Joint Spectra and Relative Spectra of structures

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26 March 2014

Degree spectra

Definition

Let $\mathfrak{A}$ be a countable structure. The *spectrum* of $\mathfrak{A}$ is the set of Turing degrees

$$\mathrm{Sp}(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \text{ computes the diagram of an isomorphic copy of } \mathfrak{A}\}.$$

Enumeration of a structure

Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a bijection from $\mathbb{N}$ onto A.
- let for any $X \subseteq A^a$
  $f^{-1}(X) = \{\langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in X\}$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$.

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The spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}(\mathfrak{A}) = \{\mathbf{a} \mid (\exists f)(d_T(f^{-1}(\mathfrak{A})) \leq_T \mathbf{a})\}$.

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The spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}(\mathfrak{A}) = \{\mathbf{a} \mid (\exists f)(d_T(f^{-1}(\mathfrak{A})) \leq_T \mathbf{a})\}$.
The k-th jump spectrum of $\mathfrak{A}$ is the set $\mathrm{Sp}_k(\mathfrak{A}) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \mathrm{Sp}(\mathfrak{A})\}$.

Joint Spectra

Let $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ be arbitrary countable abstract structures.

Definition

The Joint spectrum of $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\begin{aligned} \mathrm{JSp}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, & \mathfrak{A}_n) = \\ & \{\mathbf{a} : \mathbf{a} \in \mathrm{Sp}(\mathfrak{A}_0), \mathbf{a}' \in \mathrm{Sp}(\mathfrak{A}_1), \ldots, \mathbf{a}^{(\mathbf{n})} \in \mathrm{Sp}(\mathfrak{A}_n)\}. \end{aligned}$$

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Enumeration reducibility

1. A set X is *c.e. in* a set Y if X can be enumerated by a computable in Y function.
2. A set X is enumeration reducible to a set Y if and only if there is an effective procedure to transform an enumeration of Y to an enumeration of X.

Definition

$X \leq_e Y$ if for some e, $X = W_e(Y)$, i.e.

$$(\forall x)(x \in X \iff (\exists v)(\langle v, x\rangle \in W_e \wedge D_v \subseteq Y)).$$

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Denote by Y^+ the set $Y \oplus \overline{Y}$.

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X is c.e. in Y if and only if $X \leq_e Y^+$.
X is computable in Y if and only if $X^+ \leq_e Y^+$.

Degree structures

- The enumeration degree of set X is $d_e(X) = \{Y \mid X \equiv_e Y\}$.

  The structure of the enumeration degrees $\mathcal{D}_e$ is an upper semi-lattice with jump operation.

  The Turing degrees are embedded in to the enumeration degrees by: $\iota(d_T(X)) = d_e(X^+)$.
- This embedding agrees with the jump operation since $(K^X)^+ \equiv_e (X^+)'$.

Co-spectra of structures

Definition

Let $\mathfrak{A}$ be a countable structure and $k \in \mathbb{N}$. The k-th co-spectrum of $\mathfrak{A}$ is the set

$$\mathrm{CoSp}_k(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_e \wedge (\forall \mathbf{b} \in \mathrm{Sp}_k(\mathfrak{A}))(\mathbf{a} \leq_e \mathbf{b})\}.$$

Definition

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k \leq n}$ be a finite sequence of structures.
The k-th co-spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{CoJSp}_k(\vec{\mathfrak{A}}) = \left\{\mathbf{a} \in \mathcal{D}_e \mid \forall \mathbf{x} \in \mathrm{JSp}_k(\vec{\mathfrak{A}})(\mathbf{a} \leq_e \mathbf{x})\right\}.$$

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Co-spectra of Joint spectra of structures

For each sequence $\{f_k\}_{k\leq n}$ of enumerations of the structures $\vec{\mathfrak{A}}$ denote by: $\vec{f}^{-1}(\vec{\mathfrak{A}}) = \{f_k^{-1}(\mathfrak{A}_k)\}_{k\leq n}$ and by induction: $\mathcal{P}_0(\vec{f}^{-1}(\vec{\mathfrak{A}})) = f_0^{-1}(\mathfrak{A}_0)$ and $\mathcal{P}_{k+1}(\vec{f}^{-1}(\vec{\mathfrak{A}})) = \mathcal{P}_k(\vec{f}^{-1}(\vec{\mathfrak{A}}))' \oplus f_{k+1}^{-1}(\mathfrak{A}_{k+1})$.

Proposition

For any set $X \subseteq \mathbb{N}$ the following equivalence holds

$$d_e(X) \in \mathrm{CoJSp}_k(\vec{\mathfrak{A}}) \iff X \leq_e \mathcal{P}_k(\vec{f}^{-1}(\vec{\mathfrak{A}})) \text{ for every sequence } \{f_k\}_{k\leq n} \text{ of enumerations of } \vec{\mathfrak{A}}.$$

Corollary

For $k \leq n$:

$$\mathrm{CoJSp}_k(\vec{\mathfrak{A}}) = \mathrm{CoJSp}_k(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_k).$$

The Normal Form Theorem

Definition

The set $X \subseteq \mathbb{N}$ is *formally k-definable* on $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k \leq n}$ if there exists a computable sequence $\{\Phi^{\gamma(x)}(W_1, \dots, W_r)\}$ of Σ^+_k formulae and parameters $t_1, \dots, t_r$ such that:
$x \in X \iff \vec{\mathfrak{A}} \models \Phi^{\gamma(x)}(W_1/t_1, \dots, W_r/t_r)$.

- Σ^+_0 : $\exists \overline{Y^0}(\beta_1 \ \& \ \dots \ \& \ \beta_k)$;
- Σ^+_{k+1}: c.e. disjunction of formulae of the form $\exists \bar{Y}^0 \dots \exists \bar{Y}^{k+1} \Phi(\bar{X}^0, \dots, \bar{X}^{k+1}, \bar{Y}^0, \dots, \bar{Y}^{k+1})$ where Φ is a finite conjunction of Σ^+_k formulae and negations of Σ^+_k formulae with free variables among $\bar{Y}^0 \dots \bar{Y}^k, \bar{X}^0 \dots \bar{X}^k$ and atoms of $\mathcal{L}_{k+1}$ with variables among $\bar{X}^{k+1}, \bar{Y}^{k+1}$;

Theorem

$d_e(X) \in \mathrm{CoJSp}_k(\vec{\mathfrak{A}})$ if and only if X is formally k-definable on $\vec{\mathfrak{A}}$.

Relative Spectra of Structures

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k \leq n}$ be a finite sequence of countable structures. Denote by $A = \bigcup_k A_k$.

Definition

The relative spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{RSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid (\exists f \text{ enumeration of } A)(\forall k \leq n)(f^{-1}(\mathfrak{A}_k) \text{ is c.e. in } B^{(k)})$$

where $f^{-1}(\mathfrak{A}_k) = f^{-1}(A_k) \oplus f^{-1}(R_1^k) \oplus \cdots \oplus f^{-1}(R_{m_k}^k)$.

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The k-th jump spectrum of $\vec{\mathfrak{A}}$ is the set

$$\mathrm{RSp}_k(\vec{\mathfrak{A}}) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \mathrm{RSp}(\vec{\mathfrak{A}})\}.$$

Relative Co-spectra of Structures

Definition

The Relative co-spectrum of $\vec{\mathfrak{A}}$ is the following set of enumeration degrees:

$$\mathrm{CoRSp}(\vec{\mathfrak{A}}) = \{\mathbf{b} \in \mathcal{D}_e \mid (\forall \mathbf{a} \in \mathrm{RSp}(\vec{\mathfrak{A}}))(\mathbf{b} \leq \mathbf{a})\}.$$

The Relative kth co-spectrum of $\vec{\mathfrak{A}}$ is

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Proposition

$$\mathrm{CoRSp}_k(\vec{\mathfrak{A}}) = \mathrm{CoRSp}_k(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_k).$$

Relative Co-spectra

Let f be an enumeration of A. Denote by $f^{-1}(\vec{\mathfrak{A}}) = \{f^{-1}(\mathfrak{A}_k)\}_{k \leq n}$.

Theorem

For every $X \subseteq \mathbb{N}$, the following are equivalent:

1. $d_e(X) \in \mathrm{CoRSp}_k(\vec{\mathfrak{A}})$.
2. $X \leq_e \mathcal{P}_k(f^{-1}(\vec{\mathfrak{A}}))$, *for every enumeration f of A.*

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The set $X \subseteq \mathbb{N}$ is *formally k-definable* on $\vec{\mathfrak{A}}$ if there exists a computable sequence $\{\Phi^{\gamma(x)}(W_1, \ldots, W_r)\}$ of Σ_k^+ formulae and elements $t_1, \ldots, t_r$ of A such that:
$x \in A \iff (\vec{\mathfrak{A}}) \models \Phi^{\gamma(x)}(W_1/t_1, \ldots, W_r/t_r)$.

- Σ_0^+ : $\exists \overline{Y}(\beta_1 \ \& \ \ldots \ \& \ \beta_k)$;
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The connection with the Joint Spectra

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However at the next levels we can have a difference: there are structures $\mathfrak{A}_0$ and $\mathfrak{A}_1$ s.t. $\mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1) \neq \mathrm{CoRSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$:

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Example: Let $\mathfrak{A}_0 = (\mathbb{N}, L, R)$, $L(\langle i, j\rangle, \langle i+1, j\rangle)$, $R(\langle i, j\rangle, \langle i, j+1\rangle)$.

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Let M be a set which is Σ^0_3, but not Σ^0_2. Fix an enumeration of the elements of M, $M = \{j_0, \dots, j_i, \dots\}$.

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Finally let $\mathfrak{A}_1 = (\mathbb{N}, P)$, where $P(\langle i, j_i\rangle) \iff j_i \in M$.

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Finally let $\mathfrak{A}_1 = (\mathbb{N}, P)$, where $P(\langle i, j_i\rangle) \iff j_i \in M$.

- $d_e(M) \notin \mathrm{CoJSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$.
- $d_e(M) \in \mathrm{CoRSp}_1(\mathfrak{A}_0, \mathfrak{A}_1)$, since if $t_0 = \langle 0, 0\rangle$,

$$\begin{aligned} j \in \quad & M \iff \exists Y_0 \dots \exists Y_i \exists Z_0 \dots \exists Z_j (Y_0 = t_0 \;\&\; L(Y_0, Y_1) \;\&\; \dots \;\&\\ & L(Y_{i-1}, Y_i) \& \; Y_i = Z_0 \;\&\; R(Z_0, Z_1) \;\&\; \dots \;\&\; R(Z_{j-1}, Z_j) \;\&\; P(Z_j)). \end{aligned}$$

Minimal Pair Theorem

Theorem

For any finite sequence of structures $\vec{\mathfrak{A}}$, there exist Turing degrees $\mathbf{f}$ *and* $\mathbf{g}$ *in* $\mathrm{RSp}(\vec{\mathfrak{A}})$, *such that for any enumeration degree* $\mathbf{a}$ *and each* $k \leq n$*:*

$$\mathbf{a} \leq \mathbf{f}^{(\mathbf{k})} \ \& \ \mathbf{a} \leq \mathbf{g}^{(\mathbf{k})} \Rightarrow \mathbf{a} \in \mathrm{CoRSp}_k(\vec{\mathfrak{A}}).$$

Quasi-Minimal Degree

Definition (Soskov)

An enumeration degree $\mathbf{q_0}$ is *quasi-minimal with respect to* $\mathrm{Sp}(\mathfrak{A})$ if

- $\mathbf{q_0} \notin \mathrm{CoSp}(\mathfrak{A})$;
- for any Turing degree $\mathbf{a}$: $\iota(\mathbf{a}) \geq \mathbf{q_0} \Rightarrow \mathbf{a} \in \mathrm{Sp}(\mathfrak{A})$ and $\iota(\mathbf{a}) \leq \mathbf{q_0} \Rightarrow \iota(\mathbf{a}) \in \mathrm{CoSp}(\mathfrak{A})$.

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Theorem

For any finite sequence of structures $\vec{\mathfrak{A}}$ there exists an enumeration degree $\mathbf{q}$ *such that:*

1. $\mathbf{q} \notin \mathrm{CoRSp}(\tilde{\mathfrak{A}})$;
2. *If* $\mathbf{a}$ *is a Turing degree and* $\iota(\mathbf{a}) \geq \mathbf{q}$, *then* $\mathbf{a} \in \mathrm{RSp}(\vec{\mathfrak{A}})$;
3. *If* $\mathbf{a}$ *is a Turing degree and* $\iota(\mathbf{a}) \leq \mathbf{q}$, *then* $\iota(\mathbf{a}) \in \mathrm{CoRSp}(\vec{\mathfrak{A}})$.

ω - enumeration reducibility

Let $\mathcal{X} = \{X_n\}_{n<\omega}$ and $\mathcal{Y} = \{Y_n\}_{n<\omega}$ be some sequences of sets of natural numbers.

Definition

The *jump sequence* $\mathcal{P}(\mathcal{X}) = \{\mathcal{P}_n(\mathcal{X})\}_{n<\omega}$ of $\mathcal{X}$ is defined by induction:

(i) $\mathcal{P}_0(\mathcal{X}) = X_0$;

(ii) $\mathcal{P}_{n+1}(\mathcal{X}) = \mathcal{P}_n(\mathcal{X})' \oplus X_{n+1}$.

Definition

We say that $\mathcal{X} \leq_\omega \mathcal{Y} \iff (\forall n)(X_n \leq_e \mathcal{P}_n(\mathcal{Y})$ uniformly in n).
$\mathcal{X} \equiv_\omega \mathcal{Y} \iff \mathcal{X} \leq_\omega \mathcal{Y} \,\&\, \mathcal{Y} \leq_\omega \mathcal{X}$.

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Theorem (Selman)

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Definition

A sequence of sets $\mathcal{X} = \{X_n\}_{n<\omega}$ is *c.e. in* a set $Z \subseteq \mathbb{N}$ if for every n, X_n is c.e. in $Z^{(n)}$ uniformly in n.

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Theorem (Soskov)

$\mathcal{X} \leq_\omega \mathcal{Y}$ if and only if for every set of natural numbers Z, if $\mathcal{Y}$ is c.e. in Z then $\mathcal{X}$ is c.e. in Z.

Degree structures

- The ω-enumeration degree of a sequence $\mathcal{X}$ is $d_\omega(\mathcal{X}) = \{\mathcal{Y} = \{Y_n\}_{n<\omega} \mid \mathcal{X} \equiv_\omega \mathcal{Y}\}$

  The structure of the ω-enumeration degrees $\mathcal{D}_\omega$ is an upper semi-lattice with jump operation.

  The enumeration degrees are embedded in to the ω-enumeration degrees by: $\kappa(d_e(X)) = d_\omega(\{X^{(n)}\}_{n<\omega})$.

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- There are sets X which are not enumeration equivalent to any set of the form Y^+.

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- There are sequences $\mathcal{R} = \{R_n\}_{n<\omega}$ such that:
  - $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n.
  - $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$.

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  Sequences with this property are called *almost zero*.

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  Sequences with this property are called *almost zero*.

  To make $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$ it is sufficient to ensure $\mathcal{R} \neq \{W_e^{[n]}(\emptyset^{(n)})\}_{n<\omega}$, where $W_e^{[n]}$ is the n-th column of W_e.

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- There are sets X which are not enumeration equivalent to any set of the form Y^+.
- There are sequences $\mathcal{R} = \{R_n\}_{n<\omega}$ such that:
  - $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n.
  - $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$.

  Sequences with this property are called *almost zero*.

  To make $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$ it is sufficient to ensure $\mathcal{R} \neq \{W_e^{[n]}(\emptyset^{(n)})\}_{n<\omega}$, where $W_e^{[n]}$ is the n-th column of W_e.

$$R_n = \begin{cases} \{1\}, & \text{if } 0 \in W_n^{[n]}(\emptyset^{(n)}); \\ \{0\}, & \text{otherwise.} \end{cases} .$$

Spectra of sequences of structures

More generally let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of countable structures.

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Definition

The Joint spectrum of $\vec{\mathfrak{A}}$ is

$$\mathrm{JSp}(\vec{\mathfrak{A}}) = \{d_T(B) \mid \begin{array}{l}(\exists \{f_n\}_{n<\omega} \text{ enumerations of } \vec{\mathfrak{A}})\\ (\forall n)(f_n^{-1}(\mathfrak{A}_n) \text{ is c.e. in } B^{(n)} \text{ uniformly in } n)\},\end{array}$$

where $f_n^{-1}(\mathfrak{A}_n) = f_n^{-1}(A_n) \oplus f_n^{-1}(R_1^n) \oplus \cdots \oplus f_n^{-1}(R_{m_n}^n)$.

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If $\vec{\mathfrak{A}}$ and $\vec{\mathfrak{A}}^*$ are such that for every n $\mathfrak{A}_n \cong \mathfrak{A}_n^*$ then $\mathrm{JSp}(\vec{\mathfrak{A}}) = \mathrm{JSp}(\vec{\mathfrak{A}}^*)$.

Spectra of sequences of structures

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$$\mathrm{OCoSp}(\vec{\mathfrak{A}}) = \left\{ \mathbf{a} \in \mathcal{D}_\omega \mid \forall \mathbf{x} \in \mathrm{RSp}(\vec{\mathfrak{A}})(\mathbf{a} \leq_\omega \mathbf{x}) \right\}.$$

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For any enumeration f of A denote by $f^{-1}(\vec{\mathfrak{A}}) = \{f^{-1}(\mathfrak{A}_n)\}_{n<\omega}$.

Proposition

For every sequence of sets of natural numbers $\mathcal{X} = \{X_n\}_{n<\omega}$:
$d_\omega(\mathcal{X}) \in \mathrm{OCoSp}(\vec{\mathfrak{A}})$ *iff* $\mathcal{X} \leq_\omega \{\mathcal{P}_k(f^{-1}(\vec{\mathfrak{A}}))\}_{k<\omega}$, *for every enumeration f of A.*

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$d_\omega(\mathcal{X}) \in \mathrm{OCoSp}(\vec{\mathfrak{A}})$ *iff there exists a computable sequence* $\{\Phi^{\gamma(n,x)}(W_1, \ldots, W_r)\}$ *of* Σ^+_n *formulae and elements* $t_1, \ldots, t_r$ *of* A *s.t.:*
$x \in X_n \iff (\vec{\mathfrak{A}}) \models \Phi^{\gamma(n,x)}(W_1/t_1, \ldots, W_r/t_r)$.

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Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$, and $A = \bigcup_n A_n$. Let $R \subseteq A^m$.

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The n-th Marker's extension $\mathfrak{M}_n(R)$ of R

Let $X_0, X_1, \dots X_n$ be infinite disjoint countable - companions to $\mathfrak{M}_n(R)$.
Fix bijections: $h_0 : R \to X_0$
$h_1 : (A^m \times X_0) \setminus G_{h_0} \to X_1 \dots$
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Let $M_n = G_{h_n}$ and $\mathfrak{M}_n(R) = (A \cup X_0 \cup \dots \cup X_n; X_0, X_1, \dots X_n, M_n)$.

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Two steps (Soskov)

Lemma

For every enumeration f of $\mathfrak{M}(\vec{\mathfrak{A}})$ there is an enumeration g of $\vec{\mathfrak{A}}$:

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Theorem

Let g be an enumeration of $\vec{\mathfrak{A}}$ *and* $\mathcal{Y} \not\leq_\omega g^{-1}(\vec{\mathfrak{A}})$*. There is an enumeration f of* $\mathfrak{M}(\vec{\mathfrak{A}})$*:*

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2. $\mathcal{Y}$ *is not c.e. in* $f^{-1}(\mathfrak{M}(\vec{\mathfrak{A}}))$.

Co-spectra of Marker's extensions

Theorem (Soskov)

Fix $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *and let* $\mathfrak{M} = \mathfrak{M}(\vec{\mathfrak{A}})$.

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The least element of $\mathrm{Sp}_n(\mathfrak{M})$ if it exists is the n-th jump degree of $\mathfrak{M}$. The greatest element of $\mathrm{CoSp}_n(\mathfrak{M})$ if it exists is the n-th co-degree of $\mathfrak{M}$.

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- $\mathrm{CoSp}_n(\mathfrak{M}) = \{d_e(Y) \mid Y \leq_e \mathcal{P}_n(\mathcal{R})\}$.

Consider the *almost zero* sequence $\mathcal{R}$:

1. $\mathcal{P}_n(\mathcal{R}) \equiv_e \emptyset^{(n)}$ for every n. Hence the n-th co-degree of $\mathfrak{M}$ is $\mathbf{0}_e^{(n)}$.
2. $\mathcal{R} \not\leq_\omega \{\emptyset^{(n)}\}_{n<\omega}$. Hence $\mathfrak{M}$ has no n-th jump degree for any n.

Generalized Goncharov and Khoussainov Lemma

Proposition

Let $n \geq 0$ and R be a $\Sigma^0_{n+1}(B)$ set with an infinite computable subset. Then there exists functions $k_0 \dots k_n$ such that the graph of k_n is computable in B, uniformly in an index for R and n and
$k_0 : R \to \mathbb{N}$.
$k_1 : \mathbb{N}^2 \setminus G_{k_0} \to \mathbb{N} \dots$
$k_n : \mathbb{N}^{n+1} \setminus G_{k_{n-1}} \to \mathbb{N}$.

Lemma (Soskov, M. Soskova)

Let R be $\Sigma^0_2(X)$ and $S \subseteq R$ be infinite and computable. There exists a bijection $k : R \to \mathbb{N}$ such that $\mathbb{N}^2 \setminus G_k$ is $\Sigma^0_1(X)$ and has an infinite computable subset.

Main theorem

Denote by $\mathrm{Js}(\vec{\mathfrak{A}}) = \{\{f_n^{-1}(\mathfrak{A}_n)\}_{n<\omega} \mid f_n$ is an enumeration of $A_n\}$ and $\mathrm{Rs}(\vec{\mathfrak{A}}) = \{\{f^{-1}(\mathfrak{A}_n)\}_{n<\omega} \mid f$ is an enumeration of $A\}$.

Theorem (Soskov)

Let $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ be a sequence of structures.

1. *There exists a structure $\mathfrak{M}$ such that* $\mathrm{Sp}(\mathfrak{M}) = \left\{ d_T(B) \mid (\exists \mathcal{Y}) \in \mathrm{Rs}(\vec{\mathfrak{A}}))(\mathcal{Y} \text{ is c.e. in } B) \right\}$.
2. *There exists a structure $\mathfrak{M}$ such that* $\mathrm{Sp}(\mathfrak{M}) = \left\{ d_T(B) \mid (\exists \mathcal{Y} \in \mathrm{JSp}(\vec{\mathfrak{A}}))(\mathcal{Y} \text{ is c.e. in } B) \right\}$.

Corollary

For a finite sequence of structures $\vec{\mathfrak{A}} = \{\mathfrak{A}_k\}_{k<n}$:

1. *There exists a structure $\mathfrak{M}$ such that* $\mathrm{Sp}(\mathfrak{M}) = \mathrm{JSp}(\vec{\mathfrak{A}})$.
2. *There exists a structure $\mathfrak{M}$ such that* $\mathrm{Sp}(\mathfrak{M}) = \mathrm{RSp}(\vec{\mathfrak{A}})$.

Embedding the ω-enumeration degrees into the Muchnik degrees generated by spectra of structures

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- Recall that for every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.

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- Recall that for every enumeration g of $\vec{\mathfrak{A}}$, $\mathcal{R} \leq_\omega g^{-1}(\vec{\mathfrak{A}})$.
- By Main Theorem there is a structure $\mathfrak{M}_{\mathcal{R}}$ such that $\mathrm{Sp}(\mathfrak{M}_{\mathcal{R}}) = \left\{ d_T(B) \mid (\exists g)(g^{-1}(\vec{\mathfrak{A}}) \text{ is c.e. in } B) \right\}$.

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Consider again the structure $\vec{\mathfrak{A}}$ obtained from a sequence of sets $\mathcal{R}$. $\mathfrak{A}_0 = (\mathbb{N}; G_s, R_0)$ and for all $n \geq 1$, $\mathfrak{A}_n = (\mathbb{N}; R_n)$.

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- $\mathrm{Sp}(\mathfrak{M}_{\mathcal{R}}) = \{d_T(B) \mid \mathcal{R} \text{ is c.e. in } B\}$.

$\mathcal{R} \leq_\omega \mathcal{Q} \iff$
$\{d_T(B) \mid \mathcal{R} \text{ is c.e. in } B\} \supseteq \{d_T(B) \mid \mathcal{Q} \text{ is c.e. in } B\} \iff$
$\mathrm{Sp}(\mathfrak{M}_{\mathcal{R}}) \supseteq \mathrm{Sp}(\mathfrak{M}_{\mathcal{Q}})$.
Let $\mu(d_\omega(\mathcal{R})) = \mathrm{Sp}(\mathfrak{M}_{\mathcal{R}})$.

The main theorem for Turing degrees

Let $\mathrm{JS}(\vec{\mathfrak{A}}) = \{\{g_n^{-1}(\mathfrak{A}_n)\}_{n<\omega} \mid g_n$ is an enumeration of $A_n\}$.

Theorem

For every sequence $\vec{\mathfrak{A}} = \{\mathfrak{A}_n\}_{n<\omega}$ *there exists a structure* $\mathfrak{M}$ *such that* $\mathrm{Sp}(\mathfrak{M}) = \{d_T(B) \mid (\exists \{Y_n\}_{n<\omega} \in \mathrm{JS}(\vec{\mathfrak{A}}))(Y_n \leq_T B^{(n)}$ uniformly in $n)\}$.

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$\mathrm{Sp}(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_0)$, $\mathrm{Sp}_1(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_1), \ldots, \mathrm{Sp}_n(\mathfrak{M}) \subseteq \mathrm{Sp}(\mathfrak{A}_n) \ldots$

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Theorem (Soskov)

There is a structure $\mathfrak{M}$ *with* $\mathrm{Sp}(\mathfrak{M}) = \{\mathbf{b} \mid \forall n(\mathbf{b}^{(n)} > \mathbf{0}^{(n)})\}$.

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