A note on ω-jump inversion of degree spectra of structures

Ivan N. Soskov[1]

Faculty of Mathematics and Informatics
Sofia University

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The problem

In this paper I. Soskov provides a negative solution to the ω-jump inversion problem for degree spectra of structures.

Definition. Let $\mathfrak{A}$ be a countable structure. The *spectrum* of $\mathfrak{A}$ is the set of Turing degrees

$$Sp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \text{ computes the diagram of an isomorphic copy of } \mathfrak{A}\}.$$

For $\alpha < \omega_1^{CK}$ the α-th jump spectrum of $\mathfrak{A}$ is the set $Sp_\alpha(\mathfrak{A}) = \{\mathbf{a}^{(\alpha)} \mid \mathbf{a} \in Sp(\mathfrak{A})\}$.

Let $\alpha < \omega_1^{CK}$ and $\mathfrak{A}$ be a countable structure such that all elements of $Sp(\mathfrak{A})$ are above $\mathbf{0}^{(\alpha)}$.

Does there exist a structure $\mathfrak{M}$ such that $Sp_\alpha(\mathfrak{M}) = Sp(\mathfrak{A})$?

The jump inversion theorem

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The jump inversion theorem - the positive solutions

- *First A. Soskova and I. Soskov proved the jump inversion theorem for finite α (2007, 2009) using Marker's extensions.*
- *For successor ordinal α the idea one can see from S. Goncharov, V. Harizanov, J. Knight, C. McCoy, R. Miller and R. Solomon (2005). They did not state their result in terms of the jump inversion and they only prove the theorem for graphs, but any degree spectrum can be realized as the degree spectrum of a graph.*
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The jump of a structure

- *Another approach to jump inversion theorem for structures is given a structure $\mathfrak{A}$ that computes $\mathbf{0}^{(\alpha)}$ to find a structure $\mathfrak{B}$ such that $A \subseteq B$ and*

$$(\forall X \subseteq A)[X \in \Sigma^c_1(\mathfrak{B}) \iff X \in \Sigma^c_{\alpha+1}(\mathfrak{A})].$$

We say in this case that that $\mathfrak{B} = \mathfrak{A}^{(\alpha)}$ is an α jump of the structure $\mathfrak{A}$.
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The jump inversion theorem - a negative solution

Theorem. *[Soskov] There is a structure* $\mathfrak{A}$ *with* $Sp(\mathfrak{A}) \subseteq \{\mathbf{b} \mid \mathbf{0}^{(\omega)} \leq \mathbf{b}\}$ *for which there is no structure* $\mathfrak{M}$ *with* $Sp_\omega(\mathfrak{M}) = Sp(\mathfrak{A})$*.*

Enumeration reducibility

Definition. Given two sets of natural numbers X and Y, say that X is enumeration reducible to Y ($X \leq_e Y$) if for some e, $X = W_e(Y)$, i.e.

$$(\forall x)(x \in X \iff (\exists v)(\langle x, v\rangle \in W_e \wedge D_v \subseteq Y)).$$

Theorem. *[Selman] $X \leq_e Y$ iff for all Z if Y is c.e. in Z then X is c.e. in Z.*

Definition. Let $X \equiv_e Y$ if $X \leq_e Y$ and $Y \leq_e X$.
The enumeration degree of X is $d_e(X) = \{Y \subseteq \mathbb{N} \mid X \equiv_e Y\}$.
By $\mathcal{D}_e$ we shall denote the set of all enumeration degrees.

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Enumeration degrees

- *The degree structure* $\langle \mathcal{D}_{\mathrm{e}}, \leq \rangle$ *is defined by setting* $\mathcal{D}_{\mathrm{e}} = \{d_{\mathrm{e}}(A) \mid A \subseteq \mathbb{N}\}$, *and with partial ordering the relation* $d_{\mathrm{e}}(A) \leq d_{\mathrm{e}}(B)$ *if and only if* $A \leq_{\mathrm{e}} B$.
- *The structure* $\mathcal{D}_{\mathrm{e}}$ *is an upper semilattice with least element* $\mathbf{0}_{\mathrm{e}} = d_{\mathrm{e}}(A)$ *where* A *is any computably enumerable set.*
- *The operation of least upper bound is given by* $d_{\mathrm{e}}(A) \vee d_{\mathrm{e}}(B) = d_{\mathrm{e}}(A \oplus B)$, *where* $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

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Enumeration reducibility

Definition. Given a set $X \subseteq \mathbb{N}$, denote by $X^+ = X \oplus (\mathbb{N} \setminus X)$.
A set X is called *total* iff $X \equiv_e X^+$.

Theorem. *For any sets X and Y:*

(i) *X is c.e. in Y iff $X \leq_e Y^+$.*

(ii) *$X \leq_T Y$ iff $X^+ \leq_e Y^+$.*

Proof.

1. There is a computable function f s.t.
   if $X = W_e(Y^+)$, then $X = W^Y_{f(e)}$.
   $W^Y_{f(e)} = \{n \mid (\exists u)(\langle n, u\rangle \in W_e \ \&\ D_u \subseteq Y^+)\}$.
2. There is a computable function g s.t.
   if $X = W^Y_e$, then $X = W_{g(e)}(Y^+)$.
   $W_{g(e)} = \{\langle n, u\rangle \mid (\exists \sigma \in 2^{<\omega})(W^\sigma_e(n)\downarrow \ \&$
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Theorem. *For any sets X and Y:*

(ii) $X \leq_T Y$ *iff* $X^+ \leq_e Y^+$.

Let $\iota : \mathcal{D}_{\mathrm{T}} \to \mathcal{D}_{\mathrm{e}}$ be defined by:

$$\iota(d_{\mathrm{T}}(A)) = d_{\mathrm{e}}(A^+).$$

Remark. *The embedding ι preserves the order, the least element and the least upper bound.*

A degree is total if it is the image of a Turing degree under the standard embedding.

The enumeration jump

Definition. For any $X \subseteq \mathbb{N}$ set $J_e(X) = \{\langle a, x\rangle \mid x \in W_a(X)\}$. The *enumeration jump* X' *of* X is the set $J_e(X)^+$.

Proposition. *The standard embedding* ι *preserves the jump operation:* $J_T(X)^+ \equiv_e (X^+)'$.

Proof.

$J_T(X)^+ = (\bigoplus_a W_a^X)^+ = (\bigoplus_a W_{g(a)}(X^+))^+ \leq_m (X^+)'$.
$(X^+)' = (\bigoplus_a W_a(X^+))^+ = (\bigoplus_a W_{f(a)}^X))^+ \leq_m J_T(X)^+$. □

Remark.

- $X' \leq_T (X^+)' \leq_T J_T(X)$.
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Enumeration reducibility

Proposition. *There exists a computable function j such that for all $e \in \mathbb{N}$ and $X \subseteq \mathbb{N}$, $W_e(X)' = W_{j(e)}(X')$.*

Proof.

Consider a computable function λ such that for every a and e and for all X, $W_a(W_e(X)) = W_{\lambda(a,e)}(X)$. Then

$$2\langle a, x\rangle \in W_e(X)' \iff 2\langle \lambda(a,e), x\rangle \in X' \text{ and}$$
$$2\langle a, x\rangle + 1 \in W_e(X)' \iff 2\langle \lambda(a,e), x\rangle + 1 \in X'.$$

Let j be the computable function yielding for every e an index of the c.e. set

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Enumeration reducibility of sequences of sets

Definition. Let $\mathcal{X} = \{X_n\}_{n<\omega}$ and $\mathcal{Y} = \{Y_n\}_{n<\omega}$ be sequences of sets of natural numbers. Then $\mathcal{X}$ *is enumeration reducible to* $\mathcal{Y}$ ($\mathcal{X} \leq_e \mathcal{Y}$) if for all n, $X_n \leq_e Y_n$ uniformly in n. In other words, if there exists a computable function μ such that for all n, $X_n = W_{\mu(n)}(Y_n)$.

Definition. Let $\mathcal{X} = \{X_n\}_{n<\omega}$ be a sequence of sets of natural numbers. The *jump sequence* $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n<\omega}$ of $\mathcal{X}$ is defined by induction:

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By $\mathcal{P}_\omega(\mathcal{X})$ we shall denote the set $\bigoplus_n \mathcal{P}_n(\mathcal{X})$.
Clearly $\mathcal{X} \leq_e \mathcal{P}(\mathcal{X})$ and hence $\bigoplus_n X_n \leq_e \mathcal{P}_\omega(\mathcal{X})$.

Proposition. *For all sequences $\mathcal{X}$ of sets of natural numbers the set $\mathcal{P}_\omega(\mathcal{X})$ is total.*

Proof.

Let for all sets X, $W_{id}(X) = X$. Then

$$\begin{aligned}
&\langle n, x\rangle \notin \mathcal{P}_\omega(\mathcal{X}) \iff x \notin \mathcal{P}_n(\mathcal{X}) \iff \\
&x \notin W_{id}(\mathcal{P}_n(\mathcal{X})) \iff 2\langle id, x\rangle + 1 \in \mathcal{P}'_n(\mathcal{X}) \iff \\
&2(2\langle id, x\rangle + 1) \in \mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}'_n \oplus X_{n+1} \iff \\
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So, $\mathbb{N} \setminus \mathcal{P}_\omega(\mathcal{X}) \leq_e \mathcal{P}_\omega(\mathcal{X})$. □

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Proof.

Let $\lambda(a,b)$ be a computable function such that for all $Y \subseteq \mathbb{N}$, $W_a(Y) \oplus W_b(Y) = W_{\lambda(a,b)}(Y)$ and j be the computable function s.t. $W_e(Y)' = W_{j(e)}(Y')$.

Suppose that for all n, $X_n = W_{\mu(n)}(M^{(n)})$.

Now $P_0(\mathcal{X}) = X_0 = W_{\mu(0)}(M^{(0)})$. Suppose that $\mathcal{P}_n(\mathcal{X}) = W_a(M^{(n)})$. Then $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})' \oplus X_{n+1} = W_{j(a)}(M^{(n+1)}) \oplus W_{\mu(n+1)}(M^{(n+1)}) = W_{\lambda(j(a),\mu(n+1))}(M^{(n+1)})$. □

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Definition. Let $\mathfrak{M}$ be a countable structure and $\alpha < \omega_1^{CK}$. The α-th co-spectrum of $\mathfrak{M}$ is the set

$$CoSp_\alpha(\mathfrak{M}) = \{\mathbf{a} \mid \mathbf{a} \in D_e \wedge (\forall \mathbf{b} \in Sp_\alpha(\mathfrak{M}))(\mathbf{a} \leq_e \mathbf{b})\}.$$

Definition. Let $\alpha < \omega_1^{CK}$. A subset R of $\mathbb{N}$ is Σ^c_α definable in $\mathfrak{M}$ if there exist a computable function γ taking as values codes of computable Σ^c_α infinitary formulas $F_{\gamma(x)}$ and finitely many parameters $t_1, \ldots, t_m$ of $|\mathfrak{M}|$ such that

$$x \in R \iff \mathfrak{M} \models F_{\gamma(x)}(t_1, \ldots, t_m).$$

Theorem. *[Ash,Knight,Mannase,Slaman] Let $\alpha < \omega_1^{CK}$. Then*

1. *If $\alpha < \omega$ then $\mathbf{a} \in CoSp_\alpha(\mathfrak{M})$ if and only if all elements of $\mathbf{a}$ are $\Sigma^c_{\alpha+1}$ definable in $\mathfrak{M}$.*
2. *If $\omega \leq \alpha$ then $\mathbf{a} \in CoSp_\alpha(\mathfrak{M})$ if and only if all elements of $\mathbf{a}$ are Σ^c_α definable in $\mathfrak{M}$.*

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Computable Σ_α infinitary formulas

Definition.

(i) Let $\alpha = 1$. The Σ^c_α formula is a c.e. disjunction of formulas of the form $\exists Y_1 \ldots \exists Y_m C(X_1, \ldots, X_l, Y_1, \ldots, Y_m)$, where C is a fine conjunction of the initial (negated) predicates.

(ii) Let $\alpha = \beta + 1$. The Σ^c_α formula is a c.e. disjunction of formulae in the form

$$\exists Y_1 \ldots \exists Y_m C(X_1, \ldots, X_l, Y_1, \ldots, Y_m),$$

where C is a finite conjunction of (negated) Σ^c_β formulae.

(iii) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. The Σ^c_α formula is a c.e. disjunction of formulae in the form

$$\exists Y_1 \ldots \exists Y_l C(X_1, \ldots, X_l, Y_1, \ldots, Y_m),$$

where C is a finite conjunction of $\Sigma^c_{\alpha(p)}$ formulae.

The main property of the ω co-spectra

Theorem. *Let $\mathfrak{M}$ be a countable structure and $\mathbf{a} \in CoSp_\omega(\mathfrak{M})$. Then there exists a total enumeration degree $\mathbf{b}$ such that $\mathbf{a} \leq_e \mathbf{b}$ and $\mathbf{b} \in CoSp_\omega(\mathfrak{M})$.*

We will see that there is a structure $\mathfrak{A}$ for which all the elements of $Sp(\mathfrak{A})$ are above $\mathbf{0}^{(\omega)}$ and the greatest element of $CoSp(\mathfrak{A})$ is a non-total degree greater than $\mathbf{0}^{(\omega)}$. So for such a structure $\mathfrak{A}$ the ω-jump inversion is not possible.

Proof.

Fix an element R of $\mathbf{a} \in CoSp_\omega(\mathfrak{M})$.
R is Σ^c_ω definable in $\mathfrak{M}$ and hence there exists a computable function γ and parameters $t_1, \ldots, t_m$ of $|\mathfrak{M}|$ such that

$$x \in R \iff \mathfrak{M} \models F_{\gamma(x)}(t_1, \ldots, t_m).$$

$F_{\gamma(x)}$ is a c.e. disjunction of computable Σ_{n+1} infinitary formulas for $n \in \mathbb{N}$.
Hence there exists a computable function $\delta(n, x)$ such that for all n and x, $\delta(n, x)$ yields a code of some computable Σ^c_{n+1} infinitary formula $F_{\delta(n,x)}$ and

$$x \in R \iff (\exists n)(\mathfrak{M} \models F_{\delta(n,x)}(t_1, \ldots, t_m)).$$

□

Proof.

For each $n \in \mathbb{N}$ denote by

$$R_n = \{x \mid x \in \mathbb{N} \wedge \mathfrak{M} \models F_{\delta(n,x)}(t_1, \dots, t_m)\}.$$

Let B be the diagram of some isomorphic copy $\mathfrak{B}$ of $\mathfrak{M}$ on the natural numbers and let κ be an isomorphism from $\mathfrak{M}$ to $\mathfrak{B}$ and $x_1 = \kappa(t_1), \dots, x_m = \kappa(t_m)$. Then

$$x \in R_n \iff \mathfrak{B} \models F_{\delta(n,x)}(x_1, \dots, x_m).$$

Clearly the set of all computable Σ^c_{n+1} formulae $F_{\delta(n,x)}$ with fixed parameters $x_1, \dots, x_m$ which are satisfied in $\mathfrak{B}$ is uniformly in n enumeration reducible to $B^{(n)}$. So $R_n \leq_e B^{(n)}$ uniformly in n. We have also that $\mathcal{P}(\{R_n\}) \leq_e \{B^{(n)}\}$. Hence

$$\mathcal{P}_\omega(\{R_n\}) \leq_e B^{(\omega)}.$$

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Proof.

Set $\mathbf{b} = d_e(\mathcal{P}_\omega(\{R_n\}))$.

- $\mathbf{b} \in CoSp_\omega(\mathfrak{M})$ (we just showed).
- $\mathbf{b}$ is a total degree.
- It remains to see that $\mathbf{a} \leq_e \mathbf{b}$.
  Indeed, since $x \in R \iff (\exists n)(x \in R_n)$, $R \leq_e \bigoplus_n R_n$.
  On the other hand $\bigoplus_n R_n \leq_e \mathcal{P}_\omega(\{R_n\})$.
  Therefore $R \leq_e \mathcal{P}_\omega(\{R_n\})$.

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Lemma. *There is a set Y which is quasi-minimal above $\emptyset^{(\omega)}$, i.e. $\emptyset^{(\omega)} <_e Y$ and for every total set X such that $X \leq_e Y$ we have that $X \leq_e \emptyset^{(\omega)}$.*

Hint: Consider $Y = \emptyset^{(\omega)} \oplus G$, where G is one-generic relatively $\emptyset^{(\omega)}$(Copestake).

Corollary. *There is a set Y for which $d_e(Y)$ does not contain any total set and $\emptyset^{(\omega)} \leq_e Y$.*

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A negative solution for the ω-jump inversion problem

Let Y be a quasi-minimal above $\emptyset^{(\omega)}$ set.

- *Consider a structure $\mathfrak{A}$ such that*
  $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$.
  Then $Sp(\mathfrak{A}) \subseteq \{\mathbf{b} \mid \mathbf{0}^{(\omega)} \leq_T \mathbf{b}\}$.
- *Assume that there exists a countable structure $\mathfrak{M}$ such that*
  $Sp_\omega(\mathfrak{M}) = Sp(\mathfrak{A})$.
  Then $CoSp_\omega(\mathfrak{M}) = CoSp(\mathfrak{A})$.
- *By the main property of $CoSp_\omega(\mathfrak{M})$ there exists a total degree $\mathbf{b}$ in $CoSp_\omega(\mathfrak{M})$ such that $d_e(Y) \leq \mathbf{b}$.*
  On the other hand since $\mathbf{b} \in CoSp(\mathfrak{A})$ we have $\mathbf{b} \leq d_e(Y)$.
  This is impossible since $\mathbf{b}$ is a total degree, but $d_e(Y)$ is not.

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Theorem. *[Soskov] Every countable ideal is a co-spectrum of a structure.*

Here we will proof that every principal countable ideal is a co-spectrum of a group.

Consider a non-trivial subgroup G of the additive group of the rationales Q.

For every $a \neq 0$ element of G and every prime number p set

$$h_p(a) = \begin{cases} k & \text{if } k \text{ is the greatest number such that } p^k | a \text{ in } G, \\ \infty & \text{if } p^k | a \text{ in } G \text{ for all } k. \end{cases}$$

Let $p_0, p_1, \dots$ be the standard enumeration of the prime numbers and set

$$S_a(G) = \{\langle i, j \rangle : j \leq h_{p_i}(a)\}.$$

If a and b are non-zero elements of G, then $S_a(G) \equiv_e S_b(G)$.

Denote by $\mathbf{d}_G = d_e(S_a(G))$, for some non-zero element a of G.

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A structure $\mathfrak{A}$ with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$

Proposition. *[Coles, Downey and Slaman]*
$Sp(G) = \{\mathbf{b} \mid \mathbf{b}$ *is total* & $\mathbf{d}_G \leq_e \mathbf{b}\}$.

Corollary. $CoSp(G) = \{\mathbf{a} \mid \mathbf{a} \leq_e \mathbf{d}_G\}$.

Proof.

Clearly $\mathbf{a} \in CoSp(G)$ if and only if for all total $\mathbf{b}$,
$\mathbf{d}_G \leq_e \mathbf{b} \Rightarrow \mathbf{a} \leq_e \mathbf{b}$.
According Selman's Theorem the last is equivalent to $\mathbf{a} \leq_e \mathbf{d}_G$. □

Theorem. *[Selman]* $A \leq_e B \iff \forall C$ *if* B *is c.e. in* C *then* A *is c.e. in* C.
Equivalently $A \leq_e B \iff \forall$ *total* $C[B \leq_e C \Rightarrow A \leq_e C]$.

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A structure $\mathfrak{A}$ with $CoSp(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$

Consider the set

$$S = \{\langle i, j\rangle \mid (j = 0) \vee (j = 1 \ \&\ i \in Y)\}.$$

Clearly $S \equiv_e Y$.
Let G *be the least subgroup of* Q *containing the set*

$$\{1/p_i^j \mid \langle i, j\rangle \in S\}.$$

Then $1 \in G$ *and* $S_1(G) = S$. *So,* $\mathbf{d}_G = d_e(Y)$.

Theorem. $CoSp(G) = \{\mathbf{a} \mid \mathbf{a} \leq_e d_e(Y)\}$.

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Definition. A set G is 1-generic and for every c.e. set S of finite binary strings

$$(\exists\tau \in 2^{<\omega})(\tau \subseteq G \ \&\ ((\tau \in S) \vee (\forall\rho \supseteq \tau)(\rho \notin S))).$$

Proposition. *Every one-generic set G is quasi-minimal, i.e. $\emptyset <_{\mathrm{e}} G$ and for every total set $X \leq_{\mathrm{e}} G$ we have that X is c.e.*

Proof.

Consider first the following set:

$$S = \{\tau : \tau \in 2^{\omega} \ \&\ (\exists x \in \mathrm{dom}(\tau) \cap G))(\tau(x) = 0)\}.$$

Suppose that G is c.e.. Since G is 1-generic then there is a finite binary string $\tau \subseteq G$ such that $\tau \in S$ or $(\forall\rho)(\tau \subseteq \rho \Rightarrow \rho \notin S)$. It is clear that both cases are impossible. So G is not c.e.. □

One-generic set

Proof.

For any total set $X \subseteq \mathbb{N}$ one can construct a total function g on $\mathbb{N}$, so that $g \equiv_{\mathrm{e}} X$. To prove that G is quasi-minimal, it is sufficient to show that if g is a total function and $g \leq_{\mathrm{e}} G$, then g computable.
Let g be a total function and $g = W_e(G)$. Let
$W_e(\tau) = W_e(\{n \mid \tau(n) = 1\})$, for a binary string τ.
Consider the set:

$$S_1 = \{\tau \mid \tau \in 2^{<\omega} \;\&\; (\exists x, y_1 \neq y_2 \in \mathbb{N})(\langle x, y_1\rangle \in W_e(\tau) \;\&\; \langle x, y_2\rangle \in W_e(\tau))\}.$$

Since S_1 is c.e., we have that there exists a finite part $\tau \subseteq G$ such that either $\tau \in S_1$ or $(\forall \rho)(\rho \supseteq \tau \Rightarrow \rho \notin S_1)$.
Assume that $\tau \in S_1$. Then there exist $x, y_1 \neq y_2$ such that $\langle x, y_1\rangle \in W_e(\tau)$ and $\langle x, y_2\rangle \in W_e(\tau)$. Then $g(x) = y_1$ and $g(x) = y_2$, which is impossible. So, $(\forall \rho)(\rho \supseteq \tau \Rightarrow \rho \notin S_1)$.

□

One-generic set

Proof.

Fix x. Suppose that there exists binary strings $\mu_1 \supseteq \tau$ and $\mu_2 \supseteq \tau$, and numbers $y_1 \neq y_2$, such that $\langle x, y_1 \rangle \in W_e(\mu_1)$ and $\langle x, y_2 \rangle \in W_e(\mu_2)$.
Then define a binary string $\mu \supseteq \tau$ with length $max(|\mu_1|, |\mu_2|)$ and $\mu(z) = 1 \iff \mu_1(z) = 1 \vee \mu_2(z) = 1$. Obviously $\langle x, y_1 \rangle \in W_e(\mu)$ and $\langle x, y_2 \rangle \in W_e(\mu)$. Hence $\mu \in S_1$. A contradiction.
So g is computable.

□

Thank you!