Conservative Extensions and the Jump of a Structure

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Outline

- Degree spectra of structures
- Definability on structures
- Conservative (k, n) Extensions
- The Jump of a structure

Degree Spectra

Let $\mathfrak{A} = (A; P_1, \ldots, P_k)$ *be a denumerable structure. Enumeration of* $\mathfrak{A}$ *is every one to one mapping of* $\mathbb{N}$ *onto* A*.*

Given an enumeration f *of* $\mathfrak{A}$ *and a subset of* X *of* A^a*, let*

$$f^{-1}(X) = \{\langle x_1, \ldots, x_a \rangle : (f(x_1), \ldots, f(x_a)) \in X\}.$$

Set $f^{-1}(\mathfrak{A}) = f^{-1}(P_1) \oplus \cdots \oplus f^{-1}(P_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$*.*

Definition.[Richter] *The Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A})\}.$$

Degree Spectra

Definition.[Knight] The n-th jump spectrum of a structure $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}.$$

Proposition.*[Knight] For every automorphically nontrivial structure $\mathfrak{A}$, $DS_n(\mathfrak{A})$ is an upwards closed set of degrees.*

Degree Spectra

Theorem. *[A., Soskov] Every first jump spectrum is a spectrum of a structure, i.e. for every countable structure $\mathfrak{A}$ there is a structure $\mathfrak{B}$ such that $DS_1(\mathfrak{A}) = DS(\mathfrak{B})$.*

Theorem. *[A., Soskov] Let $\mathfrak{A}$ and $\mathfrak{C}$ be countable structures and $DS(\mathfrak{A}) \subseteq DS_1(\mathfrak{C})$. There exists a structure $\mathfrak{B}$ such that $DS(\mathfrak{A}) = DS_1(\mathfrak{B})$ and $DS(\mathfrak{B}) \subseteq DS(\mathfrak{C})$.*

Computable Σ_n^c-formulas

The computable Σ_n^c formulas are defined inductively:

- *A computable Σ_0^c (Π_0^c) formula is a finitary quantifier-free formula.*
- *A computable Σ_{n+1}^c formula $\Phi(\overline{x})$ is a disjunction of c.e. set of formulas of the form*

$$(\exists \overline{Y})\Psi(\overline{X}, \overline{Y})$$

  Ψ is a finite conjunction of Σ_n^c and Π_n^c formulas
- *Π_{n+1}^c formulas are the negations of the Σ_{n+1}^c formulas.*

Formally Σ_n^c-definable sets

Definition. A set $X \subseteq A$ is *formally* Σ_n^c*- definable on* $\mathfrak{A}$ ($X \in \Sigma_n^c(\mathfrak{A})$) if there exists a computable Σ_n^c formula $\Phi(W_1, \ldots, W_r, X)$ and elements $t_1, \ldots, t_r$ of A such that:

$$x \in X \leftrightarrow \mathfrak{A} \models \Phi(W_1/t_1, \ldots, W_r/t_r, X/x).$$

Example

Consider $\mathcal{O} = (\mathbb{N}; =)$ *and* $\mathcal{S} = (\mathbb{N}; G_{Succ}; =)$, *where* G_{Succ} *is the graph of the successor function.*

$$DS(\mathcal{O}) = DS(\mathcal{S})$$

The $\Sigma_1^c(\mathcal{O})$ *sets are all finite and co-finite sets of natural numbers. But all c.e. set are formally* Σ_1^c *definable on* $\mathcal{S}$.
So, the structure $\mathcal{S}$ *is more powerful than the* $\mathcal{O}$.

Example

Consider $\mathcal{O} = (\mathbb{N}; =)$ *and* $\mathcal{S} = (\mathbb{N}; G_{Succ}; =)$, *where* G_{Succ} *is the graph of the successor function.*

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The $\Sigma^c_1(\mathcal{O})$ *sets are all finite and co-finite sets of natural numbers.*
But all c.e. set are formally Σ^c_1 *definable on* $\mathcal{S}$.
So, the structure $\mathcal{S}$ *is more powerful than the* $\mathcal{O}$.

Definition. The pair $\alpha = (f_\alpha, R_\alpha)$ is *an enumeration* of the set $X \subseteq A$, if R_α is a set of natural numbers, f_α is a partial one-to-one mapping of $\mathbb{N}$ onto X and $\mathrm{dom}(f_\alpha) = f_\alpha^{-1}(X)$ is c.e. in R_α.
We denote this by $X \leq \alpha$.

Definition. The pair $\alpha = (f_\alpha, R_\alpha)$ is an *enumeration* of $\mathfrak{A}$ if f_α is an enumeration of A and $f_\alpha^{-1}(\mathfrak{A})$ is computable in R_α.
We denote this by $\mathfrak{A} \leq \alpha$.

Denote by $\alpha^{(n)} = (f_\alpha, R_\alpha^{(n)})$.

Reformulation

The Degree Spectrum of $\mathfrak{A}$ *is the set*

$$DS(\mathfrak{A}) = \{d_T(R_\alpha) \mid \mathfrak{A} \leq \alpha\}.$$

Theorem. *(Ash, Knigh, Manasse, Slaman, Chisholm)*
For every set $X \subseteq A$,

$$X \in \Sigma^c_{n+1}(\mathfrak{A}) \leftrightarrow (\forall \alpha)[\mathfrak{A} \leq \alpha \rightarrow X \leq \alpha^{(n)}].$$

Let $\alpha = (f_\alpha, R_\alpha)$ and $\beta = (f_\beta, R_\beta)$ be enumerations of the structures $\mathfrak{A}$ and $\mathfrak{B}$ respectively.
We write $\alpha \leq \beta$ if

(i) *$R_\alpha \leq_T R_\beta$ and*

(ii) *the set*

$$E(f_\alpha, f_\beta) = \{(x, y) \mid x \in Dom(f_\alpha) \ \&\ y \in Dom(f_\beta) \ \&\ f_\alpha(x) = f_\beta(y)\}.$$

is c.e. in R_β.

Conservative (k, n) Extensions

Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures, possibly with different signatures and $A \subseteq B$.

(i) $\mathfrak{A} \leq_n^k \mathfrak{B}$ iff for every enumeration β of $\mathfrak{B}$ there exists an enumeration α of $\mathfrak{A}$ such that $\alpha^{(k)} \leq \beta^{(n)}$.

(ii) $\mathfrak{A} \geq_n^k \mathfrak{B}$ iff for every enumeration α of $\mathfrak{A}$ there exists an enumeration β of $\mathfrak{B}$ such that $\beta^{(n)} \leq \alpha^{(k)}$.

(iii) $\mathfrak{A} \equiv_n^k \mathfrak{B}$ if $\mathfrak{A} \leq_n^k \mathfrak{B}$ and $\mathfrak{A} \geq_n^k \mathfrak{B}$. We shall say that $\mathfrak{B}$ is a (k, n)-conservative extension of $\mathfrak{A}$.

Note that the relation $\equiv_n^k$ is not symmetric.

Proposition. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A \subseteq B$.*

(i) *If* $\mathfrak{A} \leq_n^k \mathfrak{B}$ *then* $DS_n(\mathfrak{B}) \subseteq DS_k(\mathfrak{A})$;

(ii) *If* $\mathfrak{A} \geq_n^k \mathfrak{B}$ *then* $DS_k(\mathfrak{A}) \subseteq DS_n(\mathfrak{B})$;

(iii) *If* $\mathfrak{A} \equiv_n^k \mathfrak{B}$ *then* $DS_k(\mathfrak{A}) = DS_n(\mathfrak{B})$;

Corollary.

(i) $k = 1, n = 0$:
If $\mathfrak{A} \equiv_0^1 \mathfrak{B}$ *then* $DS_1(\mathfrak{A}) = DS(\mathfrak{B})$.

(ii) $k = 0, n = 1$:
If $\mathfrak{A} \equiv_1^0 \mathfrak{B}$ *then* $DS(\mathfrak{A}) = DS_1(\mathfrak{B})$.

Theorem. *Let for* $\mathfrak{A}$ *and* $\mathfrak{B}$: $A \subseteq B$. *For all* $k, n \in \mathbb{N}$,

(i) *if* $\mathfrak{A} \leq_n^k \mathfrak{B}$ *then* $(\forall X \subseteq A)[X \in \Sigma_{k+1}^c(\mathfrak{A}) \to X \in \Sigma_{n+1}^c(\mathfrak{B})]$;

(ii) *if* $\mathfrak{A} \geq_n^k \mathfrak{B}$ *then* $(\forall X \subseteq A)[X \in \Sigma_{n+1}^c(\mathfrak{B}) \to X \in \Sigma_{k+1}^c(\mathfrak{A})]$;

(iii) *if* $\mathfrak{A} \equiv_n^k \mathfrak{B}$ *then* $(\forall X \subseteq A)[X \in \Sigma_{k+1}^c(\mathfrak{A}) \leftrightarrow X \in \Sigma_{n+1}^c(\mathfrak{B})]$.

The opposite direction is not always true:

Example.
Consider $\mathcal{O}_A = (A; =)$ *and take* $\mathfrak{A} = \mathfrak{B} = \mathcal{O}_A$.
For every natural number n,
$X \subseteq A$ *is* $\Sigma^c_n(\mathcal{O}_A)$ *iff* X *is a finite or co-finite subset of A.*
Therefore $\Sigma^c_1(\mathcal{O}_A) = \Sigma^c_n(\mathcal{O}_A)$ *and*

$$(\forall n)(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathcal{O}_A) \to X \in \Sigma^c_1(\mathcal{O}_A)].$$

But $(\forall n)[\mathcal{O}_A \leq^n_0 \mathcal{O}_A]$ *is evidently not true.*

Moschovakis' extension

Let $\mathfrak{A} = (A; P_1, \ldots, P_l)$ *and* $\bar{0} \notin A$.
Set $A_0 = A \cup \{\bar{0}\}$.
Let $\langle ., . \rangle$ *be a pairing function s.t. none of the elements of* A *is a pair.*
Let $A^\star$ *be the least set containing* A_0 *and closed under* $\langle ., . \rangle$.
The decoding functions: $L(\langle s, t \rangle) = s$ & $R(\langle s, t \rangle) = t$

Moschovakis' extension

Definition. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^\star = (A^\star, P_1, \ldots, P_l, A_0, G_{\langle .,. \rangle}, G_L, G_R).$$

Proposition. *$\mathfrak{A} \equiv_n^n \mathfrak{A}^\star$ for every $n \in \mathbb{N}$.*

Proposition. *For every two structures $\mathfrak{A}$, $\mathfrak{B}$ with $A \subseteq B$ and natural numbers n, k*
$\mathfrak{A} \equiv_n^k \mathfrak{B}$ iff $\mathfrak{A}^\star \equiv_n^k \mathfrak{B}^\star$.

Theorem. *[Vatev] Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable structures with $A^\star \subseteq B$ and $k, n \in \mathbb{N}$.*
If $(\forall X \subseteq A^\star)[X \in \Sigma^c_{k+1}(\mathfrak{A}^\star) \to X \in \Sigma^c_{n+1}(\mathfrak{B})]$ then $\mathfrak{A} \leq^k_n \mathfrak{B}$.

Corollary. *For any two countable structures $\mathfrak{A}$, $\mathfrak{B}$ with $A \subseteq B$ and $n, k \in \mathbb{N}$,*

$$\mathfrak{A} \leq^k_n \mathfrak{B} \leftrightarrow (\forall X \subseteq A^\star)[X \in \Sigma^c_{k+1}(\mathfrak{A}^\star) \to X \in \Sigma^c_{n+1}(\mathfrak{B}^\star)].$$

The set $K^{\mathfrak{A}}$

A new predicate $K_{\mathfrak{A}}$ (analogue of Kleene's set).
For $e, x \in \mathbb{N}$ and finite part τ, let

$$\tau \Vdash F_e(x) \leftrightarrow x \in W_e^{\tau^{-1}(\mathfrak{A})}.$$

$$\tau \Vdash \neg F_e(x) \leftrightarrow (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x)).$$

$$K^{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

Definition. The jump of a structure $\mathfrak{A}$ is

$$\mathfrak{A}^{(1)} = (\mathfrak{A}^\star, K^{\mathfrak{A}}).$$

Theorem. $\mathrm{DS}_1(\mathfrak{A}) = \mathrm{DS}(\mathfrak{A}^{(1)})$.

Proposition.

(i) $\mathfrak{A} \equiv^1_0 \mathfrak{A}^{(1)}$;

(ii) $\mathfrak{A} \not\equiv^0_0 \mathfrak{A}^{(1)}$.

Let $\mathfrak{A} = (A; P_1, \ldots, P_k, =)$.

- *Marker's $\exists$-extension of P_i ($P_i^{\exists}$):*
- $X_i = \{x^i_{\langle a_1,\ldots,a_{r_i}\rangle} \mid P_i(a_1, \ldots, a_{r_i})\}$ *($\exists$-fellow for P_i).*
- $P_i^{\exists}(a_1, \ldots, a_{r_i}, x) \leftrightarrow$
  $a_1, \ldots, a_{r_i} \in A \;\&\; x \in X_i \;\&\; x = x^i_{\langle a_1,\ldots,a_{r_i}\rangle}$.
- $\mathfrak{A}^{\exists} = (A \cup \bigcup_{i=1}^{k} X_i, P_1^{\exists}, \ldots, P_k^{\exists}, \bar{X}_1, \ldots, \bar{X}_k, =)$.

Marker's Extensions

- *Marker's* $\forall$*-extension of* P_i $(P_i^\forall)$*:*
- $Y_i = \{y^i_{\langle a_1,\ldots,a_{r_i}\rangle} \mid \neg P_i(a_1,\ldots,a_{r_i})\}$ *(*$\forall$*-fellow for* P_i*).*
- *If* $P_i^\forall(a_1,\ldots,a_{r_i},y)$ *then* $a_1,\ldots,a_{r_i} \in A$ *and* $y \in Y_i$*;*
- *If* $a_1,\ldots,a_{r_i} \in A$ *&* $y \in Y_i$ *then*
  $\neg P_i^\forall(a_1,\ldots,a_{r_i},y) \leftrightarrow y = y^i_{\langle a_1,\ldots,a_{r_i}\rangle}$.
- $\mathfrak{A}^\forall = (A \cup \bigcup_{i=1}^k Y_i, P_1^\forall,\ldots,P_k^\forall,\bar{Y}_1,\ldots,\bar{Y}_k,=)$.

The Jump Inversion Theorem

Theorem. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be countable structures and $DS(\mathfrak{A}) \subseteq DS_1(\mathfrak{C})$. There exists a structure $\mathfrak{B} = \mathfrak{A}^{\exists\forall} \oplus \mathfrak{C}$ such that $DS(\mathfrak{A}) = DS_1(\mathfrak{B})$ and $DS(\mathfrak{B}) \subseteq DS(\mathfrak{C})$.*

Remark. *Similar results by:*

Montalban : *a different approach, keeps the domain of the structure and adds a complete set of Π^c_n formulas.*

Stukachev : *for Σ reducibility*

Stukachev proves an analogue of this theorem for the semilattices of Σ-degrees of structures with arbitrary cardinalities.

The Jump Inversion Theorem

Theorem. *If* $\mathcal{O}_A \leq_0^1 \mathfrak{A}$, *then* $\mathfrak{A} \equiv_1^0 \mathfrak{A}^{\exists\forall}$.

Remark. *Note that* $\mathcal{O}_A \leq_0^k \mathfrak{A}$ *iff the elements of* $DS(\mathfrak{A})$ *are above* $\mathbf{0}^{(k)}$.

Corollary. *If* $\mathcal{O}_A \leq_0^k \mathfrak{A}$ *for some* $k \in \mathbb{N}$ *then for each* $n \in \mathbb{N}$, *there is a structure* $\mathfrak{B}$ *such that*

$$(\forall X \subseteq A)[X \in \Sigma^c_{n+1}(\mathfrak{A}) \leftrightarrow X \in \Sigma^c_{k+1}(\mathfrak{B})].$$

Some problems

- *The definition of* $\mathfrak{A} \equiv_n^k \mathfrak{B}$ *is not symmetric since we suppose that* $A \subseteq B$. *How to define the similar relation more symmetric and for arbitrary* $\mathfrak{A}$ *and* $\mathfrak{B}$*?*
- *How to relativize the Jump Inversion Theorem for structures?*
- *The Jump inversion Theorem for structures for arbitrary constructive ordinal* α.

Thank you!

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