Structural properties of the cototal enumeration degrees

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The enumeration degrees

Definition

$A \leq_e B$ if there is a c.e. set W, such that

$$A = W(B) = \{x \mid \exists D(\langle x, D\rangle \in W \,\&\, D \subseteq B)\}\,.$$

Equivalently, $A \leq_e B$ if there is a single Turing functional which uniformly, given any enumeration of B, outputs an enumeration of A.

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- The enumeration degree of a set A is $d_e(A) = \{B \mid A \equiv_e B\}$.
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- The least element: $\mathbf{0_e} = d_e(\emptyset)$, the set of all c.e. sets.
- The least upper bound: $d_e(A) \vee d_e(B) = d_e(A \oplus B)$.
- The enumeration jump: $d_e(A)' = d_e(K_A \oplus \overline{K_A})$, where $K_A = \{\langle e, x\rangle \mid x \in W_e(A)\}$.

What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$$A \leq_T B \Leftrightarrow A \oplus \overline{A} \leq_e B \oplus \overline{B}.$$

Definition

A set A is *total* if $\overline{A} \leq_e A$, or equivalently $A \equiv_e A \oplus \overline{A}$. An enumeration degree is *total* if it contains a total set.

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Within the enumeration degrees, the total degrees are an embedded copy of the Turing degrees $\mathcal{D}_T$ via $\iota : A \to A \oplus \overline{A}$. The embedding ι preserves the order, the least upper bound and the jump operation.

Total and cototal

Definition

A set A is *cototal* if $A \leq_e \overline{A}$. A degree $\mathbf{a}$ is cototal if it contains a cototal set.

For every set A the set $A \oplus \overline{A}$ is cototal.
So, every total e-dergree is cototal.

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The cototal enumeration degrees form a proper substructure of $\mathcal{D}_e$ closed under least upper bound and the enumeration jump operator.

- The name "*total*" : for any total function f, the set $G(f) = \{\langle n, f(n)\rangle \mid n \in \omega\}$ is a total set.
- Equivalently, given a total function f, the graph-complement $\overline{G(f)}$ is cototal.
- If an enumeration degree contains a set of the form $\overline{G(f)}$, then we call it *graph-cototal*.
- So every total enumeration degree is graph-cototal, and every graph-cototal is cototal.

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3. The Turing degrees that compute elements of X are exactly the degrees that contain enumerations of L_X. So $L_X \equiv_e X$.

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3. The Turing degrees that compute elements of X are exactly the degrees that contain enumerations of L_X. So $L_X \equiv_e X$.
4. (Jaendel) If we can enumerate the set of forbidden words $\overline{L_X}$ then we can enumerate L_X. So, $L_X \leq_e \overline{L_X}$.
5. (McCarthy) If A is cototal, then $A \equiv_e L_X$ for some minimal subshift X.

Maximal independent sets

Definition

Let $G = (\mathbb{N}, E)$ be a graph and $S \subseteq \mathbb{N}$.

1. S is an *independent set* for G if $i \neq j$ are in S then $(i, j) \notin E$.
2. An independent set is *maximal* if it has no proper independent superset, i.e. for every element $i \notin S$ there is a $j \in S$ such that $(i, j) \in E$.

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Theorem

Every cototal degree contains the complement of maximal independent set for $\omega^{<\omega}$.

Joins of nontrivial $\mathcal{K}$-pairs

Definition

A $\mathcal{K}$-pair is a pair of sets $\{A, B\}$ for which there is a c.e. set W such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

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Proposition (Kalimullin)

Let $\{A, B\}$ be a $\mathcal{K}$-pair. If A and B are not c.e. then:

1. $A \leq_e \overline{B}$ *and* $\overline{A} \leq_e \emptyset' \oplus B$.
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Proof: $A \oplus B \leq_e \overline{B} \oplus \overline{A} \equiv_e \overline{A \oplus B}$.

Continuous degrees

J. Miller introduced the continuous degrees $\mathcal{D}_r$ to compare the complexity of points in computable metric spaces. A point x in a computable metric space can be described by a sequence of "rational" points that limit to it. For two points $x; y$ we say that $x \leq_r y$ if every description of y computes a description of x. The continuous degrees embed into $\mathcal{D}_e$. In fact, $D_T \subset \mathcal{D}_r \subset \mathcal{D}_e$.

Definition (J. Miller)

An e-degree is *continuous* if it contains a set of the form $A = \bigoplus_{i<\omega}(\{q \mid q < \alpha_i\} \oplus \{q \mid q > \alpha_i\})$, where $\{\alpha_i\}_{i<\omega}$ is a sequence of real numbers.

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Proposition

Continuous degrees are cototal.

$\overline{A} \equiv_e B = \bigoplus_{i<\omega}(\{q \mid q \leq \alpha_i\} \oplus \{q \mid q \geq \alpha_i\})$.
Kihara and Pauly extend Miller's idea to points in arbitrary represented topological spaces.

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Recall that $\overline{K_A} = \bigoplus_e \overline{\Gamma_e(A)}$.

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The skip of A is the set $A^\Diamond = \overline{K_A}$. The skip of a degree $\mathbf{a}$ is $\mathbf{a}^\Diamond = d_e(A^\Diamond)$.

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A degree $\mathbf{a}$ is cototal if and only if $\mathbf{a} \leq \mathbf{a}^\Diamond$ if and only if $\mathbf{a}^\Diamond = \mathbf{a}'$.

$\Rightarrow A \leq_e \overline{A} \leq_e A^\Diamond$
$\Leftarrow K_A \equiv_e A \leq_e A^\Diamond = \overline{K_A}$.
Recall that $A' = K_A \oplus \overline{K_A}$.

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We first build a table $\hat{A}$ with one empty box in each column as a set c.e. in $\emptyset'$.

The set of empty boxes will be computable from $\emptyset'$.

Then $A = \hat{A} \cup \{\langle n, s\rangle \mid n \in \overline{S}\}$.

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Then $A = \hat{A} \cup \left\{ \langle n, s \rangle \mid n \in \overline{S} \right\}$. So we have $\overline{S} \leq_e A \oplus \emptyset'$.
And we build the set A such that $S \equiv_e \overline{A} \leq_e A^{\Diamond} \leq_e \overline{A} \oplus \emptyset'$.

Skip iteration

We can define the *iterated skip operator* of an enumeration degree $\mathbf{a}$ by:

- $\mathbf{a}^{\langle 0\rangle} = \mathbf{a}$
- $\mathbf{a}^{\langle n+1\rangle} = (\mathbf{a}^{\langle n\rangle})^{\Diamond}$.

This iterated skip can exhibit exotic behavior:

Theorem

For all enumeration degrees, $\mathbf{a} \leq \mathbf{a}^{\Diamond\Diamond}$ *and* $\mathbf{a}^{\Diamond} \geq \mathbf{0}'$, *but not always* $\mathbf{a} \leq \mathbf{a}^{\Diamond}$.

$\overline{A} \leq_1 A^{\Diamond} \Rightarrow A \leq_1 \overline{A^{\Diamond}} \leq_1 A^{\Diamond\Diamond}$.

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$\overline{K_{\overline{K_A}}} \subseteq \overline{K_{\overline{K_B}}}$.

Any such enumeration degree lies above all total hyperarithmetic enumeration degrees.

Iterating the skip

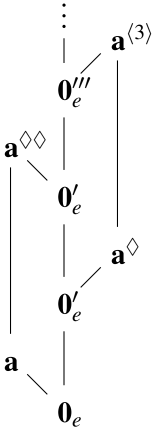

Figure: Iterated skips of a degree

Zig-zag

If $\mathbf{a}^{\langle n\rangle}$ is not cototal for every n:

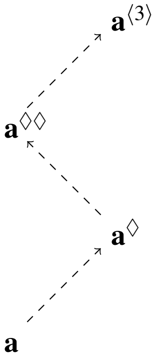

Figure: Iterated skips of a degree: the zig-zag

Generic sets

Definition

A set G is 1- *generic relative to* $\langle X \rangle$ if and only if for every $W \subseteq 2^{<\omega}$ such that $W \leq_e X$:

$$(\exists \sigma \preceq G)[\sigma \in W \vee (\forall \tau \succeq \sigma)[\tau \notin W]].$$

Proposition

If G is 1-generic relative to $\langle X \rangle$ *then:*

- $\overline{G}$ *is 1-generic relative to* $\langle X \rangle$.
- $(G \oplus X)^{\Diamond} = \overline{G} \oplus X'$.

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If G is 1-generic relative to $\langle \emptyset^{(n)} \rangle$ for every n, then the skips of G and $\overline{G}$ form a double helix.

- If n is odd then $G^{\langle n \rangle} \equiv_e \overline{G} \oplus \emptyset^{(n)}$ and $(\overline{G})^{\langle n \rangle} \equiv_e G \oplus \emptyset^{(n)}$.
- If n is even then $G^{\langle n \rangle} \equiv_e G \oplus \emptyset^{(n)}$ and $(\overline{G})^{\langle n \rangle} \equiv_e \overline{G} \oplus \emptyset^{(n)}$.

Double zig-zag

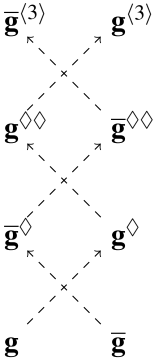

Figure: Iterated skips of a degrees of an arithmetically generic set and its complement: double zig-zag

Skips of nontrivial $\mathcal{K}$-pairs

Proposition

If $\{A,B\}$ *is a non-trivial* $\mathcal{K}$*-pair then* $A^{\Diamond} \equiv_e B \oplus \emptyset'$.
If $\{A,B\}$ *is a non-trivial* $\mathcal{K}$*-pair relative to* $\langle X \rangle$ *then* $(A \oplus X)^{\Diamond} \leq_e B \oplus X^{\Diamond}$.
The oracle X *is of cototal degree iff we have* $(A \oplus X)^{\Diamond} \equiv_e B \oplus X^{\Diamond}$ *for every nontrivial* $\mathcal{K}$*-pair relative to* $\langle X \rangle$.

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The oracle X is of cototal degree iff we have $(A \oplus X)^{\Diamond} \equiv_e B \oplus X^{\Diamond}$ for every nontrivial $\mathcal{K}$-pair relative to $\langle X \rangle$.

If $\{A, B\}$ is a non-trivial $\mathcal{K}$-pair relative to $\emptyset^{(n)}$ for every n then the iterate skips of A and B form a double zig-zag.

- if n is odd then $A^{\langle n \rangle} \equiv_e B \oplus \emptyset^{(n)}$ and $B^{\langle n \rangle} \equiv_e A \oplus \emptyset^{(n)}$, and
- if n is even then $A^{\langle n \rangle} \equiv_e A \oplus \emptyset^{(n)}$ and $B^{\langle n \rangle} \equiv_e B \oplus \emptyset^{(n)}$.

Skip iterations

Theorem (Ganchev, Sorbi)

For every enumeration degree $\mathbf{x} > \mathbf{0}_e$*, there is a degree* $\mathbf{a} \leq \mathbf{x}$ *such that* $\mathbf{a}$ *is half of a nontrivial* $\mathcal{K}$*-pair and such that* $\mathbf{a}' = \mathbf{x}'$*.*

$$A' \equiv_e A \oplus A^{\Diamond} \equiv_e A \oplus B \oplus \emptyset' \equiv_e B \oplus B^{\Diamond} \equiv_e B'.$$

Proposition

- *If* $\mathbf{x}$ *is high* ($\mathbf{x}' = \mathbf{0}''$)*:*

$$\mathbf{b}^{\Diamond} < \mathbf{b}' = \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' = \mathbf{b}^{\langle 3 \rangle} < \cdots < \mathbf{b}^{(n)} = \mathbf{b}^{\langle n+1 \rangle} < \ldots$$

Skip iterations

Theorem (Ganchev, Sorbi)

For every enumeration degree $\mathbf{x} > \mathbf{0}_e$*, there is a degree* $\mathbf{a} \leq \mathbf{x}$ *such that* $\mathbf{a}$ *is half of a nontrivial* $\mathcal{K}$*-pair and such that* $\mathbf{a}' = \mathbf{x}'$*.*

$$A' \equiv_e A \oplus A^{\Diamond} \equiv_e A \oplus B \oplus \emptyset' \equiv_e B \oplus B^{\Diamond} \equiv_e B'.$$

Proposition

- *If* $\mathbf{x}$ *is high* ($\mathbf{x}' = \mathbf{0}''$)*:*

$$\mathbf{b}^{\Diamond} < \mathbf{b}' = \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' = \mathbf{b}^{\langle 3 \rangle} < \cdots < \mathbf{b}^{(n)} = \mathbf{b}^{\langle n+1 \rangle} < \ldots$$

- *If* $\mathbf{x}$ *is intermediate:*

$$\mathbf{b}^{\Diamond} < \mathbf{b}' < \mathbf{b}^{\Diamond\Diamond} < \mathbf{b}'' < \mathbf{b}^{\langle 3 \rangle} < \cdots < \mathbf{b}^{(n)} < \mathbf{b}^{\langle n+1 \rangle} < \ldots$$

The cotetal degrees are dense

Corollary

The relation

$$SK = \left\{ (\mathbf{a}, \mathbf{a}^{\Diamond}) \mid \mathbf{a} \textit{ is half of a nontrivial } \mathcal{K}\textit{-pair} \right\}$$

is first-order definable in $\mathcal{D}_e$.

The cotatal degrees are dense

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The relation

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is first-order definable in $\mathcal{D}_e$.

Question: Is the skip operator definable in $\mathcal{D}_e$?

Good e-degrees

Definition (Lachlan, Shore)

A uniformly computable sequence of finite sets $\{A_s\}_{s<\omega}$ is *a good approximation* to a set A if:
$G1 (\forall n)(\exists s)(A \upharpoonright n \subseteq A_s \subseteq A)$
$G2 (\forall n)(\exists s)(\forall t > s)(A_t \subseteq A \Rightarrow A \upharpoonright n \subseteq A_t)$.

An enumeration degree is *good* if it contains a set with a good approximation.

- Good e-degrees cannot be tops of empty intervals.
- Total enumeration degrees and enumeration degrees of n-c.e.a. sets are good.

The cototal degrees are dense

Theorem (Harris; Miller, M. Soskova)

The good enumeration degrees are exactly the cototal enumeration degrees.

If A has a good approximation then

$$A \leq_e \{\langle x, s\rangle \mid (\forall t > s)(A_t \subseteq A \Rightarrow x \in A)\} \leq_e A^{\Diamond}.$$

Every uniformly e-pointed tree has a good approximation.

Theorem (Miller, M. Soskova)

The cototal enumeration degrees are dense.

If $V <_e U$ are cototal and U has a good approximation they build Θ such that $\Theta(U)$ is the complement of a maximal independent set and

$$V <_e \Theta(U) \oplus V <_e U.$$