x_0) = 0$. Then $k \in B_n \iff (\exists x_1) \ldots (\exists x_k)(f^{-1}(S)(x_0, x_1) \mathbin{\&} \ldots \mathbin{\&} f^{-1}(S)(x_{k-1}, x_k) \mathbin{\&} x_k \in f^{-1}(B_n))$. Then $B_n \leq_{\mathrm{e}} f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_n) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(n)}$. Let $B_0 = \emptyset'$ and let $B_n = \emptyset$ for each $n \geq 1$. Then $\emptyset' \leq_{\mathrm{e}} B_0 \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$. Thus $\mathbf{0}_{\mathrm{e}} \notin \mathrm{DS}(\mathfrak{A}, \mathcal{B})$.

Let $k \in \mathbb{N}$. *The kth ω-jump spectrum of* $\mathfrak{A}$ *with respect to* $\mathcal{B}$ is the set $\mathrm{DS}_k(\mathfrak{A}, \mathcal{B}) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})\}$.

Proposition 6. *The kth ω-jump spectrum of* $\mathfrak{A}$ *with respect to* $\mathcal{B}$ *is upwards closed with respect to total e-degrees, i.e. if* $\mathbf{b}$ *is a total e-degree,* $\mathbf{b} \geq_{\mathrm{e}} \mathbf{a}^{(k)}$ *for some* $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$ *then* $\mathbf{b} \in \mathrm{DS}_k(\mathfrak{A}, \mathcal{B})$.

Proof. Let G be a total set, $d_{\mathrm{e}}(G) = \mathbf{b}$ and let $f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ such that $f^{-1}(\mathfrak{A}) \in \mathbf{a}$. Then $f^{-1}(\mathfrak{A})^{(k)} \leq_{\mathrm{e}} G$ and $\mathcal{P}^f_k \leq_{\mathrm{e}} G$ since $\mathcal{P}^f_k \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(k)}$. By the jump inversion theorem from [7] there exists a total set F such that $G \equiv_{\mathrm{e}} F^{(k)}$, $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$ and $f^{-1}(B_i) \leq_{\mathrm{e}} F^{(i)}$ for $i \leq k$. Moreover $f^{-1}(B_{n+k}) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(n+k)} \leq_{\mathrm{e}} G^{(n)} \equiv_{\mathrm{e}} F^{(n+k)}$ uniformly in n. By Lemma 4 there is an acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathcal{B}$ so that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$. Thus $d_{\mathrm{e}}(g^{-1}(\mathfrak{A})) \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$ and $g^{-1}(\mathfrak{A})^{(k)} \equiv_{\mathrm{e}} G$. Hence $d_{\mathrm{e}}(G) \in \mathrm{DS}_k(\mathfrak{A}, \mathcal{B})$. □

For every $\mathcal{D} \subseteq \mathcal{D}_{\mathrm{e}}$ denote by $co(\mathcal{D}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \mathbin{\&} (\forall \mathbf{a} \in \mathcal{D})(\mathbf{b} \leq_\omega \mathbf{a})\}$.

Definition 7. *The ω-co-spectrum of* $\mathfrak{A}$ *with respect to* $\mathcal{B}$ is the set $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\mathrm{DS}(\mathfrak{A}, \mathcal{B}))$.

For each $\mathcal{A} \in \mathcal{S}$ it holds that $d_\omega(\mathcal{A}) \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$ if and only if $\mathcal{A} \leq_\omega \mathcal{P}^f$ for every acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathcal{B}$. Actually the ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$ is a countable ideal of ω-enumeration degrees.

The kth ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$ is the set $\mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) = co(\mathrm{DS}_k(\mathfrak{A}, \mathcal{B}))$. It is clear that $d_\omega(\mathcal{A}) \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ if and only if $\mathcal{A} \leq_\omega \{\mathcal{P}^f_{n+k}\}_{n<\omega}$ for every acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathcal{B}$. As we shall see in section 4. Corollary 22, the kth ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$ is the least ideal containing all kth ω-enumeration jumps of the elements of $\mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

In order to obtain a forcing normal form of the sequences with ω-enumeration degrees in $\mathrm{CS}(\mathfrak{A}, \mathcal{B})$ we shall define the notions of a forcing relation $\tau \Vdash_n F_e(x)$ and a relation $f \models_n F_e(x)$.

Let f be an enumeration of $\mathfrak{A}$. For every n and $e, x \in \mathbb{N}$, define the relations $f \models_n F_e(x)$ and $f \models_n \neg F_e(x)$ by induction on n:

$$\begin{aligned}
&1.\ f \models_0 F_e(x) \iff && (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq f^{-1}(\mathfrak{A}) \oplus f^{-1}(B_0));\\
&2.\ f \models_{n+1} F_e(x) \iff && (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\
& && (u = \langle 0, e_u, x_u\rangle \ \&\ f \models_n F_{e_u}(x_u)) \vee\\
& && (u = \langle 1, e_u, x_u\rangle \ \&\ f \models_n \neg F_{e_u}(x_u)) \vee\\
& && (u = \langle 2, x_u\rangle \ \&\ x_u \in f^{-1}(B_{n+1}))));\\
&3.\ f \models_n \neg F_e(x) \iff && f \not\models_n F_e(x)\ .
\end{aligned}$$

Lemma 8. (a) *Let $A \subseteq \mathbb{N}$, $n \in \mathbb{N}$. Then $A \leq_\mathrm{e} \mathcal{P}^f_n$ if and only if $A = \{x \mid f \models_n F_e(x)\}$ for some $e \in \mathbb{N}$.*

(b) *Let $\mathcal{A} = \{A_n\}_{n<\omega}$. Then $\mathcal{A} \leq_\omega \mathcal{P}^f$ if and only if there exists a computable function g such that $A_n = \{x \mid f \models_n F_{g(n)}(x)\}$ for every n.*

The forcing conditions, called *finite parts*, are finite mappings τ of $\mathbb{N}$ to $\mathbb{N}$. We will denote the finite parts by letters δ, τ, ρ.

For each n and $e, x \in \mathbb{N}$ and for every finite part τ, define the forcing relations $\tau \Vdash_n F_e(x)$ and $\tau \Vdash_n \neg F_e(x)$ following the definition of the relation "$\models_n$".

$$\begin{aligned}
&1.\ \tau \Vdash_0 F_e(x) \iff && (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq \tau^{-1}(\mathfrak{A}) \oplus \tau^{-1}(B_0));\\
&2.\ \tau \Vdash_{n+1} F_e(x) \iff && (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\
& && (u = \langle 0, e_u, x_u\rangle \ \&\ \tau \Vdash_n F_{e_u}(x_u)) \vee\\
& && (u = \langle 1, e_u, x_u\rangle \ \&\ \tau \Vdash_n \neg F_{e_u}(x_u)) \vee\\
& && (u = \langle 2, x_u\rangle \ \&\ x_u \in \tau^{-1}(B_{n+1}))));\\
&3.\ \tau \Vdash_n \neg F_e(x) \iff && (\forall \rho \supseteq \tau)(\rho \nVdash_n F_e(x))\ .
\end{aligned}$$

An enumeration f of $\mathfrak{A}$ is *k-generic with respect to $\mathcal{B}$* if for every $j < k$ and $e, x \in \mathbb{N}$ it holds that $(\exists \tau \subseteq f)(\tau \Vdash_j F_e(x) \vee \tau \Vdash_j \neg F_e(x))$.

Lemma 9. (1) *If $\tau \subseteq \rho$ then $\tau \Vdash_k (\neg)F_e(x) \Rightarrow \rho \Vdash_k (\neg)F_e(x)$.*

(2) *For every $(k+1)$-generic enumeration f of $\mathfrak{A}$ with respect to $\mathcal{B}$ $f \models_k (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_k (\neg)F_e(x))$.*

Definition 10. Let $\mathcal{A} = \{A_n\}_{n<\omega}$. The sequence $\mathcal{A}$ is *forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$* if there exist a finite part δ and a computable function g such that for every $n \in \mathbb{N}[\ x \in A_n \iff (\exists \tau \supseteq \delta)(\tau \Vdash_n F_{g(n)}(x))\ ]$.

Proposition 11. *Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be a sequence not forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$. Then there exists an enumeration f of $\mathfrak{A}$ such that $\mathcal{A} \not\leq_\omega \mathcal{P}^f$.*

Proof. The enumeration f is constructed on stages. On each stage q we find a finite part δ_q so that $\delta_q \subseteq \delta_{q+1}$ and ultimately we define $f = \bigcup_q \delta_q$. We consider three kinds of stages. On stages $q = 3r$ we ensure that the mapping f is total and surjective. On stages $q = 3r + 1$ we ensure that f is k-generic for each $k > 0$ and on stages $q = 3r + 2$ we ensure that f satisfies the omitting condition: $\mathcal{A} \not\leq_\omega \mathcal{P}^f$.

Let $g_0, g_1, \ldots$ be an enumeration of all computable functions. For each $n, e, x \in \mathbb{N}$ denote by $Y^n_{\langle e,x\rangle}$ the set of all finite parts ρ such that $\rho \Vdash_n F_e(x)$.

Let $\delta_0 = \emptyset$. Suppose that we have already defined δ_q.

(a) Case $q = 3r$. Let x_0 be the least natural number which does not belong to $\mathrm{dom}(\delta_q)$ and let t_0 be the least natural number which does not belong to the range of δ_q. Set $\delta_{q+1}(x_0) \simeq t_0$ and $\delta_{q+1}(x) \simeq \delta_q(x)$ for $x \neq x_0$.

(b) Case $q = 3\langle e, n, x\rangle + 1$. Check whether there exists a finite part $\rho \in Y^n_{\langle e,x\rangle}$ that extends δ_q. If there is such a finite part then let δ_{q+1} be the least extension of δ_q that belongs to $Y^n_{\langle e,x\rangle}$. Otherwise let $\delta_{q+1} = \delta_q$.

(c) Case $q = 3r + 2$. Consider the function g_r. For each n denote by

$$C_n = \{x \mid (\exists \tau \supseteq \delta_q)(\tau \Vdash_n F_{g_r(n)}(x))\} \ .$$

Clearly $\mathcal{C} = \{C_n\}_{n<\omega}$ is forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$ and hence $\mathcal{C} \neq \mathcal{A}$. Then $C_n \neq A_n$ for some n. Let $\langle x, n\rangle$ be the least pair such that

$$x \in C_n \ \&\ x \notin A_n \vee x \notin C_n \ \&\ x \in A_n \ .$$

(i) Suppose that $x \in C_n$. Then there exists a finite part τ such that

$$\delta_q \subseteq \tau \ \&\ \tau \Vdash_n F_{g_r(n)}(x) \ . \tag{1}$$

Let δ_{q+1} be the least τ satisfying (1).

(ii) If $x \notin C_n$ then set $\delta_{q+1}(x) \simeq \delta_q(x)$. Note that in this case we have that $\delta_{q+1} \Vdash_n \neg F_{g_r(n)}(x)$.

Let $f = \bigcup_q \delta_q$. The enumeration f is total and surjective. Let $k \in \mathbb{N}$. In order to prove that f is $(k+1)$-generic, suppose that $j \leq k$ and consider the stage $q = 3\langle e, j, x\rangle + 1$. If there is a finite part $\rho \supseteq \delta_q$ such that $\rho \Vdash_j F_e(x)$ then from the construction we have that $\delta_{q+1} \Vdash_j F_e(x)$. Otherwise $\delta_{q+1} \Vdash_j \neg F_e(x)$.

To prove that f satisfies the omitting condition suppose for a contradiction that $\mathcal{A} \leq_\omega \mathcal{P}^f$. Then there exists a computable function g_s such that for each n we have that $A_n = \{x \mid f \models_n F_{g_s(n)}(x)\}$. Since the enumeration f is $(n+1)$-generic, by Lemma 9 we have for each number x:

$$f \models_n (\neg)F_{g_s(n)}(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_n (\neg)F_{g_s(n)}(x)). \tag{2}$$

Consider the stage $q = 3s + 2$. From the construction there are numbers n and x such that one of the following two cases holds:

(i) $x \notin A_n$ & $\delta_{q+1} \Vdash_n F_{g_s(n)}(x)$. By (2) $f \models_n F_{g_s(n)}(x)$ and hence $x \in A_n$. A contradiction.

(ii) $x \in A_n$ and $(\forall \rho \supseteq \delta_q)(\rho \nVdash_n F_{g_s(n)}(x))$. Then $\delta_q \Vdash_n \neg F_{g_s(n)}(x)$. So by (2), $f \not\models_n F_{g_s(n)}(x)$ and hence $x \notin A_n$. A contradiction. $\square$

Corollary 12. *Let $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_i \dots$ be a sequence of elements of $\mathcal{S}$ such that each $\mathcal{A}_i$ is not forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$. Then there exists an enumeration f of $\mathfrak{A}$ such that $\mathcal{A}_i \not\leq_\omega \mathcal{P}^f$ for each i.*

The construction of the enumeration f is very similar to that in Proposition 11. On stages of the form $q = 3\langle r, i\rangle + 2$ we consider the computable function g_r and ensure that $\mathcal{A}_i \neq \mathcal{C}$, where the sequence $\mathcal{C}$ is defined by the same way.

Proposition 13. *Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be a sequence not forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$. Then there exists an enumeration $g \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$ such that $\mathcal{A} \not\leq_\omega \mathcal{P}^g$ and the enumeration degree of $g^{-1}(\mathfrak{A})$ is total.*

Proof. By Proposition 11 there is an enumeration f of $\mathfrak{A}$ such that $\mathcal{A} \not\leq_\omega \mathcal{P}^f$. Then by Theorem 1 there exists a total set F such that $\mathcal{P}^f \leq_\omega \{F^{(n)}\}_{n<\omega}$ and $\mathcal{A} \not\leq_\omega \{F^{(n)}\}_{n<\omega}$. By Lemma 4 there exists an acceptable enumeration g of $\mathcal{A}$ with respect to $\mathcal{B}$ such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$ and hence $g^{-1}(\mathfrak{A})^{(n)} \equiv_{\mathrm{e}} F^{(n)}$ uniformly in n. It is clear that $\mathcal{A} \not\leq_\omega \mathcal{P}^g$. □

Corollary 14. *For every sequence $\mathcal{A}$ if $d_\omega(\mathcal{A}) \in \mathrm{CS}(\mathcal{A}, \mathcal{B})$ then $\mathcal{A}$ is forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$.*

Proof. If a sequence $\mathcal{A}$ is not forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$ then by Proposition 13 there exists an acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathcal{B}$ such that $\mathcal{A} \not\leq_\omega \mathcal{P}^g$. Hence $d_\omega(\mathcal{A}) \notin \mathrm{CS}(\mathfrak{A}, \mathcal{B})$. □

Definition 15. Let $k \in \mathbb{N}$, $\mathcal{A} \in \mathcal{S}$ and let $\mathcal{A} = \{A_n\}_{n<\omega}$. The sequence $\mathcal{A}$ is *forcing k-definable* on $\mathfrak{A}$ with respect to $\mathcal{B}$ if there exist a finite part δ and a computable function g such that for every $n \in \mathbb{N}$
$[x \in A_n \iff (\exists \tau \supseteq \delta)(\tau \Vdash_{n+k} F_{g(n)}(x))]$.

Corollary 16. *For every sequence $\mathcal{A}$ if $d_\omega(\mathcal{A}) \in \mathrm{CS}_k(\mathcal{A}, \mathcal{B})$ then $\mathcal{A}$ is forcing k-definable on $\mathfrak{A}$ with respect to $\mathcal{B}$.*

We shall give an explicit form of all sequences which are forcing k-definable on $\mathfrak{A}$ with respect to $\mathcal{B}$ by means of recursive Σ_k^+ formulae. These formulae can be considered as a modification of Ash's formulae [1] appropriate for their use on abstract structures presented by Soskov and Baleva [9].

Let $\mathcal{L} = \{T_1, \dots, T_s\}$ be the first order language of the structure $\mathfrak{A}$. For each n let P_n be a new unary predicate representing the set B_n.

(1) An elementary Σ_0^+ formula with free variables among $W_1, \dots, W_r$ is an existential formula of the form $\exists Y_1 \dots \exists Y_m \Phi(W_1, \dots, W_r, Y_1, \dots, Y_m)$, where Φ is a finite conjunction of atomic formulae in $\mathcal{L} \cup \{P_0\}$;

(2) A Σ_n^+ formula is a c.e. disjunction of elementary Σ_n^+ formulae;

(3) An elementary Σ_{n+1}^+ formula is a formula of the form
$\exists Y_1 \dots \exists Y_m \Phi(W_1, \dots, W_r, Y_1, \dots, Y_m)$, where Φ is a finite conjunction of atoms of the form $P_{n+1}(Y_j)$ or $P_{n+1}(W_i)$ and Σ_n^+ formulae or negations of Σ_n^+ formulae in $\mathcal{L} \cup \{P_0\} \cup \dots \cup \{P_n\}$.

Definition 17. Let $\mathcal{A} = \{A_n\}_{n<\omega}$ and $k \in \mathbb{N}$. The sequence $\mathcal{A}$ is *formally k-definable* on $\mathfrak{A}$ with respect to $\mathcal{B}$ if there exists a recursive sequence $\{\Phi^{\gamma(n,x)}\}_{n,x<\omega}$ of formulae such that for every n, $\Phi^{\gamma(n,x)}$ is a Σ^+_{n+k} formula with free variables among $W_1, \ldots, W_r$ and elements $t_1, \ldots, t_r$ of $\mathbb{N}$ such that for every $x \in \mathbb{N}$, the following equivalence holds:
$x \in A_n \iff (\mathfrak{A}, \mathcal{B}) \models \Phi^{\gamma(n,x)}(W_1/t_1, \ldots, W_r/t_r)$.

This means that $x \in A_n$ if and only if the formula $\Phi^{\gamma(n,x)}$ is true in $\mathfrak{A}$ with all sets B_n added as new predicates, under the variable assignment v such that $v(W_1) = t_1, \ldots, v(W_n) = t_n$. With a uniform variant of the proof given in [9] we obtain the following:

Theorem 18. *If a sequence $\mathcal{A}$ is forcing k-definable on $\mathfrak{A}$ with respect to $\mathcal{B}$ then A is formally k-definable on $\mathfrak{A}$ with respect to $\mathcal{B}$.*

Corollary 19. *Let $\mathcal{A} \in \mathcal{S}$ and let $k \in \mathbb{N}$. Then the following are equivalent:*
(1) *$d_\omega(\mathcal{A}) \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$;*
(2) *$\mathcal{A}$ is forcing k-definable on $\mathfrak{A}$ with respect to $\mathcal{B}$.*
(3) *$\mathcal{A}$ is formally k-definable on $\mathfrak{A}$ with respect to $\mathcal{B}$.*

4 Properties of the ω-Degree Spectra

We prove that some properties of degree spectra from [8] remain true for ω-degree spectra. The first property follows directly from Theorem 1 and Lemma 4.

Proposition 20. $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$ & $\mathbf{a}$ *is total e-degree*$\})$.

The next property is an analogue of the minimal pair theorem for the degree spectrum of a structure $\mathfrak{A}$ by Soskov [8]: There exist $\mathbf{f}$ and $\mathbf{g}$ in $\mathrm{DS}(\mathfrak{A})$ such that
$\mathbf{a} \leq_{\mathrm{e}} \mathbf{f}^{(k)}$ & $\mathbf{a} \leq_{\mathrm{e}} \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CS}_k(\mathfrak{A})$ for every $\mathbf{a} \in \mathcal{D}_{\mathrm{e}}$ and each $k \in \mathbb{N}$.

Theorem 21. *For every structure $\mathfrak{A}$ and every sequence $\mathcal{B} \in \mathcal{S}$ there exist total enumeration degrees $\mathbf{f}$ and $\mathbf{g}$ in $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$ such that for every ω-enumeration degree $\mathbf{a}$ and $k \in \mathbb{N}$:*

$$\mathbf{a} \leq_\omega \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq_\omega \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) \ . \qquad (3)$$

Proof. First we shall construct total enumeration degrees $\mathbf{f}$ and $\mathbf{g}$ in $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$ satisfying (3) for $k = 0$. Then we will show that $\mathbf{f}$ and $\mathbf{g}$ satisfy (3) for every k.

Let f be an acceptable enumeration of $\mathfrak{A}$ with respect to $\mathcal{B}$, such that $f^{-1}(\mathfrak{A})$ is a total set. Then $\mathcal{P}^f \equiv_\omega \{f^{-1}(\mathfrak{A})^{(n)}\}_{n<\omega}$. Denote by $F = f^{-1}(\mathfrak{A})$. It is clear that $d_{\mathrm{e}}(F) \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$.

Denote by $\mathcal{X}_0, \mathcal{X}_1, \ldots \mathcal{X}_r \ldots$ all sequences ω-enumeration reducible to $\mathcal{P}^f$.

Consider the sequence $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_r \ldots$ of these elements of $\mathcal{X}_0, \mathcal{X}_1, \ldots \mathcal{X}_r \ldots$ which are not forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$. By Proposition 12 there is an enumeration h such that $\mathcal{C}_r \not\leq_\omega \mathcal{P}^h$ for all r. Then by Corollary 2 there is a total set G such that $\mathcal{P}^h \leq_\omega \{G^{(n)}\}_{n<\omega}$ and $\mathcal{C}_r \not\leq_\omega \{G^{(n)}\}_{n<\omega}$ for all r. By

Lemma 4 there is an acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathcal{B}$ such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} G$. Thus $d_{\mathrm{e}}(G) \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$.

Suppose now that $\mathcal{A}$ is a sequence such that $\mathcal{A} \leq_\omega \{F^{(n)}\}_{n<\omega}$ and $\mathcal{A} \leq_\omega \{G^{(n)}\}_{n<\omega}$. Then $\mathcal{A} = \mathcal{X}_r$ for some r. If we assume that $\mathcal{A}$ is not forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$ then $\mathcal{A} = \mathcal{C}_l$ for some l and hence $\mathcal{A} \not\leq_\omega \{G^{(n)}\}_{n<\omega}$, which is a contradiction. Thus $\mathcal{A}$ is forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$ and $d_\omega(\mathcal{A}) \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$ by Proposition 19. Then by setting $\mathbf{f} = d_{\mathrm{e}}(F)$ and $\mathbf{g} = d_{\mathrm{e}}(G)$ we obtain the desired minimal pair.

For each $\mathbf{a} \in \mathcal{D}_e$ denote by $I(\mathbf{a}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \ \&\ \mathbf{b} \leq_\omega \mathbf{a}\} = co(\{\mathbf{a}\})$ the principal ideal generated by $\mathbf{a}$. We have that $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I(\mathbf{f}) \cap I(\mathbf{g})$, since $\mathbf{f}, \mathbf{g} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$. We shall prove now that $I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)}) = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ for every k. Since $\mathbf{f}^{(k)}, \mathbf{g}^{(k)} \in \mathrm{DS}_k(\mathfrak{A}, \mathcal{B})$ it follows that $\mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) \subseteq I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$. Suppose that $\mathcal{A} = \{A_n\}_{n<\omega}$, $\mathcal{A} \leq_\omega F^{(k)} \uparrow \omega$ and $\mathcal{A} \leq_\omega G^{(k)} \uparrow \omega$. Denote by $\mathcal{C} = \{C_n\}_{n<\omega}$ the sequence such that $C_n = \emptyset$ for $n < k$, and $C_{n+k} = A_n$ for each n. Clearly $\mathcal{A} \leq_\omega \mathcal{C}^{(k)}$ and $\mathcal{C} \leq_\omega \{F^{(n)}\}_{n<\omega}$, $\mathcal{C} \leq_\omega \{G^{(n)}\}_{n<\omega}$. So $d_\omega(\mathcal{C}) \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$. Consider an arbitrary acceptable enumeration h of $\mathfrak{A}$ with respect to $\mathcal{B}$. Then $\mathcal{C} \leq_\omega \{h^{-1}(\mathfrak{A})^{(n)}\}_{n<\omega}$ and thus $\mathcal{C}^{(k)} \leq_\omega \{h^{-1}(\mathfrak{A})^{(n)}\}^{(k)}_{n<\omega}$. It follows that $\mathcal{A} \leq_\omega \{h^{-1}(\mathfrak{A})^{(n)}\}^{(k)}_{n<\omega}$ for every $h \in \mathcal{E}(\mathfrak{A}, \mathcal{B})$. Hence $d_\omega(\mathcal{A}) \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$. □

Corollary 22. $\mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ *is the least ideal containing all kth ω-jumps of the elements of* $\mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

Proof. Ganchev [5] proved that if the enumeration degrees $\mathbf{f}$ and $\mathbf{g}$ form an exact pair for a countable ideal I of ω-enumeration degrees, i.e. $I = I(\mathbf{f}) \cap I(\mathbf{g})$ then for every k the pair $\mathbf{f}^{(k)}$, $\mathbf{g}^{(k)}$ form an exact pair for the least ideal $I^{(k)}$ containing all kth ω-jumps of the elements of I, i.e. $I^{(k)} = I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$. Let $\mathbf{f}$ and $\mathbf{g}$ be the minimal pair from Theorem 21. Since $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$ is a countable ideal, $I \subseteq \mathcal{D}_\omega$ and $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I(\mathbf{f}) \cap I(\mathbf{g})$ then $I^{(k)} = I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$ for each k. On the other hand $I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)}) = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ for each k. Thus $I^{(k)} = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$. □

Soskov [8] showed that for any structure $\mathfrak{A}$, there is a quasi-minimal e-degree $\mathbf{q}$ with respect to $\mathrm{DS}(\mathfrak{A})$, i.e. $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A})$ and if $\mathbf{a}$ is a total e-degree and $\mathbf{a} \geq_{\mathrm{e}} \mathbf{q}$ then $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ and if $\mathbf{a}$ is a total e-degree and $\mathbf{a} \leq_{\mathrm{e}} \mathbf{q}$ then $\mathbf{a} \in \mathrm{CS}(\mathfrak{A})$. We can give an analogue of this theorem.

Theorem 23. *For every structure $\mathfrak{A}$ and $\mathcal{B} \in \mathcal{S}$, there exists a set $F \subseteq \mathbb{N}$ such that for $\mathbf{q} = d_\omega(F \uparrow \omega)$ it holds:*

(1) $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A}, \mathcal{B})$;

(2) *If $\mathbf{a}$ is a total e-degree and $\mathbf{a} \geq_\omega \mathbf{q}$ then $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$;*

(3) *If $\mathbf{a}$ is a total e-degree and $\mathbf{a} \leq_\omega \mathbf{q}$ then $\mathbf{a} \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.*

Proof. Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_s)$ and $\mathcal{B} = \{B_n\}_{n<\omega}$. Consider the structure $\mathfrak{A}_0 = (\mathbb{N}; R_1, \dots, R_s, B_0)$.

Soskov [8] proved that there is a partial generic enumeration f of $\mathfrak{A}_0$ such that $d_{\mathrm{e}}(f^{-1}(\mathfrak{A}_0))$ is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A}_0)$. Moreover if $i = \lambda x.x$ then $f^{-1}(\mathfrak{A}_0) \not\leq_{\mathrm{e}} i^{-1}(\mathfrak{A}_0)$. Ganchev [4] showed that there is a set F such that

$f^{-1}(\mathfrak{A}_0) <_{\mathrm{e}} F$, $f^{-1}(B_n) \leq_{\mathrm{e}} F^{(n)}$ uniformly in n and for any total set X, if $X \leq_{\mathrm{e}} F$ then $X \leq_{\mathrm{e}} f^{-1}(\mathfrak{A}_0)$. We call the set F quasi-minimal over $f^{-1}(\mathfrak{A}_0)$ with respect to $\{f^{-1}(B_n)\}_{n<\omega}$. The set F is constructed as a partial regular enumeration of $\mathfrak{A}_0$. Set $\mathbf{q} = d_\omega(F \uparrow \omega)$. We will prove that $\mathbf{q}$ has the desired properties.

Suppose for a contradiction that $\mathbf{q} \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$. Then $d_\omega(f^{-1}(\mathfrak{A}_0) \uparrow \omega) \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$ since $f^{-1}(\mathfrak{A}_0) <_{\mathrm{e}} F$. It follows that $f^{-1}(\mathfrak{A}_0) \uparrow \omega$ is forcing definable on $\mathfrak{A}$ with respect to $\mathcal{B}$. Then $f^{-1}(\mathfrak{A}_0) \leq_{\mathrm{e}} i^{-1}(\mathfrak{A}) \oplus B_0 \equiv_{\mathrm{e}} i^{-1}(\mathfrak{A}_0)$. A contradiction.

If X is a total set and $X \leq_{\mathrm{e}} F$ then $X \leq_{\mathrm{e}} f^{-1}(\mathfrak{A}_0)$ as F is quasi-minimal over $f^{-1}(\mathfrak{A}_0)$. Thus $d_{\mathrm{e}}(X) \in \mathrm{CS}(\mathfrak{A}_0)$ by the choice of $f^{-1}(\mathfrak{A}_0)$. But $\mathrm{DS}(\mathfrak{A}, \mathcal{B}) \subseteq \mathrm{DS}(\mathfrak{A}_0)$. So $d_\omega(X \uparrow \omega) \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

If X is a total set and $X \geq_{\mathrm{e}} F$ then $X \geq_{\mathrm{e}} f^{-1}(\mathfrak{A}_0)$. Since "=" is among the predicates of $\mathfrak{A}$, $\mathrm{dom}(f) \leq_{\mathrm{e}} X$ and since X is a total set, $\mathrm{dom}(f)$ is c.e. in X. Let ρ be a recursive in X enumeration of $\mathrm{dom}(f)$. Set $h = \lambda n.f(\rho(n))$. Thus $h^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} X$ and $h^{-1}(B_n) \leq_{\mathrm{e}} X^{(n)}$ uniformly in n. By Lemma 4 there is an acceptable enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} X$. And hence $d_{\mathrm{e}}(X) \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$. □

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