# Enumeration Degree Spectra of Abstract Structures

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# Outline

- Enumeration degrees
- Degree spectra and co-spectra
- Characterization of the co-spectra
- Representing the countable ideals as co-spectra
- Properties of upwards closed sets of degrees
- Selmans's theorem for degree spectra
- The minimal pair theorem
- Quasi-minimal degrees
- Jump spectra

# Enumeration reducibility

**Definition.**(Friedberg and Rogers, 1959) We say that $\Psi : 2^\omega \to 2^\omega$ is an *enumeration operator* (or e-operator) iff for some c.e. set $W_i$

$$\Psi(B) = \{x | (\exists D)[\langle x, D\rangle \in W_i \,\& D \subseteq B]\}$$

for each $B \subseteq \omega$.

**Definition.** For any sets $A$ and $B$ define $A$ is *enumeration reducible to* $B$, written $A \leq_e B$, by $A = \Psi(B)$ for some e-operator $\Psi$.

# The enumeration jump

**Definition.** Given $A \subseteq \omega$, set $A^+ = A \oplus (\omega \setminus A)$.

**Theorem.** *For any $A, B \subseteq \omega$,*

1. *$A$ is c.e. in $B$ iff $A \leq_e B^+$.*
2. *$A \leq_T B$ iff $A^+ \leq_e B^+$.*

**Definition.**(Cooper, McEvoy) Given $A \subseteq \omega$, let $E_A = \{\langle i, x\rangle | x \in \Psi_i(A)\}$. Set $J_e(A) = E_A^+$.

The enumeration jump $J_e$ is monotone and agrees with the Turing jump $J_T$ in the following sense:

**Theorem.** *For any $A \subseteq \omega$, $J_T(A)^+ \equiv_e J_e(A^+)$.*

**Definition.** A set $A$ is called *total* iff $A \equiv_e A^+$.

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# The enumeration degrees

**Definition.** Given a set $A$, let $d_e(A) = \{B \subseteq \omega | A \equiv_e B\}$.

Denote by $\mathcal{D}_T$ the partial ordering of the Turing degrees and by $\mathcal{D}_e$ the partial ordering of the enumeration degrees.

*The Rogers embedding. Define* $\iota : \mathcal{D}_T \to \mathcal{D}_e$ *by* $\iota(d_T(A)) = d_e(A^+)$. *Then* $\iota$ *is a Proper embedding of* $\mathcal{D}_T$ *into* $\mathcal{D}_e$. *The enumeration degrees in the range of* $\iota$ *are called total.*

*Let* $d_e(A)' = d_e(J_e(A))$. *The jump is always total and agrees with the Turing jump under the embedding* $\iota$.

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# Degree Spectra

*Let* $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ *be a denumerable structure. Enumeration of* $\mathfrak{A}$ *is every total surjective mapping of* $\mathbb{N}$ *onto* $\mathbb{N}$.

*Given an enumeration* $f$ *of* $\mathfrak{A}$ *and a subset of* $A$ *of* $\mathbb{N}^a$, *let*

$$f^{-1}(A) = \{\langle x_1, \ldots, x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A\}.$$

*Set* $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

**Definition.**(Richter) *The Turing Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) : f \text{ is an one to one enumeration of } \mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS_T(\mathfrak{A})$, then $\mathbf{a}$ is called the *degree of* $\mathfrak{A}$

**Definition.** *The e-Degree Spectrum of* $\mathfrak{A}$ is the set

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**Proposition.** *If* $\mathfrak{A}$ *has e-degree* $\mathbf{a}$ *then* $\mathbf{a} = d_e(f^{-1}(\mathfrak{A}))$ *for some one to one enumeration* $f$ *of* $\mathfrak{A}$.

**Proposition.** *If* $\mathbf{a} \in DS(\mathfrak{A})$, $\mathbf{b}$ *is a total e-degree and* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS(\mathfrak{A})$.

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# Total structures

**Definition.** The structure $\mathfrak{A}$ is called *total* if for every enumeration $f$ of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ is total.

**Proposition.** *If $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A}) = \iota(DS_T(\mathfrak{A}))$.*

Given a structure $\mathfrak{A} = (\mathbb{N}, R_1, \dots, R_k)$, for every $j$ denote by $R_j^c$ the complement of $R_j$ and let $\mathfrak{A}^+ = (\mathbb{N}, R_1, \dots, R_k, R_1^c, \dots, R_k^c)$.

**Proposition.** *The following are true:*

1. $\iota(DS_T(\mathfrak{A})) = DS(\mathfrak{A}^+)$.
2. *If $\mathfrak{A}$ is total then $DS(\mathfrak{A}) = DS(\mathfrak{A}^+)$.*

Clearly if $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A})$ consists of total degrees. The vice versa is not always true.

**Example.** *Let $K$ be the Kleene's set and $\mathfrak{A} = (\mathbb{N}; G_S, K)$, where $G_S$ is the graph of the successor function. Then $DS(\mathfrak{A})$ consists of all total degrees. On the other hand if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ is an c.e. set. Hence $\bar{K} \not\leq_e f^{-1}(\mathfrak{A})$. Clearly $\bar{K} \leq_e (f^{-1}(\mathfrak{A}))^+$. So $f^{-1}(\mathfrak{A})$ is not total.*

*Is it true that if $DS(\mathfrak{A})$ consists of total degrees then there exists a total structure $\mathfrak{B}$ s.t. $DS(\mathfrak{A}) = DS(\mathfrak{B})$?*

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*Is it true that if $DS(\mathfrak{A})$ consists of total degrees then there exists a total structure $\mathfrak{B}$ s.t. $DS(\mathfrak{A}) = DS(\mathfrak{B})$?*

**Definition.** Let $\mathcal{A}$ be a nonempty set of enumeration degrees the *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \&\ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_e \mathbf{a})\}.$$

**Example.** *Fix* $\mathbf{a} \in \mathcal{D}_e$ *and set* $\mathcal{A}_{\mathbf{a}} = \{\mathbf{b} \in \mathcal{D}_e : \mathbf{a} \leq_e \mathbf{b}\}$. *Then* $co(\mathcal{A}_{\mathbf{a}}) = \{\mathbf{b} \in \mathcal{D}_e : \mathbf{b} \leq_e \mathbf{a}\}$.

**Definition.** Given a structure $\mathfrak{A}$, set $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$. If $\mathbf{a}$ is the greatest element of $CS(\mathfrak{A})$ then call $\mathbf{a}$ the *co-degree* of $\mathfrak{A}$.

If $\mathfrak{A}$ has a degree $\mathbf{a}$ then $\mathbf{a}$ is also the co-degree of $\mathfrak{A}$. The vice versa is not always true.

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**Definition.** A set $A$ of natural numbers is admissible in $\mathfrak{A}$ if for every enumeration $f$ of $\mathfrak{A}$, $A \leq_e f^{-1}(\mathfrak{A})$.

Clearly $\mathbf{a} \in CS(\mathfrak{A})$ iff $\mathbf{a} = d_e(A)$ for some admissible in $\mathfrak{A}$ set $A$.

Every finite mapping of $\mathbb{N}$ into $\mathbb{N}$ is called *finite part*.

For every finite part $\tau$ and natural numbers $e, x$, let

$$\tau \Vdash F_e(x) \iff x \in \Psi_e(\tau^{-1}(\mathfrak{A})) \text{ and}$$
$$\tau \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x)).$$

**Definition.** An enumeration $f$ is *generic* if for every $e, x \in \mathbb{N}$, there exists a $\tau \subseteq f$ s.t. $\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)$.

# The admissible sets

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# Normal form of the admissible sets

**Definition.** A set $A$ of natural numbers is *forcing definable in the structure* $\mathfrak{A}$ iff there exist finite part $\delta$ and natural number $e$ s.t.

$$A = \{x | (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

**Theorem.** *Let $A \subseteq \mathbb{N}$ and $d_e(B) \in DS(\mathfrak{A})$. Then the following are equivalent:*

1. *$A$ is admissible in $\mathfrak{A}$.*
2. *$A \leq_e f^{-1}(\mathfrak{A})$ for all generic enumerations $f$ of $\mathfrak{A}$.* $(f^{-1}(\mathfrak{A}))' \equiv_e B'$.
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# Some examples

**Example.** *(Richter 1981) Let* $\mathfrak{A} = (\mathbb{N}; <)$ *be a linear ordering. Then* $DS(\mathfrak{A})$ *contains a minimal pair of degrees and hence* $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$. *Clearly* $\mathbf{0}_e$ *is the co-degree of* $\mathfrak{A}$. *Therefore if* $\mathfrak{A}$ *has a degree* $\mathbf{a}$, *then* $\mathbf{a} = \mathbf{0}_e$.

**Definition.** Let $n \geq 0$. The $n$-th jump spectrum of a structure $\mathfrak{A}$ is defined by $DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}$. Set $CS_n(\mathfrak{A}) = co(DS_n(\mathfrak{A}))$.

**Example.** *(Knight 1986) Consider again a linear ordering* $\mathfrak{A}$. *Then* $CS_1(\mathfrak{A})$ *consists of all* $\Sigma^0_2$ *sets. The first jump co-degree of* $\mathfrak{A}$ *is* $\mathbf{0}'_e$.

**Example.** *(Slaman 1998,Whener 1998) There exists an* $\mathfrak{A}$ *s.t.*

$$DS(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}.$$

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# Representing countable ideals as co-spectra

**Example.** *(based on Coles, Dawney, Slaman - 1998) Let $G$ be a torsion free Abelian group of rank 1, i.e. $G$ is a subgroup of $Q$. There exists an enumeration degree $\mathbf{s}_G$ such that*

- $DS(G) = \{\mathbf{b} : \mathbf{b}$ *is total and* $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- *The co-degree of $G$ is $\mathbf{s}_G$.*
- *$G$ has a degree iff $\mathbf{s}_G$ is total*
- *If $1 \leq n$, then $\mathbf{s}_G^{(n)}$ is the $n$-th jump degree of $G$.*

*For every $\mathbf{d} \in \mathcal{D}_e$ there exists a $G$, s.t. $\mathbf{s}_G = \mathbf{d}$. Hence every principle ideal of enumeration degrees is $CS(G)$ for some $G$.*

*Similar results on algebraic fields:*
*W. Calvert, V. Harizanov and A. Shlapentokh (2007)*
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**Example.** *Let $B_0, \dots, B_n, \dots$ be a sequence of sets of natural numbers. Set $\mathfrak{A} = (\mathbb{N}; f; \sigma)$,*

$$f(\langle i, n\rangle) = \langle i+1, n\rangle;$$
$$\sigma = \{\langle i, n\rangle : n = 2k+1 \vee n = 2k \ \&\ i \in B_k\}.$$

*Then $CS(\mathfrak{A}) = I(d_e(B_0), \dots, d_e(B_n), \dots)$*

# General Properties of Upwards Closed Sets

**Definition.** Consider a subset $\mathcal{A}$ of $\mathcal{D}_e$. Say that $\mathcal{A}$ is *upwards closed* if for every $\mathbf{a} \in \mathcal{A}$ all total degrees greater than $\mathbf{a}$ are contained in $\mathcal{A}$.

Let $\mathcal{A}$ be an upwards closed set of degrees.
Note that if $\mathcal{B} \subseteq \mathcal{A}$, then $co(\mathcal{A}) \subseteq co(\mathcal{B})$.

**Proposition.** *(Selman Theorem) Let* $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \&\ \mathbf{a} \text{ is total}\}$. *Then* $co(\mathcal{A}) = co(\mathcal{A}_t)$.

**Proposition.** *(high jumps) Let* $\mathbf{b}$ *be an arbitrary enumeration degree and* $n > 0$. *Set* $\mathcal{A}_{\mathbf{b},n} = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \&\ \mathbf{b} \leq_e \mathbf{a}^{(n)}\}$. *Then* $co(\mathcal{A}) = co(\mathcal{A}_{\mathbf{b},n})$.

**Theorem.** *(low jumps) Let* $\mathfrak{A}$ *be a structure,* $1 \le n$ *and* $\mathbf{c} \in DS_n(\mathfrak{A})$. *Then*

$$CS(\mathfrak{A}) = co(\{\mathbf{b} \in DS(\mathfrak{A}) : \mathbf{b}^{(n)} = \mathbf{c}\}).$$

**Example.** *(Upwards closed set for which the Theorem is not true)*

*Let* $B \not\le_e A$ *and* $A \not\le_e B'$. *Let*

$$\mathcal{D} = \{\mathbf{a} : d_e(A) \le_e \mathbf{a}\} \cup \{\mathbf{a} : d_e(B) \le_e \mathbf{a}\}.$$

*Set* $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \ \& \ \mathbf{a}' = d_e(B)'\}$.

- $d_e(B)$ *is the least element of* $\mathcal{A}$ *and hence* $d_e(B) \in co(\mathcal{A})$.
- $d_e(B) \not\le d_e(A)$ *and hence* $d_e(B) \notin co(\mathcal{D})$.

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*Let $B \not\leq_e A$ and $A \not\leq_e B'$. Let*

$$\mathcal{D} = \{\mathbf{a} : d_e(A) \leq_e \mathbf{a}\} \cup \{\mathbf{a} : d_e(B) \leq_e \mathbf{a}\}.$$

*Set $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \;\&\; \mathbf{a}' = d_e(B)'\}$.*

- *$d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$.*
- *$d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.*

# The minimal pair theorem

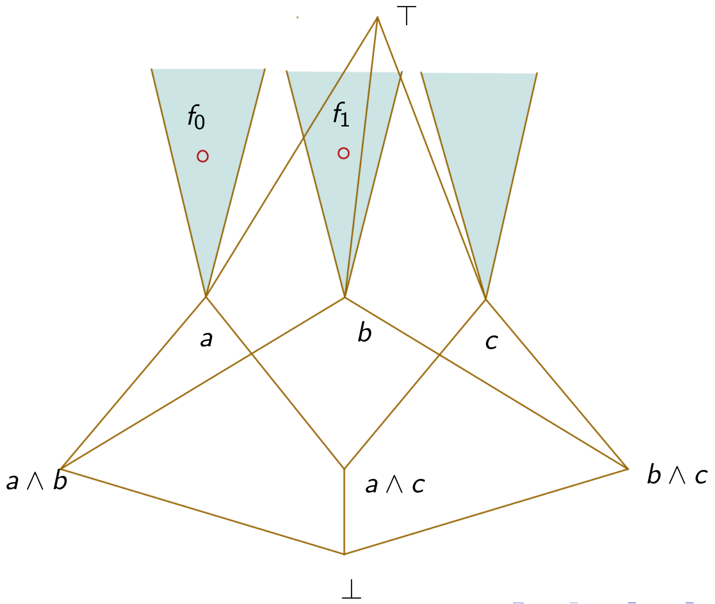
**Theorem.** *Let* $\mathbf{c} \in DS_2(\mathfrak{A})$. *There exist* $\mathbf{f},\mathbf{g} \in DS(\mathfrak{A})$ *s.t.* $\mathbf{f},\mathbf{g}$ *are total,* $\mathbf{f}'' = \mathbf{g}'' = \mathbf{c}$ *and* $CS(\mathfrak{A}) = co(\{\mathbf{f},\mathbf{g}\})$.

Notice that for every enumeration degree $\mathbf{b}$ there exists a structure $\mathfrak{A}_{\mathbf{b}}$ s. t. $DS(\mathfrak{A}_{\mathbf{b}}) = \{\mathbf{x} \in \mathcal{D}_T | \mathbf{b} <_e \mathbf{x}\}$. Hence

**Corollary.** *(Rozinas) For every* $\mathbf{b} \in \mathcal{D}_e$ *there exist total* $\mathbf{f}, \mathbf{g}$ *below* $\mathbf{b}''$ *which are a minimal pair over* $\mathbf{b}$.

*Not every upwards closed set of enumeration degrees has a minimal pair:*

# An upwards closed set with no minimal pair



# The Quasi-minimal degree

**Definition.** Let $\mathcal{A}$ be a set of enumeration degrees. The degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if:

- $\mathbf{q} \notin co(\mathcal{A})$.
- If $\mathbf{a}$ is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
- If $\mathbf{a}$ is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

**Theorem.** *If $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$, then $\mathbf{q}$ is an upper bound of $co(\mathcal{A})$.*

**Theorem.** *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

**Corollary.** *(Slaman and Sorbi) Let $I$ be a countable ideal of enumeration degrees. There exist an enumeration degree $\mathbf{q}$ s.t.*

1. *If $\mathbf{a} \in I$ then $\mathbf{a} <_e \mathbf{q}$.*
2. *If $\mathbf{a}$ is total and $\mathbf{a} \leq_e \mathbf{q}$ then $\mathbf{a} \in I$.*

**Definition.** Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

**Theorem.** *Let $\mathcal{A}$ be an upwards closed set of degrees possessing a quasi-minimal degree. Suppose that there exists a countable base $\mathcal{B}$ of $\mathcal{A}$ such that all elements of $\mathcal{B}$ are total. Then $\mathcal{A}$ has a least element.*

**Corollary.** *A total structure $\mathfrak{A}$ has a degree if and only if $DS(\mathfrak{A})$ has a countable base.*

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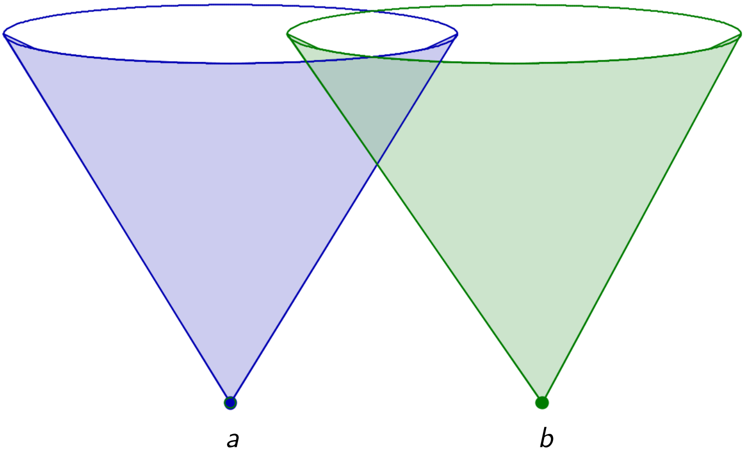
# An upwards closed set with no quasi-minimal degree



# Jump spectra

**Definition.** The *n*-th jump spectrum of a structure $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS_n(\mathfrak{A})$ then $\mathbf{a}$ is called *n*-th jump degree of $\mathfrak{A}$.

**Proposition.** *For every $\mathfrak{A}$, $DS_1(\mathfrak{A}) \subseteq DS(\mathfrak{A})$.*

*Is it true that for every $\mathfrak{A}$, $DS_1(\mathfrak{A}) \subset DS(\mathfrak{A})$? Probably the answer is "no".*

# Every jump spectrum is spectrum of a total structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_n)$.
Let $\bar{0} \notin \mathbb{N}$. Set $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$. Let $\langle ., . \rangle$ be a pairing function s.t. none of the elements of $\mathbb{N}_0$ is a pair and $N^*$ be the least set containing $\mathbb{N}_0$ and closed under $\langle ., . \rangle$.

**Definition.** *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (\mathbb{N}^*, R_1, \ldots, R_n, \mathbb{N}_0, G_{\langle ., . \rangle}).$$

**Proposition.** $DS(\mathfrak{A}) = DS(\mathfrak{A}^*)$

Let $K_{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$.
Set $\mathfrak{A}' = (\mathfrak{A}^*, K_{\mathfrak{A}}, \mathbb{N}^* \setminus K_{\mathfrak{A}})$.

**Theorem.**

1. *The structure $\mathfrak{A}'$ is total.*
2. $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$.

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**Theorem.**

1. *The structure $\mathfrak{A}'$ is total.*
2. $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$.

Consider two structures $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that

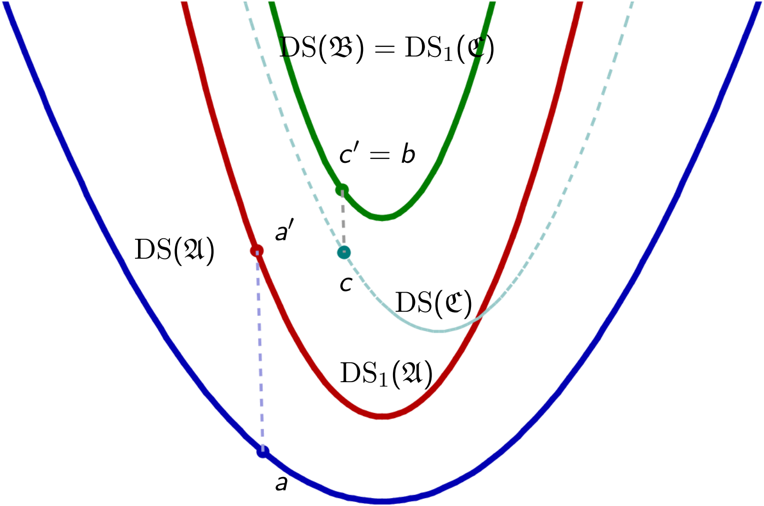
$$DS(\mathfrak{B})_t = \{\mathbf{a} | \mathbf{a} \in DS(\mathfrak{B}) \text{ and } \mathbf{a} \text{ is total}\} \subseteq DS_1(\mathfrak{A}).$$

**Theorem.** *There exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS_1(\mathfrak{C}) = DS(\mathfrak{B})_t$.*

Method: Marker's extensions.

**Corollary.** *Let $DS(\mathfrak{B}) \subseteq DS_1(\mathfrak{A})$. Then there exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS(\mathfrak{B}) = DS_1(\mathfrak{C})$.*

**Corollary.** *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}'$. Then there exists a total structure $\mathfrak{C}'$ such that $DS(\mathfrak{B}) = DS(\mathfrak{C}')$.*


$\mathrm{DS}(\mathfrak{B}) = \mathrm{DS}_1(\mathfrak{C})$
$c' = b$
$a'$
$\mathrm{DS}(\mathfrak{A})$
$c$
$\mathrm{DS}(\mathfrak{C})$
$\mathrm{DS}_1(\mathfrak{A})$
$a$

**Theorem.** *Let $n \geq 1$. Suppose that $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. There exists a structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

**Corollary.** *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}^{(n)}$. Then there exists a total structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

**Remark.** *Similar results*
*A. Montalban (2009) different approach with complete set of $\Pi^c_n$ formulas.*
*A. Stukachev (2009) for $\Sigma$ reducibility with Marker's extentions.*

**Theorem.** *Let* $n \geq 1$. *Suppose that* $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. *There exists a structure* $\mathfrak{C}$ *s.t.* $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.

**Corollary.** *Suppose that* $DS(\mathfrak{B})$ *consists of total degrees greater than or equal to* $\mathbf{0}^{(n)}$. *Then there exists a total structure* $\mathfrak{C}$ *s.t.* $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.

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**Example.** *(Ash, Jockush, Knight and Downey) Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ s.t. $\mathfrak{C}$ has a $n+1$-st jump degree $\mathbf{0}^{(n+1)}$ but has no k-th jump degree for $k \leq n$.*
*It is sufficient to construct a structure $\mathfrak{B}$ satisfying:*

1. *$DS(\mathfrak{B})$ has not least element.*
2. *$\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.*
3. *All elements of $DS(\mathfrak{B})$ are total and above $\mathbf{0}^{(n)}$.*

*Consider a set $B$ satisfying:*

1. *$B$ is quasi-minimal above $\mathbf{0}^{(n)}$.*
2. *$B' \equiv_e \mathbf{0}^{(n+1)}$.*

*Let $G$ be a subgroup of the additive group of the rationales s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{\mathbf{a} | d_e(S_G) \leq_e \mathbf{a}$ and $\mathbf{a}$ is total$\}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$.*

# Applications

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Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = \{\mathbf{a} | \mathbf{0}^{(n)} <_e \mathbf{a}\}$.
It is sufficient to construct a structure $\mathfrak{B}$ such that the elements of $DS(\mathfrak{B})$ are exactly the total e-degrees greater than $\mathbf{0}^{(n)}$.
This is done by Whener's construction using a special family of sets:

**Theorem.** *Let $n \geq 0$. There exists a family $\mathcal{F}$ of sets of natural numbers s.t. for every $X$ strictly above $\mathbf{0}^{(n)}$ there exists a recursive in $X$ set $U$ satisfying the equivalence:*

$$F \in \mathcal{F} \iff (\exists a)(F = \{x | (a, x) \in U\}).$$

*But there is no c.e. in $\mathbf{0}^{(n)}$ such $U$.*

- Questions:
  - Jump Inversion theorem for infinite recursive ordinal $\alpha$?
  - Is it true that for every structure $\mathfrak{A}$ and every sequence $\mathcal{B}$ there exists a structure $\mathfrak{B}$ such that $\mathrm{DS}(\mathfrak{B}) = \mathrm{DS}(\mathfrak{A}, \mathcal{B})$?
  - If for a countable ideal $I \subseteq \mathcal{D}_\omega$ there is an exact pair then are there a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ so that $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I$?

Thank you!