Properties of the Joint Spectra of Sequence of Structures

Alexandra A. Soskova

Department of Mathematics and Computer Science
Sofia University

PLS5 2005

Enumeration of a structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.
- for any $A \subseteq \mathbb{N}^a$ let $f^{-1}(A) = \{\langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A\}$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$.

Relatively α-intrinsic sets

1989 Ash, Knight, Manasse, Slaman, Chisholm.

- Let α be a constructive ordinal and let $A \subseteq \mathbb{N}^a$. The set A is `relatively` α`-intrinsic on` $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(A) \leq_e f^{-1}(\mathfrak{A})^{(\alpha)}$.

Relatively α-intrinsic sets on $\mathfrak{A}$

2002 Soskov, Baleva.

- Let $\{B_\xi\}_{\xi\leq\zeta}$ be a sequence of subset of $\mathbb{N}$ and ζ be a constructive ordinal.
- Add the sets B_ξ to the structure $\mathfrak{A}$ as a partial predicates which is relatively ξ-intrinsic on $\mathfrak{A}$.
- Restrict the class of all enumerations of $\mathfrak{A}$ to the class of those enumerations f of $\mathfrak{A}$ for which $f^{-1}(B_\xi) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(\xi)}$.

Relatively α-intrinsic sets on $\mathfrak{A}$

Definition

A subset A of $\mathbb{N}^a$ is **relatively α-intrinsic on $\mathfrak{A}$ with respect to the sequence** $\{B_\xi\}_{\xi\leq\zeta}$
if for every enumeration f of $\mathfrak{A}$ such that
$(\forall\xi\leq\zeta)(f^{-1}(B_\xi)\leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(\xi)})$ uniformly in ξ,
the set $f^{-1}(A)\leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(\alpha)}$.

Degree spectra of structures

Definition

- **The Degree spectrum of** $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

- **The Co-spectrum of** $\mathfrak{A}$ is the set

$$\mathrm{CS}(\mathfrak{A}) = \{\mathrm{b} : (\forall \mathrm{a} \in \mathrm{DS}(\mathfrak{A}))(\mathrm{b} \leq \mathrm{a})\}.$$

Degree spectra of structures

Definition

- **The Degree spectrum of** $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

- **The Co-spectrum of** $\mathfrak{A}$ is the set

$$\mathrm{CS}(\mathfrak{A}) = \{\mathrm{b} : (\forall \mathrm{a} \in \mathrm{DS}(\mathfrak{A}))(\mathrm{b} \leq \mathrm{a})\}.$$

Joint Spectra of Structures

Let ζ be a recursive ordinal and let $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ be a sequence of abstract structures over the natural numbers.

Definition

- **The Joint Spectrum of the sequence** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the set

$$\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{a} : (\forall\xi \leq \zeta)(\mathrm{a}^{(\xi)} \in \mathrm{DS}(\mathfrak{A}_\xi))\}.$$

- **The αth Jump Spectrum of** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the set

$$\mathrm{DS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{a}^{(\alpha)} : \mathrm{a} \in \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta})\}.$$

Joint Spectra of Structures

Let ζ be a recursive ordinal and let $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ be a sequence of abstract structures over the natural numbers.

Definition

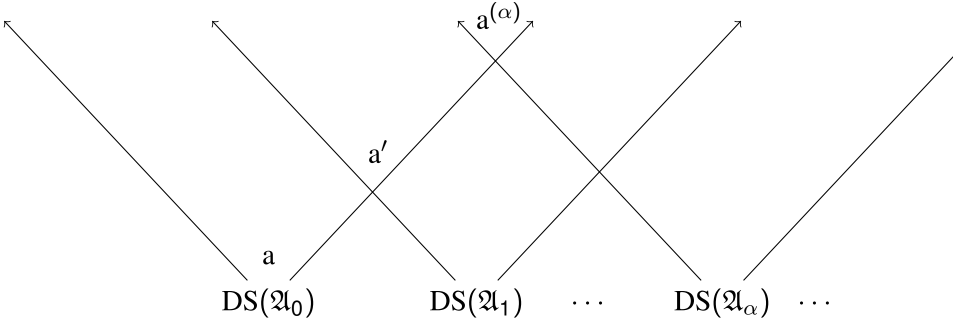
- **The Joint Spectrum of the sequence** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the set

$$\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{a} : (\forall\xi\leq\zeta)(\mathrm{a}^{(\xi)}\in\mathrm{DS}(\mathfrak{A}_\xi))\}.$$

- **The** α**th Jump Spectrum of** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the set

$$\mathrm{DS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{a}^{(\alpha)} : \mathrm{a}\in\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta})\}.$$

Joint Spectra of Structures



Cospectra of Structures

Definition

- **The Co-spectrum of** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the Co-set of $\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta})$, i.e.

$$\mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{b} : (\forall \mathrm{a} \in \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}))(\mathrm{b} \leq \mathrm{a})\}.$$

Definition

- **The** α**th Co-spectrum of** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is:

$$\mathrm{CS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{b} : (\forall \mathrm{a} \in \mathrm{DS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}))(\mathrm{b} \leq \mathrm{a})\}.$$

Cospectra of Structures

Definition

- **The Co-spectrum of** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the Co-set of $\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta})$, i.e.

$$\mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{b} : (\forall \mathrm{a} \in \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}))(\mathrm{b} \leq \mathrm{a})\}.$$

Definition

- **The** α**th Co-spectrum of** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is:

$$\mathrm{CS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{b} : (\forall \mathrm{a} \in \mathrm{DS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}))(\mathrm{b} \leq \mathrm{a})\}.$$

Cospectra of Structures

Definition

- **The Co-spectrum of** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the Co-set of $\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta})$, i.e.

$$\mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{b} : (\forall \mathrm{a} \in \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}))(\mathrm{b} \leq \mathrm{a})\}.$$

Definition

- **The αth Co-spectrum of** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is:

$$\mathrm{CS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{b} : (\forall \mathrm{a} \in \mathrm{DS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}))(\mathrm{b} \leq \mathrm{a})\}.$$

The jump set

Let f_ξ be an enumeration of $\mathfrak{A}_\xi$ and $f = \{f_\xi\}_{\xi \leq \zeta}$.

Definition

The jump set $\mathcal{P}^f_\alpha$ of the sequence $\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}$:

(i) $\mathcal{P}^f_0 = f_0^{-1}(\mathfrak{A}_0)$.

(ii) Let $\alpha = \beta + 1$. Then let $\mathcal{P}^f_\alpha = (\mathcal{P}^f_\beta)' \oplus f_\alpha^{-1}(\mathfrak{A}_\alpha)$.

(iii) Let $\alpha = \lim \alpha(p)$. Then set $\mathcal{P}^f_{<\alpha} = \{\langle p, x\rangle : x \in \mathcal{P}^f_{\alpha(p)}\}$ and let $\mathcal{P}^f_\alpha = \mathcal{P}^f_{<\alpha} \oplus f_\alpha^{-1}(\mathfrak{A}_\alpha)$.

Proposition

$$\mathrm{d_e}(A) \in \mathrm{CS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}) \iff$$
$$(\textit{ for every enumeration } f = \{f_\xi\}_{\xi \leq \zeta})(A \leq_\mathrm{e} \mathcal{P}^f_\alpha) .$$

The jump set

Let f_ξ be an enumeration of $\mathfrak{A}_\xi$ and $f = \{f_\xi\}_{\xi \le \zeta}$.

Definition

The jump set $\mathcal{P}^f_\alpha$ of the sequence $\{\mathfrak{A}_\xi\}_{\xi \le \zeta}$:

(i) $\mathcal{P}^f_0 = f_0^{-1}(\mathfrak{A}_0)$.

(ii) Let $\alpha = \beta + 1$. Then let $\mathcal{P}^f_\alpha = (\mathcal{P}^f_\beta)' \oplus f_\alpha^{-1}(\mathfrak{A}_\alpha)$.

(iii) Let $\alpha = \lim \alpha(p)$. Then set $\mathcal{P}^f_{<\alpha} = \{\langle p, x\rangle : x \in \mathcal{P}^f_{\alpha(p)}\}$ and let $\mathcal{P}^f_\alpha = \mathcal{P}^f_{<\alpha} \oplus f_\alpha^{-1}(\mathfrak{A}_\alpha)$.

Proposition

$$\mathrm{d_e}(A) \in \mathrm{CS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi \le \zeta}) \iff$$
$$(\text{ for every enumeration } f = \{f_\xi\}_{\xi \le \zeta})(A \le_{\mathrm{e}} \mathcal{P}^f_\alpha)\ .$$

Modelling

(i) $f \models_0 F_e(x)$ iff $(\exists v)(\langle v, x\rangle \in W_e \;\&\; D_v \subseteq f_0^{-1}(\mathfrak{A}_0))$;

(ii) $\alpha = \beta + 1$.

$$f \models_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)((u = \langle 0, e_u, x_u\rangle \;\&\; f \models_\beta F_{e_u}(x_u)) \vee (u = \langle 1, e_u, x_u\rangle \;\&\; f \models_\beta \neg F_{e_u}(x_u)) \vee (u = \langle 2, x_u\rangle \;\&\; x_u \in f_\alpha^{-1}(\mathfrak{A}_\alpha))));$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$f \models_\alpha F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)((u = \langle 0, p_u, e_u, x_u\rangle \;\&\; f \models_{\alpha(p_u)} F_{e_u}(x_u)) \vee (u = \langle 2, x_u\rangle \;\&\; x_u \in f_\alpha^{-1}(\mathfrak{A}_\alpha))));$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

Modelling

(i) $f \models_0 F_e(x)$ iff $(\exists v)(\langle v, x\rangle \in W_e \;\&\; D_v \subseteq f_0^{-1}(\mathfrak{A}_0))$;

(ii) $\alpha = \beta + 1$.

$$\begin{aligned} f \models_\alpha F_e(x) \iff \quad & (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\\ & (u = \langle 0, e_u, x_u\rangle \;\&\; f \models_\beta F_{e_u}(x_u)) \vee \\ & (u = \langle 1, e_u, x_u\rangle \;\&\; f \models_\beta \neg F_{e_u}(x_u)) \vee \\ & (u = \langle 2, x_u\rangle \;\&\; x_u \in f_\alpha^{-1}(\mathfrak{A}_\alpha)))); \end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned} f \models_\alpha F_e(x) \iff \quad & (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\\ & (u = \langle 0, p_u, e_u, x_u\rangle \;\&\; f \models_{\alpha(p_u)} F_{e_u}(x_u)) \vee \\ & (u = \langle 2, x_u\rangle \;\&\; x_u \in f_\alpha^{-1}(\mathfrak{A}_\alpha)))); \end{aligned}$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

Modelling

(i) $f \models_0 F_e(x)$ iff $(\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq f_0^{-1}(\mathfrak{A}_0))$;

(ii) $\alpha = \beta + 1$.

$$\begin{aligned} f \models_\alpha F_e(x) \iff \quad & (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\ & (u = \langle 0, e_u, x_u\rangle \ \&\ f \models_\beta F_{e_u}(x_u)) \vee \\ & (u = \langle 1, e_u, x_u\rangle \ \&\ f \models_\beta \neg F_{e_u}(x_u)) \vee \\ & (u = \langle 2, x_u\rangle \ \&\ x_u \in f_\alpha^{-1}(\mathfrak{A}_\alpha)))); \end{aligned}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\begin{aligned} f \models_\alpha F_e(x) \iff \quad & (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\ & (u = \langle 0, p_u, e_u, x_u\rangle \ \&\ f \models_{\alpha(p_u)} F_{e_u}(x_u)) \vee \\ & (u = \langle 2, x_u\rangle \ \&\ x_u \in f_\alpha^{-1}(\mathfrak{A}_\alpha)))); \end{aligned}$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

Modelling

(i) $f \models_0 F_e(x)$ iff $(\exists v)(\langle v, x\rangle \in W_e \;\&\; D_v \subseteq f_0^{-1}(\mathfrak{A}_0))$;

(ii) $\alpha = \beta + 1$.

$$
\begin{aligned}
f \models_\alpha F_e(x) \iff \quad & (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\\
& (u = \langle 0, e_u, x_u\rangle \;\&\; f \models_\beta F_{e_u}(x_u)) \vee \\
& (u = \langle 1, e_u, x_u\rangle \;\&\; f \models_\beta \neg F_{e_u}(x_u)) \vee \\
& (u = \langle 2, x_u\rangle \;\&\; x_u \in f_\alpha^{-1}(\mathfrak{A}_\alpha))));
\end{aligned}
$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$
\begin{aligned}
f \models_\alpha F_e(x) \iff \quad & (\exists v)(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(\\
& (u = \langle 0, p_u, e_u, x_u\rangle \;\&\; f \models_{\alpha(p_u)} F_{e_u}(x_u)) \vee \\
& (u = \langle 2, x_u\rangle \;\&\; x_u \in f_\alpha^{-1}(\mathfrak{A}_\alpha))));
\end{aligned}
$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

Forcing

Proposition

$$A \leq_{\mathrm{e}} \mathcal{P}^f_\alpha \iff (\exists e)(A = \{x : f \models_\alpha F_e(x)\}).$$

- The forcing conditions `finite parts` are sequences τ of finite mappings $\tau_\xi, \xi \leq \zeta$ from $\mathbb{N}$ to $\mathbb{N}$, so that $\bigcup_{\xi \leq \zeta} dom(\tau_\xi)$ is finite.
- If τ and ρ are finite parts, then $\tau \subseteq \rho$ if for each $\xi \leq \zeta$ $(\tau_\xi \subseteq \rho_\xi)$.

Forcing

Proposition

$$A \leq_{\mathrm{e}} \mathcal{P}^{f}_{\alpha} \iff (\exists e)(A = \{x : f \models_{\alpha} F_e(x)\}).$$

- The forcing conditions `finite parts` are sequences τ of finite mappings $\tau_\xi, \xi \leq \zeta$ from $\mathbb{N}$ to $\mathbb{N}$, so that $\bigcup_{\xi \leq \zeta} dom(\tau_\xi)$ is finite.
- If τ and ρ are finite parts, then $\tau \subseteq \rho$ if for each $\xi \leq \zeta$ $(\tau_\xi \subseteq \rho_\xi)$.

Forcing

(i) $\tau \Vdash_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq \tau_0^{-1}(\mathfrak{A}_0));$

(ii) $\alpha = \beta + 1.$

$$\tau \Vdash_\alpha F_e(x) \iff \begin{array}{l} (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\ (u = \langle 0, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta F_{e_u}(x_u)) \vee \\ (u = \langle 1, e_u, x_u\rangle \ \&\ \tau \Vdash_\beta \neg F_{e_u}(x_u)) \vee \\ (u = \langle 2, x_u\rangle \ \&\ x_u \in \tau_\alpha^{-1}(\mathfrak{A}_\alpha)))); \end{array}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then

$$\tau \Vdash_\alpha F_e(x) \iff \begin{array}{l} (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\ (u = \langle 0, p_u, e_u, x_u\rangle \ \&\ \tau \Vdash_{\alpha(p_u)} F_{e_u}(x_u)) \vee \\ (u = \langle 2, x_u\rangle \ \&\ x_u \in \tau_\alpha^{-1}(\mathfrak{A}_\alpha)))); \end{array}$$

(iv) $\tau \Vdash_\alpha \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash_\alpha F_e(x)).$

Forcing α-definable sets

The set $A \subseteq \mathbb{N}$ is `forcing` α`- definable` on $\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}$ if there exist a finite part δ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \tau \supseteq \delta)(\tau \Vdash_\alpha F_e(x)).$$

Theorem

- $A \leq_e \mathcal{P}^f_\alpha$ *for all f - enumerations of* $\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}$, *if and only if*
- *A is forcing* α*-definable on* $\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}$, *if and only if*
- $\mathrm{d_e}(A) \in \mathrm{CS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.

Forcing α-definable sets

The set $A \subseteq \mathbb{N}$ is `forcing` α- `definable` on $\{\mathfrak{A}_\xi\}_{\xi \le \zeta}$ if there exist a finite part δ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \tau \supseteq \delta)(\tau \Vdash_\alpha F_e(x)).$$

Theorem

- *$A \le_e \mathcal{P}^f_\alpha$ for all f - enumerations of $\{\mathfrak{A}_\xi\}_{\xi \le \zeta}$, if and only if*
- *A is forcing α-definable on $\{\mathfrak{A}_\xi\}_{\xi \le \zeta}$, if and only if*
- $\mathrm{d_e}(A) \in \mathrm{CS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi \le \zeta})$.

α-generic enumerations

Definition

An enumeration f of $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is α- `generic` if for every $\beta < \alpha$, $e, x \in \mathbb{N}$

$$(\forall\tau \subseteq f)(\exists\rho \supseteq \tau)(\rho \Vdash_\beta F_e(x)) \Longrightarrow (\exists\tau \subseteq f)(\tau \Vdash_\beta F_e(x))$$

Lemma

If f is an $(\alpha+1)$-generic enumeration, $\alpha < \zeta$, then

$$f \models_\alpha (\neg)F_e(x) \iff (\exists\tau \subseteq f)(\tau \Vdash_\alpha (\neg)F_e(x)) .$$

α-generic enumerations

Definition

An enumeration f of $\{\mathfrak{A}_\xi\}_{\xi<\zeta}$ is α- `generic` if for every $\beta<\alpha$, $e,x\in\mathbb{N}$

$$(\forall\tau\subseteq f)(\exists\rho\supseteq\tau)(\rho\Vdash_\beta F_e(x))\implies (\exists\tau\subseteq f)(\tau\Vdash_\beta F_e(x))$$

Lemma

If f is an $(\alpha+1)$-generic enumeration, $\alpha<\zeta$, then

$$f\models_\alpha(\neg)F_e(x)\iff(\exists\tau\subseteq f)(\tau\Vdash_\alpha(\neg)F_e(x))\ .$$

Minimal Pair Theorem

Theorem (Soskov)

There exist elements f *and* g *of DS*$(\mathfrak{A})$ *such that for any enumeration degree* a *and any* $\alpha < \zeta$

$$\mathrm{a} \leq \mathrm{f}^{(\alpha)} \;\&\; \mathrm{a} \leq \mathrm{g}^{(\alpha)} \Rightarrow \mathrm{a} \in \mathrm{CS}^{\alpha}(\mathfrak{A}).$$

Theorem

There exist elements f *and* g *of* $\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$*, such that for any enumeration degree* a *and* $\alpha < \zeta$*:*

$$\mathrm{a} \leq \mathrm{f}^{(\alpha)} \;\&\; \mathrm{a} \leq \mathrm{g}^{(\alpha)} \Rightarrow \mathrm{a} \in \mathrm{CS}^{\alpha}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}).$$

Minimal Pair Theorem

Theorem (Soskov)

There exist elements f *and* g *of DS*$(\mathfrak{A})$ *such that for any enumeration degree* a *and any* $\alpha < \zeta$

$$\mathrm{a} \leq \mathrm{f}^{(\alpha)} \ \& \ \mathrm{a} \leq \mathrm{g}^{(\alpha)} \Rightarrow \mathrm{a} \in \mathrm{CS}^{\alpha}(\mathfrak{A}).$$

Theorem

There exist elements f *and* g *of* $\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$, *such that for any enumeration degree* a *and* $\alpha < \zeta$:

$$\mathrm{a} \leq \mathrm{f}^{(\alpha)} \ \& \ \mathrm{a} \leq \mathrm{g}^{(\alpha)} \Rightarrow \mathrm{a} \in \mathrm{CS}^{\alpha}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}).$$

Minimal pair theorem

Proof.

- $g = \{g_\xi\}_{\xi \leq \zeta}$-arbitrary enumeration.
- There is a total set F, $g_\xi^{-1}(\mathfrak{A}_\xi) \leq_e F^{(\xi)}$.
- $h = \{h_\xi\}_{\xi \leq \zeta}$, $h_\xi^{-1}(\mathfrak{A}_\xi) \equiv_e F^{(\xi)}$.
- $(\alpha + 1)$-generic enumeration f:

$$A \leq_e \mathcal{P}_\alpha^f \ \&\ A \leq_e \mathcal{P}_\alpha^h \Rightarrow A \text{ is forcing } \alpha\text{-definable.}$$

- There is a total set G, $f_\alpha^{-1}(\mathfrak{A}_\alpha) \leq_e G^{(\alpha)}$ and $G^{(\alpha)}$ omits any $A \leq_e \mathcal{P}_\alpha^h$ not forcing α-definable.
- If $X \leq_e F^{(\alpha)}$ and $X \leq_e G^{(\alpha)}$ and X is a total set then $d_e(X) \in \mathrm{CS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.
- Set $d_e(F) = \mathrm{f}$ and $d_e(G) = \mathrm{g}$.

Quasi-Minimal Degree

Definition (Soskov)

An enumeration degree q_0 is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A}_0)$ if

- $\mathrm{q}_0 \not\in \mathrm{CS}(\mathfrak{A}_0)$
- for every total degree a: if $\mathrm{a} \geq \mathrm{q}_0$, then $\mathrm{a} \in \mathrm{DS}(\mathfrak{A}_0)$
- if $\mathrm{a} \leq \mathrm{q}_0$, then $\mathrm{a} \in \mathrm{CS}(\mathfrak{A}_0)$.

Theorem

There exists an enumeration degree q *such that:*

1. $\mathrm{q}^{(\alpha)} \in \mathrm{DS}(\mathfrak{A}_\alpha), \alpha < \zeta$, $\mathrm{q} \not\in \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi<\zeta})$;
2. *If* a *is a total degree and* $\mathrm{a} \geq \mathrm{q}$*, then* $\mathrm{a} \in \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi<\zeta})$;
3. *If* a *is a total degree and* $\mathrm{a} \leq \mathrm{q}$*, then* $\mathrm{a} \in \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi<\zeta})$.

Quasi-Minimal Degree

Definition (Soskov)

An enumeration degree q_0 is $\mathrm{q}$uasi-minimal with respect to $\mathrm{DS}(\mathfrak{A}_0)$ if

- $\mathrm{q}_0 \not\in \mathrm{CS}(\mathfrak{A}_0)$
- for every total degree a: if $\mathrm{a} \geq \mathrm{q}_0$, then $\mathrm{a} \in \mathrm{DS}(\mathfrak{A}_0)$
- if $\mathrm{a} \leq \mathrm{q}_0$, then $\mathrm{a} \in \mathrm{CS}(\mathfrak{A}_0)$.

Theorem

There exists an enumeration degree q *such that:*

1. $\mathrm{q}^{(\alpha)} \in \mathrm{DS}(\mathfrak{A}_\alpha), \alpha < \zeta$, $\mathrm{q} \not\in \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$;
2. *If* a *is a total degree and* $\mathrm{a} \geq \mathrm{q}$, *then* $\mathrm{a} \in \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$;
3. *If* a *is a total degree and* $\mathrm{a} \leq \mathrm{q}$, *then* $\mathrm{a} \in \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.

Quasi-Minimal Degree

Proof.

- Let q_0 be a quasi-minimal degree q_0 with respect to $\mathrm{DS}(\mathfrak{A}_0)$ [Soskov].
- Let $B_0 \subseteq \mathbb{N}$, $d_\mathrm{e}(B_0) = \mathrm{q}_0$, and $\{f_\xi\}_{\xi \leq \zeta}$ be fixed total enumerations of $\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}$.
- There is quasi-minimal over B_0 set F, such that
  - $B_0 <_\mathrm{e} F$,
  - $f_\alpha^{-1}(\mathfrak{A}_\alpha) \leq_\mathrm{e} F^{(\alpha)}$, $\alpha < \zeta$
  - if $A \leq_\mathrm{e} F$, then $A \leq_\mathrm{e} B_0$, for any total set A.
- Then $\mathrm{q} = d_\mathrm{e}(F)$ is quasi-minimal with respect to $\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.

□

Quasi-Minimal Degree

continued.

- Since q_0 is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A}_0)$, $\mathrm{q}_0 \notin \mathrm{CS}(\mathfrak{A}_0)$.
- But $\mathrm{q}_0 < \mathrm{q}$ and thus $\mathrm{q} \notin \mathrm{CS}(\mathfrak{A}_0)$. Hence $\mathrm{q} \notin \mathrm{CS}\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.
- $\mathrm{q}^{(\alpha)} \in \mathrm{DS}(\mathfrak{A}_\alpha)$.
- X — total, $X \geq_e F$. Then $d_e(X) \geq \mathrm{q}_0$. But q_0 is quasi-minimal, thus $d_e(X) \in \mathrm{DS}(\mathfrak{A}_0)$. Since $X^{(\alpha)} \geq_e F^{(\alpha)} \geq_e f_\alpha^{-1}(\mathfrak{A}_\alpha)$, and $X^{(\alpha)}$ is a total, $d_e(X^{(\alpha)}) \in \mathrm{DS}(\mathfrak{A}_\alpha)$, and hence $d_e(X) \in \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.
- X — total, $X \leq_e F$. Then, $X \leq_e B_0$. From the quasi-minimality of q_0, $d_e(X) \in \mathrm{CS}(\mathfrak{A}_0) = \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.
- The existence of the set F — quasi-minimal over B_0, uses the technique of partial regular enumerations.

Properties of joint spectra of sequence of structures

- The Minimal pair theorem.
- The Quasi-minimal degree.

- Questions:
  - Another specific properties of Joint spectra of structures?
  - Do there exist a structure $\mathfrak{A}$ such that $\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$?

C. J. Ash, J. F. Knight, M. Manasse, and T. Slaman,
Generic copies of countable structures.
Ann. Pure Appl. Logic 42 : 195–205 1989.

J. Chisholm,
Effective model theory vs. recursive model theory
J. Symbolic Logic, 55:1168–1191 1990.

I. N. Soskov and V. Baleva
Ash's Theorem for abstract structures.
Proceedings of Logic Colloquium'2002, 2002.

I. N. Soskov,
Degree spectra and co-spectra of structures.
Ann. Univ. Sofia, 96:45–68, 2003.