Enumeration Degree Spectra

Definability in Computable Structures May 12-13, 2012, Chicago

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[1]Supported by David and Lucile Packard Foundation and BNSF Grant No. D002-258/18.12.08; Sofia University Science Fund, contract 131/09.05.2012

Enumeration reducibility

Definition. We say that $\Gamma : 2^\omega \to 2^\omega$ is an *enumeration operator* iff for some c.e. set W_i for each $B \subseteq \omega$

$$\Gamma(B) = \{x | (\exists D)[\langle x, D\rangle \in W_i \ \& D \subseteq B]\}.$$

Definition. The set A is *enumeration reducible to* the set B ($A \leq_e B$), if $A = \Gamma(B)$ for some e-operator Γ.
The enumeration degree of A is $d_e(A) = \{B \subseteq \omega | A \equiv_e B\}$.
The set of all enumeration degrees is denoted by $\mathcal{D}_e$.

The enumeration jump

Definition. Given a set A, denote by $A^+ = A \oplus (\omega \setminus A)$.

Theorem. *For any sets A and B:*

1. *A is c.e. in B iff $A \leq_e B^+$.*
2. *$A \leq_T B$ iff $A^+ \leq_e B^+$.*

Definition. For any set A let $K_A = \{\langle i, x\rangle | x \in \Gamma_i(A)\}$. Set $A' = K_A^+$.

Definition. A set A is called *total* iff $A \equiv_e A^+$.

Let $d_e(A)' = d_e(A')$. The enumeration jump is always a total degree and agrees with the Turing jump under the standard embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$.

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Degree spectra

Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ be a countable structure. An enumeration of $\mathfrak{A}$ is every surjective (partial) mapping of ω onto A.

Given an enumeration f of $\mathfrak{A}$ and a subset of B of A^a, let

$$f^{-1}(B) = \{\langle x_1, \ldots, x_a \rangle \mid x_1, \ldots, x_a \in dom(f) \ \& \ (f(x_1), \ldots, f(x_a)) \in B\}.$$

$$f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq).$$

Definition.[Richter] *The Turing degree spectrum of $\mathfrak{A}$* is the set

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) \mid f \text{ is a one-to-one total enum. of } \mathfrak{A}\}.$$

If **a** is the least element of $DS_T(\mathfrak{A})$, then **a** is called the *degree of* $\mathfrak{A}$.

Definition.[Soskov] *The enumeration degree spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is a total enumeration of } \mathfrak{A}\}.$$

If **a** is the least element of $DS(\mathfrak{A})$, then **a** is called the *e-degree of* $\mathfrak{A}$.

Proposition. *The enumeration degree spectrum is closed upwards with respect to total e-degrees, i.e. if* $\mathbf{a} \in DS(\mathfrak{A})$, $\mathbf{b}$ *is a total e-degree* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS(\mathfrak{A})$.

Total structures

Definition. The structure $\mathfrak{A}$ is called *total* if for every total enumeration f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ is total.

Proposition. *If $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A}) = \iota(DS_T(\mathfrak{A}))$.*

Given a structure $\mathfrak{A} = (A, R_1, \ldots, R_k)$, for every j denote by R_j^c the complement of R_j and let $\mathfrak{A}^+ = (A, R_1, \ldots, R_k, R_1^c, \ldots, R_k^c)$.

Proposition.

- $\iota(DS_T(\mathfrak{A})) = DS(\mathfrak{A}^+)$.
- *If $\mathfrak{A}$ is total then $DS(\mathfrak{A}) = DS(\mathfrak{A}^+)$.*

Proposition. *If $\mathfrak{A}$ has e-degree* **a** *then* $\mathbf{a} = d_e(f^{-1}(\mathfrak{A}))$ *for some one-to-one enumeration f of $\mathfrak{A}$.*

Definition. Let $\mathcal{A}$ be a nonempty set of enumeration degrees. The *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_e \mathbf{a})\}.$$

Definition. Given a structure $\mathfrak{A}$, set $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$. If $\mathbf{a}$ is the greatest element of $CS(\mathfrak{A})$ then we call $\mathbf{a}$ the *co-degree* of $\mathfrak{A}$.

If $\mathfrak{A}$ has a degree $\mathbf{a}$ then $\mathbf{a}$ is also the co-degree of $\mathfrak{A}$. The vice versa is not always true.

Co-spectra

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Definition. A set B of natural numbers is admissible in $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $B \leq_e f^{-1}(\mathfrak{A})$.

Clearly $\mathbf{a} \in CS(\mathfrak{A})$ *iff* $\mathbf{a} = d_e(B)$ *for some admissible in* $\mathfrak{A}$ *set* B.

Every finite mapping of ω *into* A *is called a finite part.*
For every finite part τ *and natural numbers* e, x, *let*

$$\tau \Vdash F_e(x) \iff x \in \Gamma_e(\tau^{-1}(\mathfrak{A})) \text{ and}$$
$$\tau \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x)).$$

Definition. An enumeration f of $\mathfrak{A}$ is *generic* if for every $e, x \in \omega$, there exists a $\tau \subseteq f$ s.t. $\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)$.

The admissible in $\mathfrak{A}$ sets

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Forcing definable in $\mathfrak{A}$ sets

Definition. A set B of natural numbers is *forcing definable in the structure* $\mathfrak{A}$ iff there exist a finite part δ and a natural number e s.t.

$$B = \{x | (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

Theorem. *Let $B \subseteq \omega$ and $d_e(C) \in DS(\mathfrak{A})$. Then the following are equivalent:*

1. *B is admissible in $\mathfrak{A}$.*
2. *$B \leq_e f^{-1}(\mathfrak{A})$ for all generic enumerations f of $\mathfrak{A}$ s.t. $(f^{-1}(\mathfrak{A}))' \equiv_e C'$.*
3. *B is forcing definable on $\mathfrak{A}$.*

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The formally definable sets on $\mathfrak{A}$

Definition. A Σ^c_1 formula with free variables among $W_1, \ldots, W_r$ is a c.e. disjunction of existential formulae of the form $\exists Y_1 \ldots \exists Y_k \theta(\bar{Y}, \bar{W})$, where θ is a finite conjunction of atomic and negated atomic formulae.

Definition. A set $B \subseteq \omega$ is *formally definable* on $\mathfrak{A}$ if there exists a recursive function $\gamma(x)$, such that $\bigvee_{x \in \omega} \Phi_{\gamma(x)}$ is a Σ^c_1 formula with free variables among $W_1, \ldots, W_r$ and elements $t_1, \ldots, t_r$ of A such that the following equivalence holds:

$$x \in B \iff \mathfrak{A} \models \Phi_{\gamma(x)}(W_1/t_1, \ldots, W_r/t_r) \ .$$

Theorem. *Let $B \subseteq \omega$. Then*

1. *B is admissible in $\mathfrak{A}$ ($d_e(B) \in CS(\mathfrak{A})$) iff*
2. *B is forcing definable on $\mathfrak{A}$ iff*
3. *B is formally definable on $\mathfrak{A}$.*

The partial case

Definition. *The partial enumeration degree spectrum of* $\mathfrak{A}$ is the set

$$DS^p(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is a partial enumeration of } \mathfrak{A}\}.$$

Lemma. *If f is a partial enumeration of $\mathfrak{A}$ then* $\mathrm{dom}(f) \leq_e f^{-1}(\mathfrak{A})$.

Proposition. *The partial enumeration degree spectrum is closed upwards with respect to enumeration degrees, i.e. if* $\mathbf{a} \in DS^p(\mathfrak{A})$ *and* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS^p(\mathfrak{A})$.

Theorem. *Let $B \subseteq \omega$. The following are equivalent:*

1. *$B \leq_e f^{-1}(\mathfrak{A})$ for all partial enumerations of $\mathfrak{A}$ ($d_e(B) \in CS^p(\mathfrak{A})$)*
2. *B is formally definable on $\mathfrak{A}$.*

Definition. The nth jump spectrum of $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})^{(n)}) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

If $\mathbf{a}$ is the least element of $DS_n(\mathfrak{A})$, then $\mathbf{a}$ is called the *nth jump degree of* $\mathfrak{A}$.

Definition. The co-set $CS_n(\mathfrak{A})$ of the nth jump spectrum of $\mathfrak{A}$ is called nth jump co-spectrum of $\mathfrak{A}$.
If $CS_n(\mathfrak{A})$ has a greatest element then it is called the *nth jump co-degree of* $\mathfrak{A}$.

Some examples

Example. *[Richter] Let* $\mathfrak{A} = (A; <)$ *be a linear ordering.* $DS(\mathfrak{A})$ *contains a minimal pair of degrees and hence* $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$. $\mathbf{0}_e$ *is the co-degree of* $\mathfrak{A}$. *So, if* $\mathfrak{A}$ *has a degree* $\mathbf{a}$, *then* $\mathbf{a} = \mathbf{0}_e$.

Example. *[Knight] For a linear ordering* $\mathfrak{A}$, $CS_1(\mathfrak{A})$ *consists of all e-degrees of* Σ^0_2 *sets. The first jump co-degree of* $\mathfrak{A}$ *is* $\mathbf{0}'_e$.

Example. *[Slaman,Whener] There exists a structure* $\mathfrak{A}$ *s.t.*

$$DS(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}.$$

Clearly the structure $\mathfrak{A}$ *has co-degree* $\mathbf{0}_e$ *but has not a degree.*

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Representing the principle countable ideals as co-spectra

Example. *[Coles, Downey, Slaman] Let G be a torsion free abelian group of rank 1, i.e. G is a subgroup of Q.*
There exists an enumeration degree $\mathbf{s}_G$ such that

- $DS(G) = \{\mathbf{b} : \mathbf{b}$ *is total and* $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- *The co-degree of G is $\mathbf{s}_G$.*
- *G has a degree iff $\mathbf{s}_G$ is a total e-degree.*
- *If $1 \leq n$, then $\mathbf{s}_G^{(n)}$ is the n-th jump degree of G.*

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, s.t. $\mathbf{s}_G = \mathbf{d}$.

Corollary. *Every principle ideal of enumeration degrees is $CS(G)$ for some G.*

Remark. *If we consider the partial enumeration degree spectra, then the abelian groups of rank 1 always have an e-degree.*

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Remark. *If we consider the partial enumeration degree spectra, then the abelian groups of rank 1 always have an e-degree.*

Example. *Let* $B_0, \dots, B_n, \dots$ *be a sequence of sets of natural numbers. Set* $\mathfrak{A} = (\mathbb{N}; f; \sigma)$,

$$f(\langle i, n\rangle) = \langle i+1, n\rangle;$$
$$\sigma = \{\langle i, n\rangle : n = 2k+1 \vee n = 2k \ \&\ i \in B_k\}.$$

Then $CS(\mathfrak{A}) = I(d_e(B_0), \dots, d_e(B_n), \dots)$

Spectra with a countable base

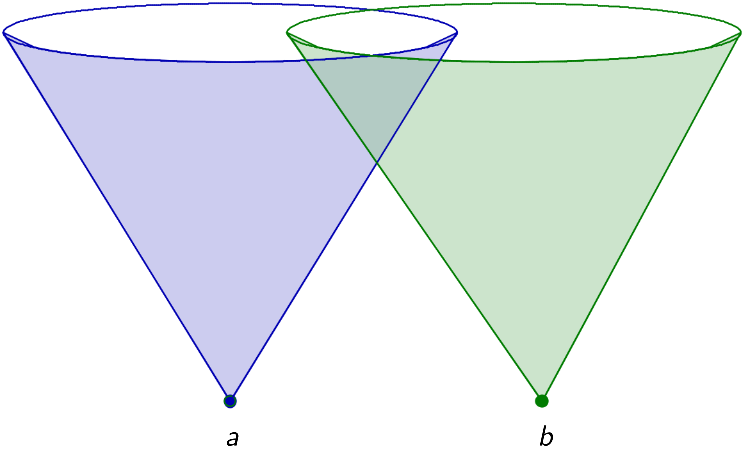
Definition. Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

Theorem. *A structure $\mathfrak{A}$ has an e-degree if and only if $DS(\mathfrak{A})$ has a countable base.*

Suppose that the sequence of e-degrees $\{\mathbf{b}_i\}_i$ is a base for $DS(\mathfrak{A})$. Assume that no $\mathbf{b}_i$ is an e-degree of $\mathfrak{A}$. Then for every i, $\mathbf{b}_i \notin CS(\mathfrak{A})$.
Let $B_i \in \mathbf{b}_i$ for every $i \in \omega$. Then all the sets B_i have no forcing normal form.
We can construct a generic enumeration f of $\mathfrak{A}$, omitting all B_i, i.e. $B_i \not\leq_e f^{-1}(\mathfrak{A})$.
This contradicts with fact that $\{\mathbf{b}_i\}_i$ is a base for $DS(\mathfrak{A})$.

An upwards closed set of degrees which is not a degree spectra of a structure



General properties of upwards closed sets

Definition. Consider a subset $\mathcal{A}$ of $\mathcal{D}_e$. Say that $\mathcal{A}$ is *upwards closed* if for every $\mathbf{a} \in \mathcal{A}$ all total degrees greater than $\mathbf{a}$ are contained in $\mathcal{A}$.

Theorem. *[Selman] $\mathbf{a} \leq_e \mathbf{b}$ iff for all total $\mathbf{c}$ $(\mathbf{b} \leq_e \mathbf{c} \Rightarrow \mathbf{a} \leq_e \mathbf{c})$.*

Proposition. *Let $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A}\ \&\ \mathbf{a}$ is total$\}$. Then $co(\mathcal{A}) = co(\mathcal{A}_t)$.*

Proposition. *Let $\mathbf{b}$ be an arbitrary enumeration degree and $n > 0$. Set $\mathcal{A}_{\mathbf{b},n} = \{\mathbf{a} : \mathbf{a} \in \mathcal{A}\ \&\ \mathbf{b} \leq_e \mathbf{a}^{(n)}\}$. Then $co(\mathcal{A}) = co(\mathcal{A}_{\mathbf{b},n})$.*

Specific properties of the degree spectra

Theorem. *Let $\mathfrak{A}$ be a structure, $1 \leq n$ and $\mathbf{c} \in DS_n(\mathfrak{A})$. Then*

$$CS(\mathfrak{A}) = co(\{\mathbf{b} \in DS(\mathfrak{A}) : \mathbf{b}^{(n)} = \mathbf{c}\}).$$

Example. *(Upwards closed set for which the Theorem is not true)*

Let $B \not\leq_e A$ and $A \not\leq_e B'$. Let

$$\mathcal{D} = \{\mathbf{a} : d_e(A) \leq_e \mathbf{a}\} \cup \{\mathbf{a} : d_e(B) \leq_e \mathbf{a}\}.$$

Set $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \;\&\; \mathbf{a}' = d_e(B)'\}$.

- *$d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$.*
- *$d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.*

Specific properties of the degree spectra

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- *$d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.*

The minimal pair theorem

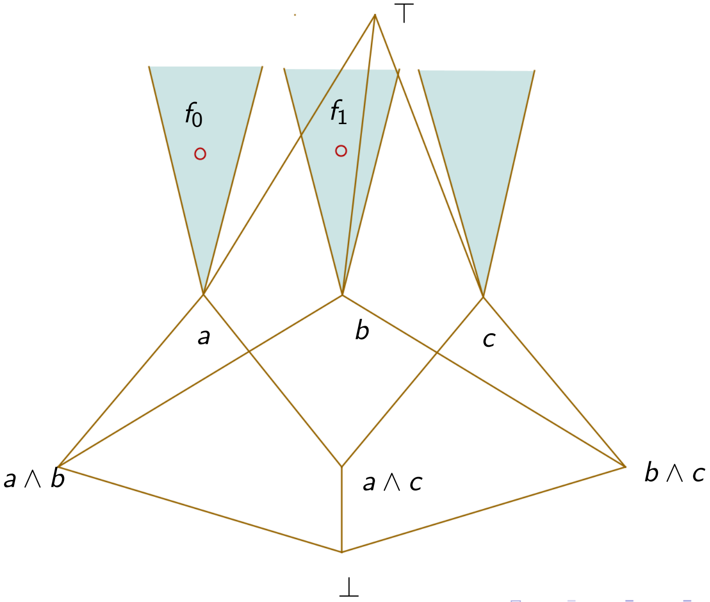
Theorem. *Let* $\mathbf{c} \in DS_2(\mathfrak{A})$*. There exist* $\mathbf{f},\mathbf{g} \in DS(\mathfrak{A})$ *s.t.* $\mathbf{f},\mathbf{g}$ *are total,* $\mathbf{f}'' = \mathbf{g}'' = \mathbf{c}$ *and* $CS(\mathfrak{A}) = co(\{\mathbf{f},\mathbf{g}\})$*.*

Notice that for every enumeration degree $\mathbf{b}$ *there exists a structure* $\mathfrak{A}_\mathbf{b}$ *s. t.* $DS(\mathfrak{A}_\mathbf{b}) = \{\mathbf{x} \in \mathcal{D}_T | \mathbf{b} <_e \mathbf{x}\}$*. Hence*

Corollary. *[Rozinas] For every* $\mathbf{b} \in \mathcal{D}_e$ *there exist total* $\mathbf{f}, \mathbf{g}$ *below* $\mathbf{b}''$ *which are a minimal pair over* $\mathbf{b}$*.*

Not every upwards closed set of enumeration degrees has a minimal pair:

An upwards closed set with no minimal pair



The quasi-minimal degree

Definition. Let $\mathcal{A}$ be a set of enumeration degrees. The degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if:

- $\mathbf{q} \notin co(\mathcal{A})$.
- If $\mathbf{a}$ is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
- If $\mathbf{a}$ is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

From Selman's theorem it follows that if $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$, then $\mathbf{q}$ is an upper bound of $co(\mathcal{A})$.

Theorem. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

Partial generic enumerations

Let $\perp \notin A$.

Definition. A *partial finite part* is a finite mapping of ω into $A \cup \{\perp\}$.

Let τ be a partial finite part and let f be a partial enumeration, by $\tau \subseteq f$ we denote that for all x in $\mathrm{dom}(\tau)$ either $\tau(x) = \perp$ and $f(x)$ is not defined or $\tau(x) \in A$ and $f(x) = \tau(x)$.

Definition. A subset B of ω is *partially forcing definable* on $\mathfrak{A}$ if there exist an $e \in \omega$ and a partial finite part δ such that for all natural numbers x,

$$x \in B \iff (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x)).$$

Lemma. *Let $B \subseteq \omega$ be partially forcing definable on $\mathfrak{A}$. Then $d_e(B) \in CS(\mathfrak{A})$.*

The quasi-minimal degree

Proposition.

1. *For every partial generic f, $f^{-1}(\mathfrak{A}) \not\leq_e D(\mathfrak{A})$. Hence $d_e(f^{-1}(\mathfrak{A})) \notin CS(\mathfrak{A})$.*
2. *There exists a partial generic enumeration $f \leq_e D(\mathfrak{A})'$ such that $f^{-1}(\mathfrak{A}) \leq_e D(\mathfrak{A})'$.*
3. *If $B \leq_e f^{-1}(\mathfrak{A})$ for all partial generic enumerations f, then B is partially forcing definable on $\mathfrak{A}$.*

Theorem. *Let f be a partial generic enumeration of $\mathfrak{A}$. Then $d_e(f^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $DS(\mathfrak{A})$.*

Corollary. *[Slaman and Sorbi] Let I be a countable ideal of enumeration degrees. There exists an enumeration degree $\mathbf{q}$ s.t.*

1. *If $\mathbf{a} \in I$ then $\mathbf{a} <_e \mathbf{q}$.*
2. *If $\mathbf{a}$ is total and $\mathbf{a} \leq_e \mathbf{q}$ then $\mathbf{a} \in I$.*

Proposition. *For every countable structure $\mathfrak{A}$ there exist continuum many quasi-minimal degrees with respect to $DS(\mathfrak{A}$.*

Suppose that all quasi-minimal degrees with respect to $DS(\mathfrak{A})$ are $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_n, \ldots$ and let $X_i \in \mathbf{q}_i$, for all $i \in \omega$. Then all $\mathbf{q}_i$ are not in $CS(\mathfrak{A})$ and hence every X_i is not forcing definable on $\mathfrak{A}$.
We could build a partial generic enumeration f of $\mathfrak{A}$ such that $X_i \not\leq_e f^{-1}(\mathfrak{A})$.
Thus $d_e(f^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $DS(\mathfrak{A})$ and not in $\{\mathbf{q}_i\}$.

Jumps of quasi-minimal degrees

Lemma. *Let* $\mathbf{a} \in DS_1(\mathfrak{A})$ *and* g *be an enumeration of* $\mathfrak{A}$ *such that* $g^{-1}(\mathfrak{A})' \in \mathbf{a}$. *There exists a partial generic enumeration* f *such that* $f^{-1}(\mathfrak{A})' \equiv_e g^{-1}(\mathfrak{A})'$.

Proposition. *The first jump spectrum of every structure* $\mathfrak{A}$ *consists exactly of the enumeration jumps of the quasi-minimal degrees.*

Corollary. *[McEvoy] For every total e-degree* $\mathbf{a} \geq_e \mathbf{0}'_e$ *there is a quasi-minimal degree* $\mathbf{q}$ *with* $\mathbf{q}' = \mathbf{a}$.

Splitting a total set

Proposition. *[Jockusch] For every total e-degree* $\mathbf{a}$ *there are quasi-minimal degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$*.*

Proposition. *For every element* $\mathbf{a}$ *of the jump spectrum of a structure* $\mathfrak{A}$ *there exists quasi-minimal with respect to* $\mathfrak{A}$ *degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$*.*

A method of splitting a total set

Suppose that $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_n)$.
Denote by Δ *the set of all finite parts.*
For each $\tau \in \Delta$ *and* $x \in \mathbb{N}$ *by* $\tau * x$ *we denote an extension of* τ *such that* $\tau * x(lh(\tau)) = x$.
Let $f : \Delta \to \Delta$ *and* $\{y_n\}_n$ *be a sequence of natural numbers.*
If $\tau_0 = \emptyset$, $\tau_{n+1} = f(\tau_n * y_i)$, *then we denote by* $f(\{y_n\}_n) = \bigcup_n \tau_n$.
Let P *be a set of enumerations of* $\mathfrak{A}$.

Lemma.*[Ganchev] If* f *is computable in the total set* Q *and such that for every sequence* $\{y_n\}_n$ *computable in* Q, $f(\{y_n\}_n) \in P$, *then there exist enumerations* $g, h \in P$ *of* $\mathfrak{A}$ *such that* $Q \equiv_e \langle g \rangle \oplus \langle h \rangle$.

A method of splitting a total set

Let q be an enumeration of Q such that $\langle q \rangle \leq_e Q$. We construct two sequences of finite parts $\{\tau_n\}_n$ and $\{\sigma_n\}_n$ by the following rule:

1. $\tau_0 = \sigma_0 = \emptyset$;
2. $y_n = \langle lh(\sigma_n), q(2n) \rangle$;
3. $\tau_{n+1} = f(\tau_n * y_n)$;
4. $z_n = \langle lh(\tau_n), q(2n+1) \rangle$;
5. $\sigma_{n+1} = f(\sigma_n * z_n)$.

Define $g = f(\{y_n\}_n)$ and $h = f(\{z_n\}_n)$.

Let $\mathbf{a} \in DS_1(\mathfrak{A})$ and $B' \in \mathbf{a}$. $Q = B'$ is a total set. Let P be the class of all partial generic enumerations of $\mathfrak{A}$. Applying the lemma we have that $\mathbf{p} = d_e(\langle g \rangle)$ and $\mathbf{q} = d_e(\langle h \rangle)$ are quasi-minimal for $DS(\mathfrak{A})$ and $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Every jump spectrum is the spectrum of a total structure

Let $\mathfrak{A} = (A; R_1, \ldots, R_n)$.
Let $\bar{0} \notin A$. *Set* $A_0 = A \cup \{\bar{0}\}$. *Let* $\langle ., . \rangle$ *be a pairing function s.t. none of the elements of* A_0 *is a pair and* A^* *be the least set containing* A_0 *and closed under* $\langle ., . \rangle$.

Definition. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (A^*, R_1, \ldots, R_n, A_0, G_{\langle ., . \rangle}).$$

Let $K_{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$.
Set $\mathfrak{A}' = (\mathfrak{A}^*, K_{\mathfrak{A}}, A^* \setminus K_{\mathfrak{A}})$.

Theorem.

1. *The structure* $\mathfrak{A}'$ *is total.*
2. $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$.

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The jump inversion theorem

Consider two structures $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that

$$DS(\mathfrak{B})_t = \{\mathbf{a} | \mathbf{a} \in DS(\mathfrak{B}) \text{ and } \mathbf{a} \text{ is total}\} \subseteq DS_1(\mathfrak{A}).$$

Theorem. *There exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS_1(\mathfrak{C}) = DS(\mathfrak{B})_t$.*

Method: Marker's extensions.

Corollary. *Let $DS(\mathfrak{B}) \subseteq DS_1(\mathfrak{A})$. Then there exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS(\mathfrak{B}) = DS_1(\mathfrak{C})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}'$. Then there exists a total structure $\mathfrak{C}$ such that $DS(\mathfrak{B}) = DS(\mathfrak{C})$.*

The jump inversion theorem

Theorem. *Let $n \geq 1$. Suppose that $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. There exists a structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}^{(n)}$. Then there exists a total structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Remark.

2009 *Montalban, Notes on the jump of a structure, Mathematical Theory and Computational Practice, 372–378.*

2009 *Stukachev, A jump inversion theorem for the semilattices of Sigma-degrees, Siberian Electronic Mathematical Reports, v. 6, 182 – 190*

2012 *Montalban, Rice Sequences of Relations, to appear in the Philosophical Transactions A.*

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Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}^{(n)}$. Then there exists a total structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

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Example. *[Ash, Jockusch, Knight and Downey] Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ s.t. $\mathfrak{C}$ has a $n+1$-th jump degree $\mathbf{0}^{(n+1)}$ but has no k-th jump degree for $k \leq n$.*

It is sufficient to construct a structure $\mathfrak{B}$ satisfying:

1. *$DS(\mathfrak{B})$ has not a least element.*
2. *$\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.*
3. *All elements of $DS(\mathfrak{B})$ are total and above $\mathbf{0}^{(n)}$.*

Consider a set B satisfying:

1. *B is quasi-minimal above $\mathbf{0}^{(n)}$.*
2. *$B' \equiv_e \mathbf{0}^{(n+1)}$.*

Let G be a subgroup of the additive group of the rationals s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{\mathbf{a} \mid d_e(S_G) \leq_e \mathbf{a}$ and $\mathbf{a}$ is total$\}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$.

Applications

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Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = \{\mathbf{a} \mid \mathbf{0}^{(n)} <_e \mathbf{a}\}$.
It is sufficient to construct a structure $\mathfrak{B}$ such that the elements of $DS(\mathfrak{B})$ are exactly the total e-degrees greater than $\mathbf{0}^{(n)}$.
This could be done by Whener's construction using a special family of sets:

Theorem. *Let $n \geq 0$. There exists a family $\mathcal{F}$ of sets of natural numbers s.t. for every X strictly above $\mathbf{0}^{(n)}$ there exists a computable in X set U satisfying the equivalence:*

$$F \in \mathcal{F} \iff (\exists a)(F = \{x | (a, x) \in U\}).$$

But there is no such U c.e. in $\mathbf{0}^{(n)}$.

Recent examples of spectra

2004 *Soskov, Degree spectra and co-spectra of structures, Ann. Univ. Sofia, 96, 45–68.*

2007 *Calvert, Harizanov and Shlapentokh, Turing degrees of isomorphism types of algebraic objects, Journal of the London Math. Soc., 73, 273–286.*

2007 *Kalimullin, Some notes on degree spectra of the structures. CiE 2007, LNCS 4497, 389–397*

2009 *Frolov, Kalimullin and Miller, Spectra of algebraic fields and subfields, CiE 2009, LNCS, 5635, 232–241.*

2010 *Frolov, Harizanov, Kalimullin, Kudinov and Miller, Spectra of* $high_n$ *and* $nonlow_n$ *degrees, Journal of Logic and Computation; doi:10.1093/logcom/exq041.*

2012 *Andersen, Kach, Melnikov and Solomon, Jump degrees of torsion free abelian groups, submitted.*

Degree spectra

- Questions:
  - Describe the sets of Turing degrees (enumeration degrees) which are equal to $DS(\mathfrak{A})$ for some structure $\mathfrak{A}$.
  - Is the set of all Muchnik degrees containing some degree spectra definable in the lattice of the Muchnik degrees?

Thank you!