Quasi-minimal degrees for degree spectra

Spring Scientific Conference

16.03.2013

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[1]This research was partially supported by Sofia University Science Fund and a NSF grant DMS- 1101123.

Enumeration reducibility

Definition. We say that $\Gamma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is an *enumeration operator* iff for some c.e. set W_i for each $B \subseteq \mathbb{N}$

$$\Gamma(B) = \{x | (\exists D)[\langle x, D\rangle \in W_i \ \& D \subseteq B]\}.$$

Definition. The set A is *enumeration reducible to* the set B ($A \leq_e B$), if $A = \Gamma(B)$ for some e-operator Γ.
The enumeration degree of A is $d_e(A) = \{B \subseteq \mathbb{N} | A \equiv_e B\}$.
The set of all enumeration degrees is denoted by $\mathcal{D}_e$.

- $\mathbf{0_e} = d_e(\emptyset) = \{W \mid W \textit{ is c.e.}\}$.
- $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- $\mathcal{D}_e = \langle \mathcal{D}_e; \leq; \oplus; \mathbf{0_e}\rangle$ *is an upper semi-lattice with least element.*

The enumeration reducibility

Definition. Given a set A, denote by $A^+ = A \oplus (\mathbb{N} \setminus A)$.
A set A is called *total* iff $A \equiv_e A^+$.

Theorem. *For any sets A and B:*

1. *A is c.e. in B iff $A \leq_e B^+$.*
2. *$A \leq_T B$ iff $A^+ \leq_e B^+$.*

Theorem. *[Selman]* $\mathbf{a} \leq_e \mathbf{b}$ *iff for all total* $\mathbf{c}$ $(\mathbf{b} \leq_e \mathbf{c} \Rightarrow \mathbf{a} \leq_e \mathbf{c})$.

The enumeration jump

Definition. For any set A let $K_A = \{\langle i, x\rangle | x \in \Gamma_i(A)\}$. Set $A' = K_A^+$.

- *Let* $d_e(A)' = d_e(A')$.
- *The enumeration jump is always a total degree and agrees with the Turing jump under the standard embedding* $\iota : \mathcal{D}_T \to \mathcal{D}_e$ *by* $\iota(d_T(A)) = d_e(A^+)$.
- A *is* Σ^B_{n+1} *if* $A \leq_e (B^+)^{(n)}$.

Theorem. *[Soskov] For every* $\mathbf{x} \in \mathcal{D}_e$ *there exists a total e-degree* $\mathbf{a} \geq \mathbf{x}$, *such that* $\mathbf{a}' = \mathbf{x}'$.

Degree spectra

Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ be a countable structure. An enumeration of $\mathfrak{A}$ is every one to one mapping of $\mathbb{N}$ onto A.

Definition. *The degree spectrum of $\mathfrak{A}$* is the set of all Turing degrees which computes the diagram of an isomorphic copy of $\mathfrak{A}$.

Given an enumeration f of $\mathfrak{A}$ and a subset of B of A^a, let

$$f^{-1}(B) = \{\langle x_1, \ldots, x_a\rangle \mid (f(x_1), \ldots, f(x_a)) \in B\}.$$

$f^{-1}(\mathfrak{A}) = f^{-1}(R_1)^+ \oplus \cdots \oplus f^{-1}(R_k)^+$.

Degree spectra

Definition. *The degree spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_T \ \& \ (\exists f)(d_T(f^{-1}(\mathfrak{A})) \leq_T \mathbf{a})\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathfrak{A})$ then we call $\mathbf{a}$ the *degree* of $\mathfrak{A}$.

Co-spectra

Definition.[Soskov] The *co-spectrum of* $\mathfrak{A}$ is the set

$$CS(\mathfrak{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in DS(\mathfrak{A}))(\mathbf{b} \leq_e \mathbf{a})\}.$$

If $\mathbf{a}$ *is the greatest element of* $CS(\mathfrak{A})$ *then we call* $\mathbf{a}$ *the co-degree of* $\mathfrak{A}$.

Soskov proved that every countable ideal of enumeration degrees is a co-spectrum of a structure.

The admissible in $\mathfrak{A}$ sets

Definition. A set B of natural numbers is admissible in $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $B \leq_e f^{-1}(\mathfrak{A})$.

Clearly $\mathbf{a} \in CS(\mathfrak{A})$ *iff* $\mathbf{a} = d_e(B)$ *for some admissible in* $\mathfrak{A}$ *set* B.

Every finite one-to-one mapping of $\mathbb{N}$ *into* A *is called a finite part. For every finite part* τ *and natural numbers* e, x, *let*

$$\tau \Vdash F_e(x) \iff x \in \Gamma_e(\tau^{-1}(\mathfrak{A})) \text{ and}$$
$$\tau \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x)).$$

Definition. An enumeration f of $\mathfrak{A}$ is *generic* if for every $e, x \in \mathbb{N}$, there exists a $\tau \subseteq f$ s.t. $\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)$.

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Forcing definable in $\mathfrak{A}$ sets

Definition. A set B of natural numbers is *forcing definable in the structure* $\mathfrak{A}$ iff there exist a finite part δ and a natural number e s.t.

$$B = \{x | (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

Denote by $D(\mathfrak{A})$ the diagram of $\mathfrak{A}$.

Proposition. *Let* $\{B_i\}_{i \in \mathbb{N}}$ *be subsets of* $\mathbb{N}$ *be not forcing definable on* $\mathfrak{A}$. *There exists a* 1*-generic enumeration* f *of* $\mathfrak{A}$ *satisfying the following conditions:*

1. $f \leq_e D(\mathfrak{A})'$.
2. $f^{-1}(\mathfrak{A})' \leq_e f \oplus D(\mathfrak{A})'$.
3. $B_i \not\leq_e f^{-1}(\mathfrak{A})$ *for every* $i \in \mathbb{N}$.

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Proposition. *Let $\{B_i\}_{i \in \mathbb{N}}$ be subsets of $\mathbb{N}$ be not forcing definable on $\mathfrak{A}$. There exists a 1-generic enumeration f of $\mathfrak{A}$ satisfying the following conditions:*

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3. $B_i \not\leq_e f^{-1}(\mathfrak{A})$ *for every* $i \in \mathbb{N}$.

The formally definable sets on $\mathfrak{A}$

Definition. A Σ^c_1 formula with free variables among $W_1, \ldots, W_r$ is a c.e. disjunction of existential formulae of the form $\exists Y_1 \ldots \exists Y_k \theta(\bar{Y}, \bar{W})$, where θ is a finite conjunction of atomic and negated atomic formulae.

Definition. A set $B \subseteq \mathbb{N}$ is *formally definable* on $\mathfrak{A}$ if there exists a recursive function $\gamma(x)$, such that $\bigvee_{x \in \mathbb{N}} \Phi_{\gamma(x)}$ is a Σ^c_1 formula with free variables among $W_1, \ldots, W_r$ and elements $t_1, \ldots, t_r$ of A such that the following equivalence holds:

$$x \in B \iff \mathfrak{A} \models \Phi_{\gamma(x)}(W_1/t_1, \ldots, W_r/t_r) \ .$$

Normal form

Theorem. *Let* $B \subseteq \mathbb{N}$. *Then*

1. B *is admissible in* $\mathfrak{A}$ ($d_e(B) \in CS(\mathfrak{A})$) *iff*
2. B *is forcing definable on* $\mathfrak{A}$ *iff*
3. B *is formally definable on* $\mathfrak{A}$.

Corollary. *If* $\mathfrak{B}$ *is an isomorphic structure of* $\mathfrak{A}$ *then a set* $X \subseteq \mathbb{N}$ *is forcing definable on* $\mathfrak{A}$ *if and only if* X *is forcing definable on* $\mathfrak{B}$.

Jump spectra

Definition. The nth jump spectrum of $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} \mid \in DS(\mathfrak{A})\}.$$

Definition. The nth jump co-spectrum $CS_n(\mathfrak{A})$ of $\mathfrak{A}$ is the set

$$CS_n(\mathfrak{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in DS_n(\mathfrak{A}))(\mathbf{b} \leq \mathbf{a})\}.$$

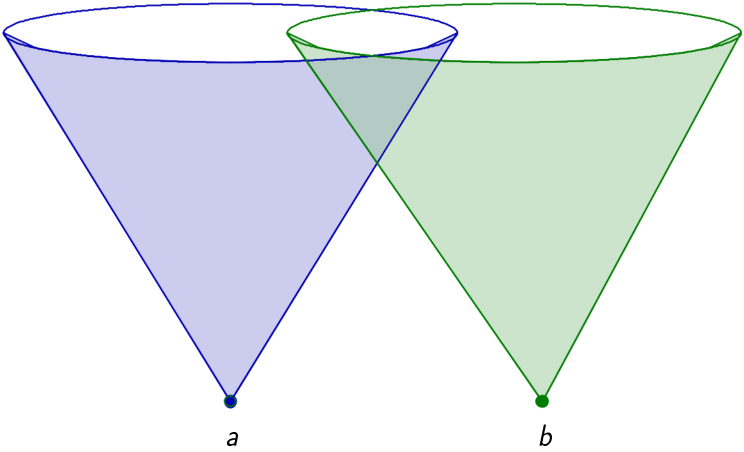
Definition. Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

Theorem. *A structure $\mathfrak{A}$ has a degree if and only if $DS(\mathfrak{A})$ has a countable base.*

Suppose that the sequence of e-degrees $\{\mathbf{b}_i\}_i$ is a base for $DS(\mathfrak{A})$. Assume that no $\mathbf{b}_i$ is an e-degree of $\mathfrak{A}$. Then for every i,
$\mathbf{b}_i \notin CS(\mathfrak{A})$.
Let $B_i \in \mathbf{b}_i$ for every $i \in \mathbb{N}$. Then all the sets B_i have no forcing normal form.
We can construct a generic enumeration f of $\mathfrak{A}$, omitting all B_i, i.e. $B_i \not\leq_e f^{-1}(\mathfrak{A})$.
This contradicts with fact that $\{\mathbf{b}_i\}_i$ is a base for $DS(\mathfrak{A})$.

An upwards closed set of degrees which is not a degree spectra of a structure



The minimal pair theorem

Theorem. *[Soskov] There exist* $\mathbf{f},\mathbf{g} \in DS(\mathfrak{A})$ *such that*

$$(\forall \mathbf{b} \in \mathcal{D}_e)(\mathbf{b} \leq \mathbf{f} \ \& \ \mathbf{b} \leq \mathbf{g} \Rightarrow \mathbf{b} \in CS(\mathfrak{A})).$$

Definition. [Medevdev (1955)] An e-degree $\mathbf{a}$ is said to be quasi–minimal if

- $\mathbf{a} \neq \mathbf{0}_e$;
- $(\forall \text{ total } \mathbf{b})[\mathbf{b} \leq \mathbf{a} \to \mathbf{b} = \mathbf{0}_e]$.

Definition.[Slaman, Sorbi] Given any $I \subseteq \mathcal{D}_e$, we say that an e-degree $\mathbf{a}$ is I-quasi– minimal if

- $(\forall \mathbf{c} \in I)[\mathbf{c} < \mathbf{a}]$;
- $(\forall \text{ total } \mathbf{c})[\mathbf{c} \leq a \iff (\exists \mathbf{b} \in I)[\mathbf{c} \leq \mathbf{b}]$.

Definition. Let $\mathcal{A}$ be a set of enumeration degrees. The degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if:

- $\mathbf{q} \notin co(\mathcal{A})$.
- If $\mathbf{a}$ is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
- If $\mathbf{a}$ is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

From Selman's theorem it follows that if $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$, then $\mathbf{q}$ is an upper bound of $co(\mathcal{A})$.

Theorem. *[Soskov] For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

Partial generic enumerations

Let $\perp \notin A$.

Definition. A *partial finite part* is a finite mapping of $\mathbb{N}$ into $A \cup \{\perp\}$.

Let τ be a partial finite part and let f be a partial enumeration, by $\tau \subseteq f$ we denote that for all x in $\mathrm{dom}(\tau)$ either $\tau(x) = \perp$ and $f(x)$ is not defined or $\tau(x) \in A$ and $f(x) = \tau(x)$.

Definition. A subset B of $\mathbb{N}$ is *partially forcing definable* on $\mathfrak{A}$ if there exist an $e \in \mathbb{N}$ and a partial finite part δ such that for all natural numbers x,

$$x \in B \iff (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x)).$$

Lemma. *Let $B \subseteq \mathbb{N}$ be partially forcing definable on $\mathfrak{A}$. Then $d_e(B) \in CS(\mathfrak{A})$.*

The quasi-minimal degree

Proposition.

1. *For every partial generic* f, $f^{-1}(\mathfrak{A}) \not\leq_e D(\mathfrak{A})$. *Hence* $d_e(f^{-1}(\mathfrak{A})) \notin CS(\mathfrak{A})$.
2. *There exists a partial generic enumeration* $f \leq_e D(\mathfrak{A})'$ *such that* $f^{-1}(\mathfrak{A}) \leq_e D(\mathfrak{A})'$.
3. *If* $B \leq_e f^{-1}(\mathfrak{A})$ *for all partial generic enumerations* f, *then* B *is partially forcing definable on* $\mathfrak{A}$.

Theorem. *Let* f *be a partial generic enumeration of* $\mathfrak{A}$. *Then* $d_e(f^{-1}(\mathfrak{A}))$ *is quasi-minimal with respect to* $DS(\mathfrak{A})$.

Corollary. *[Slaman and Sorbi] Let* I *be a countable ideal of enumeration degrees. There exists an enumeration degree* $\mathbf{q}$ *s.t.*

1. *If* $\mathbf{a} \in I$ *then* $\mathbf{a} <_e \mathbf{q}$.
2. *If* $\mathbf{a}$ *is total and* $\mathbf{a} \leq_e \mathbf{q}$ *then* $\mathbf{a} \in I$.

Proposition. *For every countable structure $\mathfrak{A}$ there exist continuum many quasi-minimal degrees with respect to $DS(\mathfrak{A}$.*

Suppose that all quasi-minimal degrees with respect to $DS(\mathfrak{A})$ are $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_n, \ldots$ and let $X_i \in \mathbf{q}_i$, for all $i \in \mathbb{N}$. Then all $\mathbf{q}_i$ are not in $CS(\mathfrak{A})$ and hence every X_i is not forcing definable on $\mathfrak{A}$.
Then we could build a partial generic enumeration f of $\mathfrak{A}$ such that $X_i \not\leq_e f^{-1}(\mathfrak{A})$.
Thus $d_e(f^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $DS(\mathfrak{A})$ and not in $\{\mathbf{q}_i\}$.

Jumps of quasi-minimal degrees

Theorem. *[Ganchev] Let $B \subseteq \mathbb{N}$ and Q be a total set such that $B' \leq Q$. There exists a partial set F called quasi-minimal over B. with the following properties:*

1. $B < F$;
2. $F' \equiv Q$.
3. *for every total* $X \leq F$ *we have that* $X \leq B$.

Lemma. *There exists a patrial 1-generic enumeration f of $\mathfrak{A}$, such that $f^{-1}(\mathfrak{A})' \leq D(\mathfrak{A})'$ and $\langle f \rangle \leq D(\mathfrak{A})'$.*

Theorem. *The first jump spectrum of every structure $\mathfrak{A}$ consists exactly of the enumeration jumps of the quasi-minimal degrees.*

Corollary. *[McEvoy] For every total e-degree $\mathbf{a} \geq_e \mathbf{0}'_e$ there is a quasi-minimal degree $\mathbf{q}$ with $\mathbf{q}' = \mathbf{a}$.*

Proof

Proof.

- Let $g^{-1}(\mathfrak{A})' \in DS_1(\mathfrak{A})$. Denote by $B = g^{-1}(\mathfrak{A})$.
- $\mathfrak{B} = (\mathbb{N}, g^{-1}(R_1), \dots, g^{-1}(R_n))$.
- There is a partial 1-generic enumeration f of $\mathfrak{B}$ such that $f^{-1}(\mathfrak{B})' \leq B'$.
- There is a partial set F, such that $f^{-1}(\mathfrak{B}) < F$, $F' \equiv B'$, $(\forall \text{ total } X)(X \leq F \Rightarrow X \leq f^{-1}(\mathfrak{B}))$.
- Set $\mathbf{q} = d_e(F)$.
- $\mathbf{q}$ is a quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A})$.

□

Splitting a total set

Proposition.[Jockusch] *For every total e-degree* $\mathbf{a}$ *there are quasi-minimal degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Theorem. *For every element* $\mathbf{a}$ *of the jump spectrum of a structure* $\mathfrak{A}$ *there exists quasi-minimal with respect to* $DS(\mathfrak{A})$ *degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

A method of splitting a total set

Suppose that $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_n)$.
Denote by Δ *the set of all finite parts.*
For each $\tau \in \Delta$ *and* $x \in \mathbb{N}$ *by* $\tau * x$ *we denote an extension of* τ *such that* $\tau * x(lh(\tau)) = x$.
Let $f : \Delta \to \Delta$ *and* $\{y_n\}_n$ *be a sequence of natural numbers.*
If $\tau_0 = \emptyset$, $\tau_{n+1} = f(\tau_n * y_i)$, *then we denote by* $f(\{y_n\}_n) = \bigcup_n \tau_n$.
Let P *be a set of enumerations of* $\mathfrak{A}$.

Lemma.*[Ganchev] If* f *is computable in the total set* Q *and such that for every sequence* $\{y_n\}_n$ *computable in* Q, $f(\{y_n\}_n) \in P$, *then there exist enumerations* $g, h \in P$ *of* $\mathfrak{A}$ *such that* $Q \equiv_e \langle g \rangle \oplus \langle h \rangle$.

A method of splitting a total set

Let q be an enumeration of Q such that $\langle q\rangle \leq_e Q$. We construct two sequences of finite parts $\{\tau_n\}_n$ and $\{\sigma_n\}_n$ by the following rule:

1. $\tau_0 = \sigma_0 = \emptyset$;
2. $y_n = \langle lh(\sigma_n), q(2n)\rangle$;
3. $\tau_{n+1} = f(\tau_n * y_n)$;
4. $z_n = \langle lh(\tau_n), q(2n+1)\rangle$;
5. $\sigma_{n+1} = f(\sigma_n * z_n)$.

Define $g = f(\{y_n\}_n)$ and $h = f(\{z_n\}_n)$.

Theorem. *For every element* $\mathbf{a}$ *of the jump spectrum of a structure* $\mathfrak{A}$ *there exists quasi-minimal with respect to* $DS(\mathfrak{A})$ *degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Proof.

- Let $\mathbf{a} = \mathbf{d}_T(g^{-1}(\mathfrak{A})') \in DS_1(\mathfrak{A})$. Denote by $B = g^{-1}(\mathfrak{A})$.
- $\mathfrak{B} = (\mathbb{N}, g^{-1}(R_1), \ldots, g^{-1}(R_n))$.
- Construct a partial 1-generic enumeration f of $\mathfrak{B}$ such that $f^{-1}(\mathfrak{B})' \leq B'$.
- Let P be the class of all partial generic enumerations g of $\mathfrak{A}$, s.t. $\langle g \rangle$ is quasi-minimal over $f^{-1}(\mathfrak{B})$, i.e $f^{-1}(\mathfrak{B}) < \langle g \rangle$, $\langle g \rangle' \equiv B'$, $(\forall \text{ total } X)(X \leq \langle g \rangle \Rightarrow X \leq f^{-1}(\mathfrak{B}))$.
- Applying the lemma there are $\mathbf{p} = d_e(\langle g \rangle)$ and $\mathbf{q} = d_e(\langle h \rangle)$ are quasi-minimal over $f^{-1}(\mathfrak{B})$ and hence quasi-minimal for $DS(\mathfrak{A})$ and $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

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Thank you!