Turing reducibility and Enumeration reducibility

Alexandra A. Soskova [1]

Faculty of Mathematics and Computer Science
Sofia University

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Outline

- Relative computability
- Turing reducibility
- Turing jump
- Genericity and forcing
- Jump inversion
- Enumeration reducibility
- Quasi-minimal degree
- Selmans's theorem
- The minimal pair theorem

Enumeration of the partial computable functions

We will consider only partial functions on the set of the natural numbers $\mathbb{N}$.
Let $\{\varphi_i^{(n)}\}_{i\in\omega},\}$ be the standard listings of the Turing computable functions on n arguments. Here i is the code of the Turing machine M_i which computes $\varphi_i^{(n)}$.

Fact. *The Turing computable functions coincides with the μ-recursive ones.*

Definition. We say that the function f is μ-recursive (partial recursive) if it can be obtain from the basic O, S, and I_m^n by the operations superpositions primitive recursion and μ operation appliaied finitely many times.

Denote by $W_i^{(n)} = \mathrm{dom}(\varphi_i^{(n)})$ the r.e (c.e) set -the domain of $\varphi_i^{(n)}$.
We know that the set is c.e. iff it is a domain of a partial computable function.

Properties of the p.c. functions and c.e. sets

Theorem. *(Turing, 1936) There exists a Turing machine U the Universal Turing Machine which if given input (e, x) simulates the eth Turing machine with input x:*

$$U(e, x) = \varphi_e(x).$$

Theorem. *[Normal form theorem] There exists a primitive recursive function T_n s.t.*

1. $\downarrow \varphi_e^{(n)}(\bar{x}) \iff \exists z[T_n(e, \bar{x}, z) = 0]$
2. $\varphi_e^{(n)}(\bar{x}) = L(\mu z[T_n(e, \bar{x}, z) = 0])$.

Corollary. *Normal form of the c.e sets:* $W_e = \{x \mid \exists z[T_1(e, x, z) = 0]\}$.

Definition. Kleene' set $K = \{x \mid x \in W_x\} = \bar{L_d}$

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Oracle Turing machines

Let $A \subseteq \mathbb{N}$.

Definition. The oracle TM (OTM) with an oracle A is a TM M with a query tape and special states $q_?; q_{Yes}$ and q_{No}, such that: M runs as a usual TM, but when moving to state $q_?$ the oracle A is consulted with query y (on a separate tape) and if $y \in A$, M is restarted at state q_{Yes} else at state q_{No}

Definition. A function ψ is A-Turing computable if ψ is computable by an oracle Turing machine with oracle A.

Definition. A set B is said to be A-Turing computable, or Turing reducible to A ($B \leq_T A$) if B is A-Turing computable, i.e. the characteristic function c_B is A-Turing computable.

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Register machines with oracle

Let $A \subseteq \mathbb{N}$.

Definition. Register machines with oracle (RMO): The same as register machines with an additional command $O(n)$ to the basics: $Z(n), S(n), T(m, n), J(m, n, q)$, which asks the oracle with the contents of the nth register. If the oracle says "yes" then it writes 1 in the nth reg otherwise writes 0.

Proposition. *A function ψ is RM computable with an oracle A $\iff$ it is A- Turing computable.*

Definition. A function is p.c in A $\iff$ it can be obtained from the basic functions O, S, I^n_k and c_A by superposition, primitive recursion and μ-operation, applied finitely many times.

Let $A \subseteq \mathbb{N}$.
Denote by φ_e^A or $\{e\}^A$ the Turing computable function by the TM with code e with oracle A.

1. If A is decidable, then $(\forall B \subseteq \mathbb{N})(A \leq_T B)$, i.e. $c_A \leq_T B$ for an arbitrary oracle B.
2. $A \leq_T \mathbb{N} \Rightarrow A$ is decidable.

Definition. $A \equiv_T B \iff (A \leq_T B \ \&\ B \leq_T A)$.

Proposition. $(\forall A \subseteq \mathbb{N})(A \equiv_T \overline{A})$.

Easy- change only $0/1$ in the oracle answers.

m-reducibility

Definition. Let $A, B \subseteq \mathbb{N}$. A is *m-reducible* to B

$$A \leq_m B \iff (\exists h \text{ - total computable})(x \in A \iff h(x) \in B).$$

Proposition. *If A is c.e, then $A \leq_m K$.*

Proof.

$$g(x,y) \simeq \begin{cases} 0 & , x \in A \\ \neg\downarrow & , x \notin A. \end{cases}$$

By S^m_n theorem, there is a pr. rec function h, s.t. $\varphi_{h(x)}(y) \simeq g(x,y)$. Then $x \in A \iff \downarrow \varphi_{h(x)}(h(x)) \iff h(x) \in K$. □

Theorem. *(S^m_n-theorem)* $(\forall m)(\forall n)(\exists S^m_n)$ *a primitive recursive function:*

$$(\forall a)(\forall \bar{x})(\forall \bar{y})(\forall A)(\varphi_a^{A,(m+n)}(\bar{x}, \bar{y}) \simeq \varphi^{A,(n)}_{S^m_n(a,\bar{x})}(\bar{y})).$$

Proposition. $A \leq_m B, B \leq_m C \Rightarrow A \leq_m C$.

Proof.

Let $x \in A \iff h(x) \in B$ and $x \in B \iff g(x) \in C$, where g and h are computable. Then $x \in A \iff g(h(x)) \in C$, i.e. $A \leq_m C$. □

Proposition. *If* $A \leq_m B$*, then* $A \leq_T B$*.* $\bar{K} \leq_T K$, $\bar{K} \not\leq_m K$.

Definition. A is *computable enumerable in B*:

$$A \leq_{c.e.} B \iff A = \mathrm{dom}(\{a\}^B)$$

for some TM with code a.

1. If A is c.e., then $(\forall B)(A \leq_{c.e.} B)$.
2. If $A \leq_{c.e.} \mathbb{N}$, then A is c.e.

Proposition. $A \leq_m B, B \leq_{c.e.} C \Rightarrow A \leq_{c.e.} C$.

Proof.

Let h be a computable: $x \in A \iff h(x) \in B$ and e: $B = \mathrm{dom}(\varphi_e^C)$. Let $g(x) \simeq \varphi_e^C(h(x))$. Then $x \in A \iff h(x) \in B \iff \downarrow g(x)$. Hence $A = dom(g)$ but $g \leq_T C$, then $A \leq_{c.e.} C$. □

c.e.-reducibility

Proposition. $A \leq_T B \Rightarrow A \leq_{c.e.} B$.

Proof.

Let $c_A = \{a\}^B$. Construct a new TM:

1. execute $\{a\}^B(x)$ with output y
2. if $y = 1$ stop
3. if $y = 0$ infinite loop

□

Definition. $W_a^B = \mathrm{dom}(\{a\}^B)$, $K_B = \{a \mid a \in W_a^B\}$.

$K_B \leq_{c.e.} B$ and $\overline{K_B} \not\leq_{c.e.} B$, $K_B \not\leq_T B$.

Proposition. $A \leq_{c.e.} B, B \leq_T C \Rightarrow A \leq_{c.e.} C$.

Proposition. $A \leq_T B, B \leq_T C \Rightarrow A \leq_T C$.

From $A \leq_{c.e.} B, B \leq_{c.e.} C$ it does not follow that $A \leq_{c.e.} C$. Since $\overline{K} \leq_{c.e.} K$ and $K \leq_{c.e.} \emptyset$, (K is c.e.), then $\overline{K} \leq_{c.e.} \emptyset \leq_T \mathbb{N}$. Thus $\overline{K}$ is c.e., a contradiction.

Kleene - Post theorem

Proposition.

$$A \leq_T B \iff A \leq_{c.e.} B \;\&\; \overline{A} \leq_{c.e.} B.$$

Proof.

Let $A \leq_T B$. Then $A \leq_{c.e.} B$, and $\overline{A} \leq_{c.e.} B$ ($\overline{A} \leq_T B$)
Let $A \leq_{c.e.} B$ and $\overline{A} \leq_{c.e.} B$. Then there are TM P and Q,such that $\{P\}^B = \chi_A$, $\{Q\}^B = \chi_{\overline{A}}$. Then we construct a TM PQ, which computes P and Q with two tapes, step by step, and gives in output 1 if P halts, and 0 if Q ends.

$$(\forall x)(\downarrow \{P\}^B(x) \vee \downarrow \{Q\}^B(x))$$
$$\Rightarrow (\forall x) \downarrow \{PQ\}^B(x)).$$

So $A \leq_T B$.

□

Turing degrees

The relation $\equiv_T$ is an equivalence relation.

Definition. *The Turing degree of the set A* is the equivalence class containing A:

$$d_T(A) = \{B \mid B \equiv_T A\}.$$

Definition. $d_T(A) \leq d_T(B) \iff A \leq_T B$

The Turing upper semi-lattice

Let D_T be the set of all Turing degrees. $(D_T, \leq)$ is a partial order.

Definition.[The $\oplus$ operation - join]
$A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

Proposition. *$d_T(A \oplus B)$ is the least upper bound of $d_T(A)$ and $d_T(B)$.*

Proof.

1. $x \in A \iff 2x \in A \oplus B \Rightarrow A \leq_m A \oplus B \Rightarrow A \leq_T A \oplus B$, i.e. $A \oplus B$ is a upper bound of A and B.
2. Let $c_A = \{a\}^C, c_B = \{b\}^C$. Then $c_{A \oplus B}$ is comp.rel to C

$$c_{A\oplus B}(x) \simeq \begin{cases} \varphi_a^C([x/2]) & \text{, if } x \text{ is odd,} \\ \varphi_b^C([x/2]) & \text{, if } x \text{ is even.} \end{cases}$$

□

The Turing jump

$\mathcal{D}_T = (\mathcal{D}_T, \leq, \oplus, \mathbf{0}_T)$ is a upper semi-lattice, where
$\mathbf{0}_T = d_T(\emptyset) = d_T(R)$, R - decidable.

Definition. The Turing jump of the set A:
$A' = K_A = \{x \mid x \in \mathrm{dom}(\varphi_x^A)\}$.

Proposition.

1. $K_A \leq_{c.e.} A$;
2. $B \leq_{c.e.} A \Rightarrow B \leq_m K_A$. *Hint:*

$$g(x,y) \simeq \begin{cases} 0 & , x \in B \\ \neg\downarrow & , x \notin B. \end{cases}$$

   By S^m_n *th.* $\varphi^A_{h(x)}(y) \simeq g(x,y)$. *Then*
   $x \in B \iff \downarrow \varphi^A_{h(x)}(h(x)) \iff h(x) \in K_A$.
3. $A <_T K_A$, *since* $\overline{K^A} \not\leq_{c.e.} A$.

Monotonicity of the Turing jump

Proposition. $A \leq_T B \iff A' \leq_m B'$.

Proof.

($\Rightarrow$) Let $A \leq_T B$. We have $A' \leq_{c.e.} A$ and then $A' \leq_{c.e.} B$. Thus $A' \leq_m B'$ (by(2)).
($\Leftarrow$) Let $A' \leq_m B'$. We have $A \leq_{c.e.} A \Rightarrow A \leq_m A' \leq_m B'$ and $\overline{A} \leq_{c.e.} A \Rightarrow \overline{A} \leq_m A' \leq_m B'$. Then $A \leq_m B', \overline{A} \leq_m B'$, But by (1) $B' \leq_{c.e.} B$ and then $A \leq_{c.e.} B, \overline{A} \leq_{c.e.} B$. By Kleene-Post th. $A \leq_T B$. □

Corollary. *[Monotonicity of the jump]* $A \leq_T B \Rightarrow A' \leq_T B'$.

Definition. $(d_T(A))' = d_T(A')$.

Since $A <_T K_A$,then $d_T(A) < d_T(A')$.

Finite parts

Definition. τ is *a finite part*, if $\tau : [0; n-1] \longrightarrow \mathbb{N}$ is a finite function. denote by $|\tau| = n$ the length of the interval, where τ is defined.

$$(\tau * a)(x) \simeq (\tau * n \to a)(x) \simeq \begin{cases} \tau(x) & \text{if } 0 \leq x < n, \\ a & \text{if } x = n. \end{cases}$$

If A is a set, we write $\tau \subseteq A$ instead of $\tau \subseteq c_A$.

Genericity

Definition. The set A is *generic*, if for every c.e. set S of finite parts:

$$(\exists \alpha \subseteq A)\underbrace{(\alpha \in S \vee (\forall \beta \supseteq \alpha)(\beta \notin S))}_{\alpha \text{ decides } S}.$$

Equivalently:

Definition. S is dense in A, if $(\forall \alpha \subseteq A)(\exists \beta \in S)(\alpha \subseteq \beta)$. Then A is generic, if any time when S is dense in A, then A meets S, i.e. $(\exists \alpha \subseteq A)(\alpha \in S)$.

Constructing generic sets

The c.e. sets of finite parts we can list in a sequence, and moreover There is a total computable function h, s.t. $S_e = W_{h(e)}$.

The construction:

- We construct on steps finite parts α_n, which will approximate c_A.
- We start with $\alpha_0 = \emptyset$.
- For α_{n+1} we ask if there is an extension of α_n in S_n. If there is set α_{n+1} to be the least one. If there is not let $\alpha_{n+1} = \alpha_n$.

It is clear that the construction assures that A is generic.

Generic sets - some Properties

Let A be a generic set.

Proposition. *A not a finite set.*

Proof.

Assume that it is. $\exists n$, s.t. $x \in A \Rightarrow x < n$. Let $S = \{\alpha \mid (\exists m > n)(\alpha(m) \simeq 1)\}$ - c.e. Since A is generic then $(\exists \alpha)(\alpha \in S \vee (\forall \beta \supseteq \alpha)(\beta \notin S))$. It is clear that $\alpha \notin S$. Then $(\forall \beta \supseteq \alpha)(\forall m > n)(\beta(m) \not\simeq 1)$, which is impossible. Hence A is infinite. □

Proposition. *If $V \subseteq A$ is c.e., then V is finite.*

Proof.

Let $S = \{\alpha \mid (\exists x)(\alpha(x) \simeq 0 \,\&\, x \in V)\}$ - c.e. Since A is generic then $\exists\alpha$ s.t.$\alpha \in S \vee (\forall\beta \supseteq \alpha)(\beta \notin S)$. $\alpha \notin S$. Then $(\forall\beta \supseteq \alpha)(\forall x)(\beta(x) \simeq 0 \Rightarrow x \notin V)$. Let $n \geq |\alpha|$, then for every $\beta \supseteq \alpha$, with $|\beta| = n, \beta(n) = 0$ hence $n \notin V$. Thus V is finite. □

Corollary. *A is not c.e., since $A \subseteq A$ is infinite.*

Proposition. *If $V \leq_T A$ is c.e., then V is decidable.*

Proof.

We know $\overline{V} \leq_T V \leq_T A$, hence there is an a, s.t. $\overline{V} = dom(\{a\}^A)$. Let $S = \{\alpha \mid (\exists x \in V)(\downarrow \{a\}^\alpha(x))\}$. Since S is c.e. and A generic there is $\alpha \subseteq A$, s.t. $\alpha \in S \vee (\forall \beta \supseteq \alpha)(\beta \notin S)$. If $\alpha \in S$, then $(\exists x \in V)(\downarrow \{a\}^A(x) \iff x \in \overline{V})$, a contradiction. Then $(\forall \beta \supseteq \alpha)(\forall x \in V)(\neg \downarrow \{a\}^\beta(x))$.

$$x \in \overline{V} \iff (\exists \beta \supseteq \alpha)(\downarrow \{a\}^\beta(x)),$$

i.e. $\overline{V}$ is c.e., V is c.e., therefore V is decidable . □

Forcing relation

Definition. *A models* the formula $F_e(x)$:

$$A \models F_e(x) \iff \{e\}^A(x) \iff x \in W_e^A.$$

Definition. The finite part α *forces* formula $F_e(x)$:

$$\alpha \Vdash F_e(x) \iff \downarrow \{e\}^\alpha(x).$$

1. $\alpha \subseteq A \,\&\, \alpha \Vdash F_e(x) \Rightarrow A \models F_e(x)$.
2. $\alpha \subseteq \beta \,\&\, \alpha \Vdash F_e(x) \Rightarrow \beta \Vdash F_e(x)$.
3. $A \models F_e(x) \Rightarrow (\exists \alpha \subseteq A)(\alpha \Vdash F_e(x))$

Lemma. *The set* $\{(\alpha, e, x) \mid \alpha \Vdash F_e(x)\}$ *is c.e.*

Forcing relation

Definition.

$$A \models \neg F_e(x) \iff A \not\models F_e(x) \iff \neg \downarrow \{e\}^A(x)$$

Definition.

$$\alpha \Vdash \neg F_e(x) \iff (\forall \beta \supseteq \alpha)(\beta \nVdash F_e(x)).$$

Theorem. *Let A be generic. Then*

$$A \models \neg F_e(x) \iff (\exists \alpha \subseteq A)(\alpha \Vdash \neg F_e(x)).$$

Proof.

$(\Leftarrow)$ Let $\alpha \subseteq A \,\&\, \alpha \Vdash \neg F_e(x)$.Suppose that $A \not\models \neg F_e(x)$, i.e. $A \models F_e(x) \Rightarrow (\exists \beta \subseteq A)(\beta \Vdash F_e(x))$. Let $\gamma = \alpha \cup \beta$.Then $\gamma \supseteq \beta \Rightarrow \gamma \Vdash F_e(x)$, but $\gamma \supseteq \alpha, \alpha \Vdash \neg F_e(x) \Rightarrow \gamma \nVdash F_e(x)$ - a contradiction.
$(\Rightarrow)$ Let $A \models \neg F_e(x)$. We search for $\alpha \subseteq A$, $\alpha \Vdash \neg F_e(x)$, i.e. no extension of α does not forces $F_e(x)$. A is generic. Indeed suppose that $(\forall \alpha \subseteq A)(\alpha \nVdash \neg F_e(x)) \iff (\forall \alpha \subseteq A)(\exists \beta \supseteq \alpha)(\beta \Vdash F_e(x))$. Set $S_{e,x} = \{\beta \mid \beta \Vdash F_e(x)\}$. $S_{e,x}$ is c.e. and dense in A, then there is $\alpha \subseteq A, \alpha \in S_{e,x} \iff \alpha \Vdash F_e(x) \Rightarrow A \models F_e(x)$, a contradiction. So, $(\exists \alpha \subseteq A)(\alpha \Vdash \neg F_e(x))$. □

Truth lemma

Corollary. *[Truth lemma] If A is generic, then*

$$A \models (\neg)F_e(x) \iff (\exists \alpha \subseteq A)(\alpha \Vdash (\neg)F_e(x)).$$

Notice that $\{(\alpha, e, x) \mid \alpha \Vdash \neg F_e(x)\} \leq_T \emptyset'$.

Corollary. *For A generic $A' \equiv_T A \oplus \emptyset'$.*

Proof.

1. A' is a upper bound of $\emptyset'$ and $A \Rightarrow \emptyset' \oplus A \leq_T A'$.
2. $A' = K_A = \{x \mid x \in W_x^A\} \leq_{c.e.} A$, then there is e, s.t. $x \in K_A \iff \downarrow \{e\}^A(x) \iff A \models F_e(x) \iff (\exists \alpha \subseteq A)(\alpha \Vdash F_e(x)) \leq_T A \oplus \emptyset'$. A is generic then $x \in \overline{K_A} \iff \neg \downarrow \{e\}^A(x) \iff A \not\models F_e(x) \iff (\exists \alpha \subseteq A)(\alpha \Vdash \neg F_e(x)) \leq_T A \oplus \emptyset'$. Thus $K_A = A' \leq_T A \oplus \emptyset'$.

□

Theorem. *[Jump inversion theorem, Fridberg] Let $\emptyset' \leq_T B$. There exists a generic A, s.t. $A' \equiv_T B$.*

Proof.

We construct A by steps, so that $A \leq_T B$ and A - generic. Then $A' \equiv_T \emptyset \oplus A \Rightarrow A' \leq_T B$. For the other direction we code B in $A \oplus \emptyset'$. On each step n we define a finite part α_n of c_A.
Let $\alpha_0 = \emptyset$. If α_n is constructed then we ask: Is it true that: $(\exists \beta \supseteq \alpha_n)(\beta \in S_n)$?". Since the set $V = \{(\alpha, n) \mid (\exists \beta \supseteq \alpha)(\beta \in S_n)\}$ is c.e, then $V \leq_T K = \emptyset'$. If **yes**, set α_n^* will be the minimal such β, if **no**, then $\alpha_n^* = \alpha_n$. Thus assures that A is generic. Set $\alpha_{n+1} = \alpha_n^* * c_B(n)$. □

Jump inversion theorem

Proof.

1. $A \leq_T B$. Since $|\alpha_{x+1}| \geq x$, $x \in A \iff x \in \alpha_{x+1}$. But $\alpha_n \leq_T B \oplus \emptyset' \leq_T B$.
2. A is generic, since α_n^* assures genericity with respect to S_n.
3. $B \leq_T A \oplus \emptyset'$. We have $k \in B \iff \alpha_{k+1}(|\alpha_k^*|) = 1$. We can construct B repeating the construction, changing $c_B(n)$ with $c_A(|\alpha_n^*|)$. So, using oracle A and $\emptyset'$ we have $B \leq_T A \oplus \emptyset'$.

Thus A is generic and $A' \equiv_T B$. □

Corollary. *There exists* $A \not\equiv_T \emptyset$ *- generic s.t.* $A' \equiv_T \emptyset'$.

Corollary. *There exists* A *- generic, s.t.* $\emptyset \lneq_T A \lneq_T A' \equiv_T \emptyset'$.

Enumeration reducibility

Definition. The operator $\Gamma : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$ is an *enumeration operator*, if:

1. $x \in \Gamma(A) \iff (\exists D \subseteq A)(x \in \Gamma(D)\,\&\, D$ - finite) (Γ is compact),
2. There is a total computable function h, s.t. $\Gamma(W_a) = W_{h(a)}$ (Γ is effective).

Definition. The set A is *enumeration reducible* to B:

$$A \leq_e B \iff (\exists \Gamma \text{ - e-operator})(A = \Gamma(B)).$$

Proposition. $A \leq_e B, B \leq_e C \Rightarrow A \leq_e C$.

Enumeration operator

Proposition. Γ *is e-operator* $\iff$ *there exists c.e.* W, *s.t.:*

$$\Gamma(A) = \{x \mid (\exists v)(\langle x, v\rangle \in W \,\&\, D_v \subseteq A)\}$$

Proof.

$(\Leftarrow)$ Let $x \in \Gamma(A) \iff (\exists v)(\langle x, v\rangle \in W \,\&\, D_v \subseteq A)$.

1. (compact)
   $x \in \Gamma(A) \Rightarrow (\exists v)(\langle v, x\rangle \in W \,\&\, D_v \subseteq A) \Rightarrow x \in \Gamma(D)$.
2. (monotone).
3. (effective) $x \in \Gamma(W_a) \iff (\exists v)(\langle v, x\rangle \in W \,\&\, D_v \subseteq W_a)$.

$$R = \{(a, x) \mid \underbrace{(\exists v)(\langle v, x\rangle \in W \,\&(\forall y \in D_v)(y \in W_a))}_{\text{c.e. condition}}\}.$$

Let $R = W_e$, $h(a) = S^1_1(e, a)$.
$x \in \Gamma(W_a) \iff (a, x) \in R \iff x \in W_{h(a)}$.

Proof.

($\Rightarrow$) Let Γ is compact and effective. Then

$$\begin{aligned} x \in \Gamma(A) &\iff (\exists D \text{ - finite})(D \subseteq A \,\&\, x \in \Gamma(D)) \\ &\iff (\exists v)(D_v \subseteq A \,\&\, x \in \Gamma(D_v)) \\ &\iff (\exists v)(D_v \subseteq A \,\&\, x \in \Gamma(W_{\lambda(v)})) \\ \text{(effective)} &\iff (\exists v)(D_v \subseteq A \,\&\, x \in W_{h(\lambda(v))}) \\ &\iff (\exists v)(D_v \subseteq A \,\&\, \langle x, v\rangle \in W), \end{aligned}$$

where $W = \{\langle x, v\rangle \mid x \in W_{h(\lambda(v))}\}$.

□

Examples

$A \leq_e B$ if there exists an effective procedure that, given any enumeration of B, computes an enumeration A.

Example

- $A \leq_e A$ via the c.e. set $W = \{ \langle n, \{n\} \rangle \mid n \in \mathbb{N} \}$.
- If A is c.e. and B is any set, then $A \leq_e B$ via the c.e. set $W = \{ \langle n, \emptyset \rangle \mid n \in A \}$.
- If f is computable and $A = f^{-1}(B)$ (i.e. $A \leq_m B$), then $A \leq_e B$ via the c.e. set $W = \{ \langle n, \{f(n)\} \rangle \mid n \in stN \}$.
- More generally, if A is c.e in B, $A = \{ x \mid \varphi_e^B(x) \downarrow \}$ then $A \leq_e B \oplus \bar{B}$ via the

$$W = \{ \langle x, D \oplus E \rangle \mid \varphi_e^D(x) \downarrow \ \& \ Q^-(\varphi_e^D, x) = E \}.$$

Totality

$\langle f \rangle$ denotes the graph of the function f, i.e
$\langle f \rangle = \{\langle x, y \rangle \mid f(x) = y\}$.

Example

If f is total, then $\langle \bar{f} \rangle \leq_e \langle f \rangle$ via the set

$$W = \{\langle\langle x, y\rangle, \{\langle x, z\rangle\}\rangle \mid y \neq z\}.$$

- Y is *total* if $\bar{Y} \leq_e Y$. $X \oplus \bar{X}$ is total for any X and if f is total then $\langle f \rangle$ is total.
- If Y is total then, for any X: $X \leq_e Y \iff X$ is c.e. in Y.
- Consequence $A \leq_T B \iff A \oplus \bar{A} \leq_e B \oplus \bar{B}$.

Enumeration reducibility of functions

Definition. $\varphi \leq_e \psi \iff \langle\varphi\rangle \leq_e \langle\psi\rangle$.

Proposition. $\varphi \leq_T \psi \Rightarrow \varphi \leq_e \psi$.

Proof.

Let $\varphi \leq_T \psi$ and $\{e\}^\psi = \varphi$. We look for a c.e. W:

$$x \in \Gamma(\langle\psi\rangle) = \langle\varphi\rangle \iff (\exists v)(\langle v, x\rangle \in W \,\&\, \langle\theta_v\rangle \subseteq \langle\psi\rangle),$$

Let $W = \{\langle\langle x, y\rangle, v\rangle \mid \{e\}^{\theta_v}(x) \simeq y\}$. W is c.e.

$$\begin{aligned} \langle x, y\rangle \in \Gamma(\langle\psi\rangle) &\iff (\exists v)(\{e\}^{\theta_v}(x) \simeq y \,\&\, \langle\theta_v\rangle \subseteq \langle\psi\rangle) \\ &\iff \{e\}^\psi(x) \simeq y \\ &\iff \varphi(x) \simeq y \\ &\iff \langle x, y\rangle \in \langle\varphi\rangle . \end{aligned}$$

Thus $\varphi \leq_e \psi$.

Enumeration reducibility of functions

The enumeration reducibility is weaker than Turing reducibility. We will prove that $c_K \leq_e \chi_{\overline{K}}$, but $c_K \not\leq_T \chi_{\overline{K}}$.

1. Suppose that $c_K \leq_T \chi_{\overline{K}}$. Then there is an e, s.t. $\{e\}^{\chi_{\overline{K}}} = c_K$ - total. But $\downarrow \chi_{\overline{K}}(x) \simeq y \iff y = 1$. Swich all oracle questions with 1. Then we can compute c_K, a contradiction.
2. Let

$$W = \{\langle\langle x, 0\rangle, v\rangle \mid x \in \mathbb{N} \,\&\, D_v = \{\langle x, 1\rangle\}\} \cup \{\langle\langle x, 1\rangle, v\rangle \mid x \in K \,\&\, D_v = \emptyset\}.$$

   Then W defines an e-operator and $W(\chi_{\overline{K}}) = c_K$.

Definition. $A \equiv_e B \iff A \leq_e B \,\&\, B \leq_e A$.

$\equiv_e$ is an equvalence relation, and teh classes of equvalences we call enumeration degrees.

Definition. *Enumertion degree* of A is the class of equvalence of A with respect to $\equiv_e$:

$$d_e(A) = \{B \mid B \equiv_e A\}.$$

Definition. $d_e(A) \leq d_e(B) \iff A \leq_e B$

Denote by D_e the set of all enumeration degrees. $(D_e, \leq)$ is a partial ordered set. The operation $\oplus$ gives a least upper bound of two e. degrees.

The upper semilattice $\mathcal{D}_e = (D_e, 0_e, \oplus, \leq)$

Proposition. *$d_e(A \oplus B)$ is the least upper bound of $d_e(A)$ $d_e(B)$.*

Proof.

1. $x \in A \iff 2x \in A \oplus B \Rightarrow A \leq_m A \oplus B \Rightarrow A \leq_T A \oplus B \Rightarrow A \leq_e A \oplus B$. And $x \in B \iff 2x + 1 \in A \oplus B \Rightarrow B \leq_e A \oplus B$, i.e. $A \oplus B$ is a upper bound of A and B.
2. Let $A \leq_e C, B \leq_e C$, i.e. $A = W'(C), B = W''(C)$. Let $W = \{\langle x, v\rangle \mid (\exists v' v'')(\langle x, v'\rangle \in W') \& (\langle x, v''\rangle \in W'') \& D_v = D_{v'} \oplus D_{v''}\}$. W is c.e.. And $W(C) = A \oplus B$, i.e. $C \leq_e A \oplus B$.

□

Denote by $0_e = \{W \mid W \text{ is c.e.}\}$, $0_e \leq a$ for an arbitrary a. . $\mathcal{D}_e = (D_e, 0_e, \oplus, \leq)$ is a upper semi-lattice.

Total enumeration degrees

A is *total*, if $A \equiv_e A \oplus \overline{A} = A^+$.

Proposition.

1. $A^{++} \equiv_e A^+$, *i.e.. A^+ is total.*
2. $A^+ \equiv_e \langle c_A \rangle$.
3. *A is total* $\iff$ $A \equiv_e \langle \chi_A \rangle$.
4. *If f is total, then $\langle f \rangle$ is total.*

Every decidable set is total. K is not total since $\overline{K} \not\leq_e K$.

Definition. $a \in D_e$ is *total*, if there is a total $A \in a$.

Remark 0_e is total e-degree, but $K \in 0_e$ is not total.

The total degrees in D_e form an upper semi-lattice isomorphic to D_T.

Definition.[Rogers, Michael] $\varkappa : D_T \to D_e$ is *Rogers embedding*, defined as $\varkappa(d_T(A)) = d_e(A^+)$.

Proposition.

1. *(correctness)* $A \equiv_T B \iff A^+ \equiv_e B^+$;
2. $Range(\varkappa) = Tot = \{a \mid a \text{ is total e-degree}\}$. *Indeed if* $A \in a \in Tot$, *then since* $A \equiv_e A^+$ *we have* $A^+ \in a \Rightarrow \varkappa(d_T(A)) = a$ *and thus* $a \in Range(\varkappa)$.*But if* $a \in Range(\varkappa)$, *then* $a = d_e(A^+) \Rightarrow a \in Tot$.
3. *(injective) Let* $\varkappa(d_T(A)) = \varkappa(d_T(B)) \Rightarrow A^+ \equiv_e B^+ \Rightarrow A \equiv_T B \Rightarrow d_T(A) = d_T(B)$.
4. *(isomorphic embeding)* $A \leq_T B \Rightarrow \varkappa(d_T(A)) = A^+ \leq_e B^+ = \varkappa(d_T(B))$.

So, $\varkappa$ is an isomrphic embedding of D_T in the total degrees in D_e. To see that it is strong embeding: $Range(\varkappa) = Tot \subsetneq D_e$, we have to show that there are nontotal degrees.

Definition.

$$A \models_e F_a(x,y) \iff \langle x,y\rangle \in W_a(A),$$
$$\alpha \Vdash_e F_a(x,y) \iff \langle x,y\rangle \in W_a(\alpha^+), \text{ where } \alpha^+ = \{x \mid \alpha(x) \simeq 1\}.$$

Properties:

1. $\alpha \subseteq A \,\&\, \alpha \Vdash_e F_a(x,y) \Rightarrow A \models_e F_a(x,y)$ (monotonicity).
2. $\alpha \subseteq \beta \,\&\, \alpha \Vdash_e F_a(x,y) \Rightarrow \beta \Vdash_e F_a(x,y)$, since $\alpha^+ \subseteq \beta^+$;
3. $A \models_e F_a(x,y) \Rightarrow (\exists \alpha \subseteq A)(\alpha \Vdash_e F_a(x,y))$ (compactness)

Proposition. *Let A be generic and $\varphi \leq_e A$. There exists a computable function ψ, s.t. $\varphi \subseteq \psi$.*

Nontotal degrees

Proof.

Let A be a generic and $\varphi \leq_e A$, $\langle\varphi\rangle = W_a(A)$. Then $\langle x, y\rangle \in \langle\varphi\rangle \iff A \models_e F_a(x, y)$. Consider $S = \{\alpha \mid (\exists x)(\exists y_1)(\exists y_2)(\alpha \Vdash_e F_a(x, y_1) \,\&\, \alpha \Vdash_e F_a(x, y_2) \,\&\, y_1 \neq y_2)\}$.

There is $\alpha \subseteq A$, $\alpha \notin S$ and then $(\forall\beta \supseteq \alpha)(\beta \notin S)$. Define $\psi(x) \simeq y \iff (\exists\beta \supseteq \alpha)(\beta \Vdash_e F_a(x, y))$. ψ is a function and ψ is computable.

Let $\varphi(x) \simeq y$. Then $A \models_e F_a(x, y) \Rightarrow (\exists\beta \supseteq \alpha)(\beta \Vdash_e F_a(x, y)) \Rightarrow \psi(x) \simeq y$.

Thus $\varphi \subseteq \psi$. □

Corollary. *$d_e(A)$ is nontotal for A - generic.*

Corollary. *If A is generic, X total and $X \leq_e A$, then $X \leq_e \emptyset$.*

Quasi-minimal degree

Definition. A is *quasi-minimal* if

1. $A \not\leq_e \emptyset$;
2. If $X \leq_e A$ and X is total, then $X \leq_e \emptyset$.

From the previous Proposition:

Proposition. *Each generic set is quasi-minimal.*

Regular enumerations

Definition. Let $B \subseteq \mathbb{N}$. The total function $f : \mathbb{N} \longrightarrow \mathbb{N}$ is *regular enumeration* of B, if $f(2\mathbb{N}+1) = B$.

If $B = \emptyset$, we consider enumerations of $\mathbb{N} = \overline{\emptyset} \equiv_e \emptyset$.
If f is a regular enumeration of B then

$$\chi_B(x) \simeq sg(\mu n[x = f(2n+1)]).$$

And $c_B \leq_T f$, then $B \leq_e f$.

Definition. *B-regular finite part* is a function $\tau : [0; 2q+1] \longrightarrow \mathbb{N}$, s.t.

$$2x+1 \in dom(\tau) \Rightarrow \tau(2x+1) \in B.$$

If τ is a B-regular finite part, then

$$(\exists f \supseteq \tau)(f \text{ is a reg. enum of } B).$$

Forcing

Definition.

$$f \models F_e(x) \iff x \in W_e(\langle f \rangle),$$
$$\tau \Vdash F_e(x) \iff x \in W_e(\langle \tau \rangle).$$

$$\tau \Vdash F_e(x) \iff (\exists v)(\langle x, v \rangle \in W_e \,\&\, D_v \subseteq \langle \tau \rangle),$$

So $S_\tau = \{\langle e, x \rangle \mid \tau \Vdash F_e(x)\}$ is c.e..
Properties:

1. $\tau \subseteq f \,\&\, \tau \Vdash F_e(x) \Rightarrow f \models F_e(x)$
2. $\tau \subseteq \rho \,\&\, \tau \Vdash F_e(x) \Rightarrow \rho \Vdash F_e(x)$
3. $f \models F_e(x) \Rightarrow (\exists \tau \subseteq f)(\tau \Vdash F_e(x))$

Regular enumerations

Proposition. *Let $A \not\leq_e B$. There exists a regular enumeration f of B, s.t. $A \not\leq_e f$.*

Proof.

We construct a sequence of B regular finite parts

$$\tau_0 \subseteq \tau_1 \subseteq \dots \subseteq \tau_q \subseteq \dots$$

Let $\tau_0(0) = 0, \tau_0(1) = z_0 \in B$. If τ_q is constructed:
1. $q = 2e$. Let $z_0 = \mu z[z \in B \,\&\, z \notin \tau_q(2\mathbb{N}+1)]$. Set $\tau_{q+1} = \tau_q * 0 * z_0$.
2. $q = 2e+1$.

$$C = \{x \mid (\exists \rho \supseteq \tau_q)(\rho \text{ is a B-regular finite part } \& \\ \rho(|\tau_q|) = x \,\&\, \rho \Vdash F_e(|\tau_q|))\}.$$

Since $C \leq_e B$, then $C \neq A$.

Regular enumerations

Proof.

2 (a). $(\exists x)(x \in C \,\&\, x \notin A)$. Then τ_{q+1} is the minimal ρ from C.
2 (b). $(\exists x)(x \notin C \,\&\, x \in A)$.Then $\tau_{q+1} = \tau_q * x * z_0$ for some $z_0 \in B$.
Let $f = \bigcup_q \tau_q$ - a regular enumerartion of B.
Suppose that $A \leq_e f$, i.e. $A = W_e(\langle f \rangle)$. Then $f^{-1}(A) = \{x \mid f(x) \in A\} \leq_e f$ and there is e, s.t. $n \in f^{-1}(A) \iff f \models F_e(n)$.
Consider the step $q = 2e + 1$. Let $n = |\tau_q| = 2q + 1$.
Case1. $n \in f^{-1}(A) \Rightarrow f(n) \in A \Rightarrow (\exists \rho \supseteq \tau_q)(\rho \Vdash F_e(n) \,\&\, \rho(n) = f(n) \in A)$. Then $f(n) \in C \cap A$, A contradiction.
Case 2. $n \notin f^{-1}(A) \Rightarrow f(n) \notin A \Rightarrow (\forall \rho \supseteq \tau_q)(\rho \nVdash F_e(n) \,\&\, \rho(n) = f(n) \notin A)$, then $f(n) \notin C$ - a contradiction.
So, $A \not\leq_e f$. □

Selman's theorem

Theorem. *[Selman]*

$$A \leq_e B \iff (\forall X \text{ - } total)(B \leq_e X \Rightarrow A \leq_e X).$$

Proof.

($\Rightarrow$) From the transitivity of $\leq_e$.
($\Leftarrow$) Suppose that $A \not\leq_e B$. Then we construct a B-regular enumeration f, s.t. $A \not\leq_e \langle f \rangle$, but $\langle f \rangle$ is total and $B \leq_e \langle f \rangle$, a contradiction. □

Corollary. *If $a, b \in D_e$, then*

$$a \leq b \iff (\forall x \text{ - } total)(b \leq x \Rightarrow a \leq x).$$

Minimal pair

Definition. The sets F and G form a *minimal pair for* B, if

1. $B \lneq_e F, B \lneq_e G$;
2. $A \lneq_e F, A \lneq_e G \Rightarrow A \leq_e B$,

i.e. B is an greatest lower bound for F and G.

Definition. We call f *generic regular enumeration* of B, if f is a regular enumeration of B and for every $S \leq_e B$, containing only B-regular fnite parts.

$$(\exists \tau \subseteq f)(\tau \in S \vee (\forall \rho \supseteq \tau)(\rho \notin S)).$$

Proposition.
Let $B \subseteq \mathbb{N}$, and $\{A_n\}$ is a sequence of sets, s.t. $(\forall n)(A_n \nleq_e B)$. Then there exists a generic regular enumeration f of B, s.t. $(\forall n)(A_n \nleq_e f)$.

Minimal pair

Proof.

We construct a monotone incresing sequence of B-regular finite parts $\tau_q : [0; 2q+1] \longrightarrow \mathbb{N}$. Let $\tau_0(0) = \tau_0(1) = b_0 \in B$. Suppose that we have constructed τ_q.

1. $q = 3e$. Set $\tau_{q+1} = \tau_q * 0 * b$, where b is the first nonenumerated element of B.

2. $q = 3e + 1$. Genericity of f. Consider $S_e = W_e(B) \cap R_B$, $R_B = \{\tau \mid \tau \text{ is B-reg fin part}\}$. Let τ_{q+1} be the least $\tau \supseteq \tau_q$, $\tau \in S_e$, if there is. If not $\tau_{q+1} = \tau_q$.

3. $q = 3e + 2$. Let $e = \langle n, k \rangle$. we will assure $f^{-1}(A_n) \neq W_k(\langle f \rangle)$. ($\Longrightarrow f^{-1}(A_n) \not\leq_e f$ and $A_n \not\leq_e f$). □

Minimal pair

Proof.

$n_q = |\tau_q|$

$C_q = \{x \mid x(\exists \tau \supseteq \tau_q)(\underbrace{\tau \text{ is } B\text{-reg.fin, part)}}_{\leq_e B} \,\&\, \underbrace{\tau(n_q) \simeq x}_{\text{effective}} \,\&\, \underbrace{\tau \Vdash F_k(n_q)}_{\text{effective}}\}$

$C_q \leq_e B$ and then $C_q \neq A_n$.
3.(a) $(\exists x)(x \in C_q \,\&\, x \notin A_n)$. We get the minimal such x τ_{q+1} teh minimal such τ.
3.(b) $(\exists x)(x \notin C_q \,\&\, x \in A_n)$. $\tau_{q+1} = \tau_q * x * b$, where $b \in B$.
Define f as follows:

$$f(n) \simeq x \iff (\exists q)(\tau_q(n) \simeq x)$$

□

Minimal pair

Proof.

By construction f is generic regular enumeration of B. We will show that $f^{-1}(A_n) \not\leq_e f$.

Suppose $f^{-1}(A_n) \equiv_e W_k(\langle f \rangle)$ for some n and k. Consider step $q = 3\langle n, k\rangle + 2$. We know $f(n_q) \simeq x \in A_n \triangle C_q$.

1. $x \in C_q \,\&\, x \notin A_n$. Then

$$f \models F_k(n_q) \Rightarrow n_q \in f^{-1}(A_n) \Rightarrow f(n_q) = x \in A_n \text{ - a contradiction.}$$

2. $x \notin C_q \,\&\, x \in A_n$. Then

$$n_q \in f^{-1}(A) \Rightarrow (\exists \tau \supseteq \tau_q)(\tau(n_q) \simeq x \,\&\, \tau \Vdash F_k(n_q)) \Rightarrow x \in C_q \text{ - a contr.}$$

Then $f^{-1}(A_n) \not\leq_e f \Rightarrow A_n \not\leq_e f$. □

Minimal pai theorem

Theorem. *Let $B \subseteq \mathbb{N}$. There is a minimal pair F and G for B.*

Proof.

Let f be an arbitrary generic regular enumeration of B. Let $\{A_n\}$ is a sequence of those sets that $\leq_e f$ and $\not\leq_e B$ (countable). By the last proposition we can construct g , such that g is generic and $(\forall n)(A_n \not\leq_e g)$. Set $F = \langle f \rangle , G = \langle g \rangle$. By the next lemma since $\langle f \rangle , \langle g \rangle$ are generic regular enumerations of B $B \lneq_e F, B \lneq_e G$. Let $A \leq_e F, A \leq_e G$. Then $A \notin \{A_n\}$, otherwise $A \not\leq_e G$. Since $A \notin \{A_n\}$, then $A \leq_e B$. □

Lemma. *If is generic regular enumaration of B, then $f \not\leq_e B$.*

Proof.

Suppose that $f \leq_e B$. Consider

$$S = \{\tau \text{ - } B\text{-regular finite part} \mid (\exists x)(\downarrow \tau(x) \,\&\, \tau(x) \not\simeq f(x))\}.$$

$S \leq_e B \oplus \langle f \rangle$, but $\langle f \rangle \leq_e B \Rightarrow S \leq_e B$. By genericity of f we have

$$(\exists \tau \subseteq f)(\underbrace{\tau \in S}_{\tau \not\subseteq f} \vee \underbrace{(\forall \rho \supseteq \tau)(\rho \notin S)}_{\text{not true } f \not\supseteq \rho \supseteq \tau}).$$

In both cases we have a contradiction. □