A Jump Inversion Theorem for the Degree Spectra

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Outline

- Degree spectra and jump spectra
- Every jump spectrum is spectrum
- Marker's extensions
- Jump inversion theorem for the degree spectra
- Some applications
- Joint spectra of structures
- Relative spectra of structures

Enumeration of a Structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k, =)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.
- For each predicate R of $\mathfrak{A}$:

$$f^{-1}(R) = \begin{array}{l} \{\langle x_1, \dots, x_r, 0\rangle \mid R(f(x_1), \dots, f(x_r)\} \cup \\ \{\langle x_1, \dots, x_r, 1\rangle \mid \neg R(f(x_1), \dots, f(x_r))\}. \end{array}$$

- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=)$.

Degree Spectra

Definition

The degree spectrum of $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- L. Richter [1981], J. Knight [1986].
- Let ι be the Roger's embedding of the Turing degrees into the enumeration degrees and $\mathfrak{A}$ is a total structure. Then

$$\mathrm{DS}(\mathfrak{A}) = \{\iota(d_T(f^{-1}(\mathfrak{A}))) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- The degree spectra are upwards closed with respect to the total degrees:

$$\mathbf{a} \in \mathrm{DS}(\mathfrak{A}), \mathbf{b} \text{ is total and } \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathrm{DS}(\mathfrak{A}).$$

- The jump spectrum of $\mathfrak{A}$ is the set $\mathrm{DS}_1(\mathfrak{A}) = \{\mathbf{a}' \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A})\}$.

For any countable structures $\mathfrak{A}$ and $\mathfrak{B}$ define the relation

$$\mathfrak{B} \preceq \mathfrak{A} \iff \mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}(\mathfrak{B}) \; .$$

- $\mathfrak{A} \equiv \mathfrak{B}$ if $\mathfrak{A} \preceq \mathfrak{B}$ and $\mathfrak{B} \preceq \mathfrak{A}$.
- $\mathfrak{B}' \preceq \mathfrak{A}$ if $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{B})$.
- $\mathfrak{A} \preceq \mathfrak{B}'$ if $\mathrm{DS}_1(\mathfrak{B}) \subseteq \mathrm{DS}(\mathfrak{A})$.
- $\mathfrak{A} \equiv \mathfrak{B}'$ if $\mathfrak{A} \preceq \mathfrak{B}'$ and $\mathfrak{B}' \preceq \mathfrak{A}$.

Theorem (Soskov)

Each jump spectrum is degree spectrum of a structure, i.e. for every structure $\mathfrak{A}$ there exists a structure $\mathfrak{B}$ such that $\mathfrak{A}' \equiv \mathfrak{B}$.

Definition

Moschovakis' extension

- $\bar{0} \notin \mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$.
- A pairing function $\langle ., . \rangle$, $\mathrm{range}(\langle ., . \rangle) \cap \mathbb{N}_0 = \emptyset$.
- The least set $\mathbb{N}^* \supseteq \mathbb{N}_0$, closed under $\langle ., . \rangle$.
- Moschovakis' extension of $\mathfrak{A}$ is the structure $\mathfrak{A}^* = (\mathbb{N}^*, R_1, \ldots, R_n, =, \mathbb{N}_0, G_{\langle ., . \rangle})$.
- $\mathfrak{A} \equiv \mathfrak{A}^*$.
- A new predicate $K_{\mathfrak{A}}$ (analogue of Kleene's set).
- For $e, x \in \mathbb{N}$ and finite part τ, let $\tau \Vdash F_e(x) \iff x \in \Gamma_e(\tau^{-1}(\mathfrak{A}))$.
- $K_{\mathfrak{A}} = \{\langle \delta^*, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$.
- $\mathfrak{B} = (\mathfrak{A}^*, K_{\mathfrak{A}})$.
- $\mathrm{DS}_1(\mathfrak{A}) = \mathrm{DS}(\mathfrak{B})$.

Question (Inverting the jump)

Given a set of enumeration degrees $\mathcal{A}$ does there exist a structure $\mathfrak{C}$ such that $\mathrm{DS}_1(\mathfrak{C}) = \mathcal{A}$?

1. Each element of $\mathcal{A}$ should be a jump of a degree.
2. $\mathcal{A}$ should be upwards closed (since each jump spectrum is a spectrum and the spectrum is upwards closed).

Problem

Not any upwards closed set of enumeration degrees is a spectrum of a structure and hence a jump spectrum.

A subset $\mathcal{B}$ of $\mathcal{A}$ is called *base* of $\mathcal{A}$ if for every element $\mathbf{a}$ of $\mathcal{A}$ there exists an element $\mathbf{b} \in \mathcal{B}$ such that $\mathbf{b} \leq \mathbf{a}$.

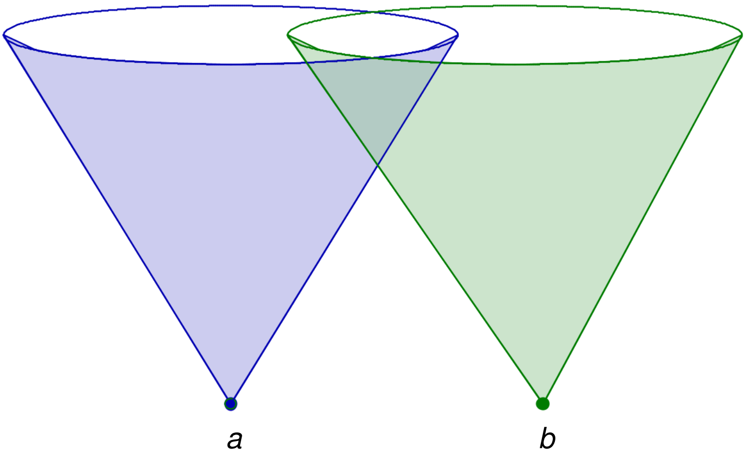
Proposition (Soskov)

If $\mathrm{DS}(\mathfrak{A})$ *has a countable base of total enumeration degrees, then* $\mathrm{DS}(\mathfrak{A})$ *has a least element.*

Example

Let $\mathbf{a}$ and $\mathbf{b}$ be incomparable enumeration degrees. Then there does not exist a structure $\mathfrak{A}$ such that:

$$\begin{aligned}\mathrm{DS}(\mathfrak{A}) = &\{\mathbf{c} : \mathbf{c} \text{ is total } \& \ \mathbf{c} \geq \mathbf{a}\} \cup \\ &\{\mathbf{c} : \mathbf{c} \text{ is total } \& \ \mathbf{c} \geq \mathbf{b}\}.\end{aligned}$$


a
b

- The set $\mathcal{A}$ should be a degree spectrum of a structure $\mathfrak{A}$.
- $\mathrm{DS}(\mathfrak{A})$ should contain only jumps of enumeration degrees.

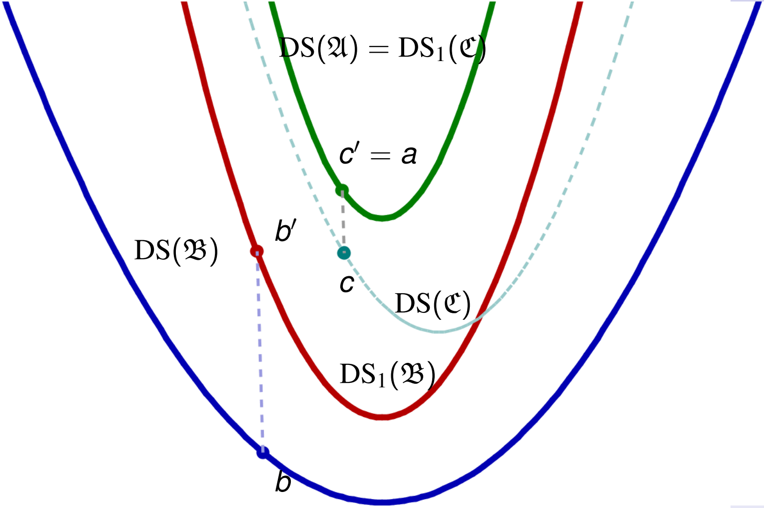
More generally:

Theorem (Jump Inversion Theorem)

If $\mathfrak{A}$ and $\mathfrak{B}$ are structures and $\mathfrak{B}' \preceq \mathfrak{A}$ then there exists a structure $\mathfrak{C}$ such that $\mathfrak{B} \preceq \mathfrak{C}$ and $\mathfrak{C}' \equiv \mathfrak{A}$.

- The structure $\mathfrak{C}$ we shall construct as a Marker's extension of $\mathfrak{A}$.
- We code the structure $\mathfrak{B}$ in $\mathfrak{C}$.
- In our construction we use also the relativized representation lemma for Σ^0_2 sets proved by Goncharov and Khoussainov

Theorem $\mathfrak{B}' \preceq \mathfrak{A} \Longrightarrow (\exists \mathfrak{C})(\mathfrak{B} \preceq \mathfrak{C} \ \&\ \mathfrak{C}' \equiv \mathfrak{A})$.



Marker's Extensions

Let $\mathfrak{A} = (A; R_1, \ldots, R_s, =)$.
$R^\exists$ — Marker's $\exists$-extension of R:

- $\exists$-fellow for R — $X = \{x_{\langle a_1,\ldots,a_r\rangle} \mid R(a_1, \ldots, a_r)\}$.
- $R^\exists(a_1, \ldots, a_r, x) \iff$
  $a_1, \ldots, a_r \in A \;\&\; x \in X \;\&\; x = x_{\langle a_1,\ldots,a_r\rangle}$.
- $\mathfrak{A}^\exists = (A \cup \bigcup_{i=1}^s X_i, R_1^\exists, \ldots, R_s^\exists, \bar{X}_1, \ldots, \bar{X}_s, =)$.

$R^\forall$ — Marker's $\forall$-extension of R:

- $\forall$-fellow for R — $Y = \{y_{\langle a_1,\ldots,a_r\rangle} \mid \neg R(a_1, \ldots, a_r)\}$.
-
  1. If $R^\forall(a_1, \ldots, a_r, y)$ then $a_1, \ldots, a_r \in A$ and $y \in Y$;
  2. If $a_1, \ldots, a_r \in A \;\&\; y \in Y$ then
     $\neg R^\forall(a_1, \ldots, a_r, y) \iff y = y_{\langle a_1,\ldots,a_r\rangle}$.
- $\mathfrak{A}^\forall = (A \cup \bigcup_{i=1}^s Y_i, R_1^\forall, \ldots, R_s^\forall, \bar{Y}_1, \ldots, \bar{Y}_s, =)$.

Properties of $\mathfrak{A}^{\exists\forall}$

Definition

The structure $\mathfrak{A}^{\exists\forall}$ is obtained from $\mathfrak{A}$ as $(\mathfrak{A}^{\exists})^{\forall}$.

1. $R(a_1, \ldots, a_r) \iff$
   $(\exists x \in X)(\forall y \in Y) R^{\exists\forall}(a_1, \ldots, a_r, x, y)$;
2. $(\forall y \in Y)(\exists$ a unique sequence
   $a_1, \ldots, a_r \in A \;\&\; x \in X)(\neg R^{\exists\forall}(a_1, \ldots, a_r, x, y))$;
3. $(\forall x \in X)(\exists$ a unique sequence
   $a_1, \ldots, a_r \in A)(\forall y \in Y) R^{\exists\forall}(a_1, \ldots, a_r, x, y)$.

Join of Two Structures

Let $\mathfrak{A} = (A; R_1, \ldots, R_s, =)$ and $\mathfrak{B} = (B; P_1, \ldots, P_t, =)$ be countable structures.

The join of the structures $\mathfrak{A}$ and $\mathfrak{B}$ is the structure $\mathfrak{A} \oplus \mathfrak{B} = (A \cup B; R_1, \ldots, R_s, P_1, \ldots, P_t, \bar{A}, \bar{B}, =)$

(a) the predicate $\bar{A}$ is true only over the elements of A and similarly $\bar{B}$ is true only over the elements of B;

(b) the predicate R_i is defined on the elements of A as in the structure $\mathfrak{A}$ and false on all elements not in A and the predicate P_j is defined similarly.

Lemma

$\mathfrak{A} \preceq \mathfrak{A} \oplus \mathfrak{B}$ *and* $\mathfrak{B} \preceq \mathfrak{A} \oplus \mathfrak{B}$.

One-to-one Representation of $\Sigma_2^0(D)$ Sets

Let $D \subseteq \mathbb{N}$.
A set $M \subseteq \mathbb{N}$ is in $\Sigma_2^0(D)$ if there exists a computable in D predicate Q such that
$n \in M \iff \exists a \forall b Q(n, a, b)$.

Definition

If $M \in \Sigma_2^0(D)$ then M is *one-to-one representable* if there is a computable in D predicate Q with the following properties:

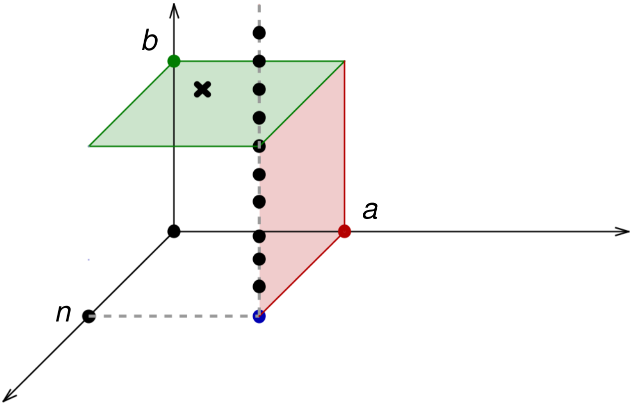
1. $n \in M \Leftrightarrow (\exists$ a unique $a)(\forall b) Q(n, a, b)$;
2. $(\forall b)(\exists$ a unique pair $\langle n, a \rangle)(\neg Q(n, a, b))$;
3. $(\forall a)(\exists$ a unique $n)(\forall b) Q(n, a, b)$.

Lemma (Goncharov and Khoussainov)

If M is a coinfinite $\Sigma_2^0(D)$ subset of $\mathbb{N}$ which has an infinite computable in D subset S such that $M \setminus S$ is infinite then M has an one-to-one representation.

One-to-one Representation of $\Sigma^0_2(D)$ Sets

1. $n \in M \Leftrightarrow (\exists \text{ a unique } a)(\forall b)Q(n, a, b)$;
2. $(\forall b)(\exists \text{ a unique pair } \langle n, a\rangle)(\neg Q(n, a, b))$;
3. $(\forall a)(\exists \text{ a unique } n)(\forall b)Q(n, a, b)$.



Theorem (Jump Inversion Theorem)

Let $\mathfrak{B}' \preceq \mathfrak{A}$. Then there exists a structure $\mathfrak{C}$ such that $\mathfrak{B} \preceq \mathfrak{C}$ and $\mathfrak{C}' \equiv \mathfrak{A}$.

- The structure $\mathfrak{C}$ is constructed as

$$\mathfrak{C} = \mathfrak{B} \oplus \mathfrak{A}^{\exists\forall}.$$

- $\mathrm{DS}_1(\mathfrak{C}) \subseteq \mathrm{DS}(\mathfrak{A})$.
  For each enumeration h of $\mathfrak{C}$ we construct an enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} h^{-1}(\mathfrak{C})'$.
- $\mathrm{DS}(\mathfrak{A}) \subseteq \mathrm{DS}_1(\mathfrak{C})$.
  For each enumeration $\bar{f}$ of $\mathfrak{A}$ there is a bijective enumeration f of $\mathfrak{A}$ such that $f^{-1}(\mathfrak{A}) \leq_e \bar{f}^{-1}(\mathfrak{A})$.
  We construct an enumeration h of $\mathfrak{C}$ such that $h^{-1}(\mathfrak{C})' \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$, using the one-to-one representation lemma.
- We use the fact that the degree spectra and the jump spectra are upwards closed with respect to total degrees

Some Applications

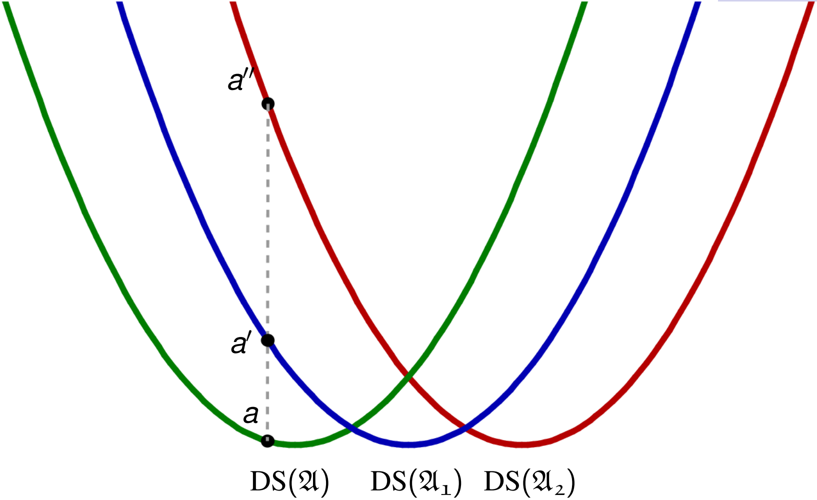
Definition

If **a** is the least element of $\mathrm{DS}(\mathfrak{A})$ then **a** is called the *degree of* $\mathfrak{A}$.

Proposition

Let $\mathfrak{B}' \preceq \mathfrak{A}$ *and suppose that the structure* $\mathfrak{A}$ *has a degree. Then there exists a torsion free abelian group* $\mathfrak{G}$ *of rank 1 which has a degree as well and such that* $\mathfrak{B} \preceq \mathfrak{G}$ *and* $\mathfrak{G}' \equiv \mathfrak{A}$.

The joint spectrum of $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$



The Joint Spectrum of Structures

Let $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ be countable structures.

Definition

The joint spectrum of $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set
$\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) =$
$\{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}), \mathbf{a}' \in \mathrm{DS}(\mathfrak{A}_1), \dots, \mathbf{a}^{(\mathbf{n})} \in \mathrm{DS}(\mathfrak{A}_n)\}$.

Corrolary

Let $\mathfrak{B}' \preceq \mathfrak{A}$. There exists a structure $\mathfrak{C} \succeq \mathfrak{B}$ such that $\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \mathrm{DS}_1(\mathfrak{C}, \mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

Relative Spectra of Structures

Definition

An enumeration f of $\mathfrak{A}$ is *n-acceptable with respect to the structures* $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, if $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} (f^{-1}(\mathfrak{A}))^{(i)}$ for each $i \leq n$.

Definition

The relative spectrum of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set
$\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) =$
$\{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f$ is a n-acceptable enumeration of $\mathfrak{A}\}$.

Proposition

Let $\mathfrak{B}' \preceq \mathfrak{A}$. *There exists a structure* $\mathfrak{C} \succeq \mathfrak{B}$ *such that* $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \mathrm{RS}_1(\mathfrak{C}, \mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

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